Thompson's Group is Probably **NOT** Amenable

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The aim of this work is to compute as much of the cogrowth series of Thompson's group as we can, then analyse the sequence to determine whether Thompson's group F seems to be amenable.

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If the sequence $a_1, a_2, \ldots$ behaves like $a_n \sim \mu^n n^g$, then the ratios $r_n = a_{n+1}/a_n$ will behave like

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This is the case for nice groups such as $\mathbb{Z}^k$.

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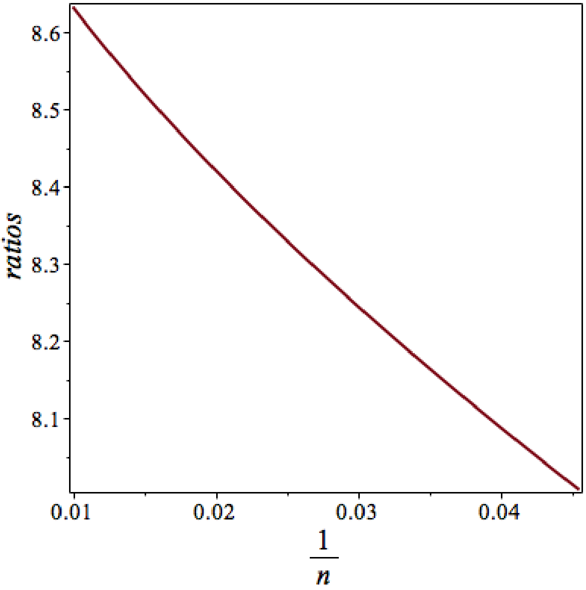

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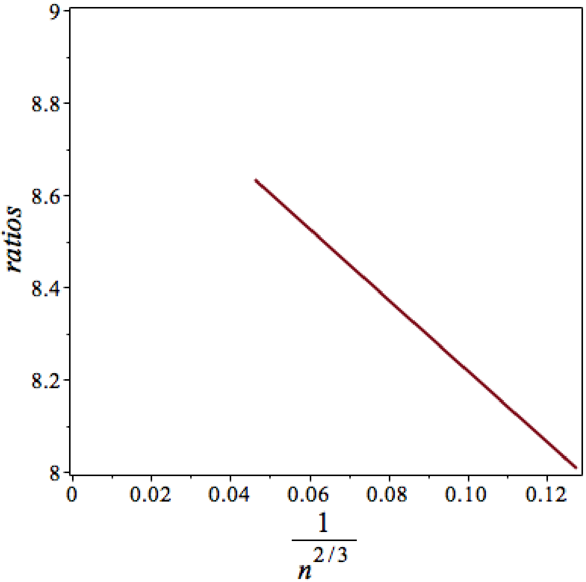

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We did the same analysis for the Baumslag-Solitar group $BS(1, 2)$, and it is equally straight forward to detect the behaviour of the cogrowth sequence.

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As with the lamplighter group, we plot the ratios against $\frac{1}{n^{1-\sigma}}$ for different value of σ:

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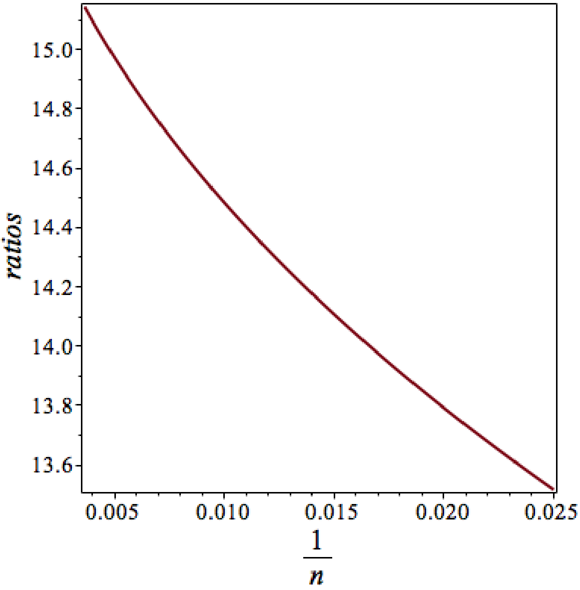

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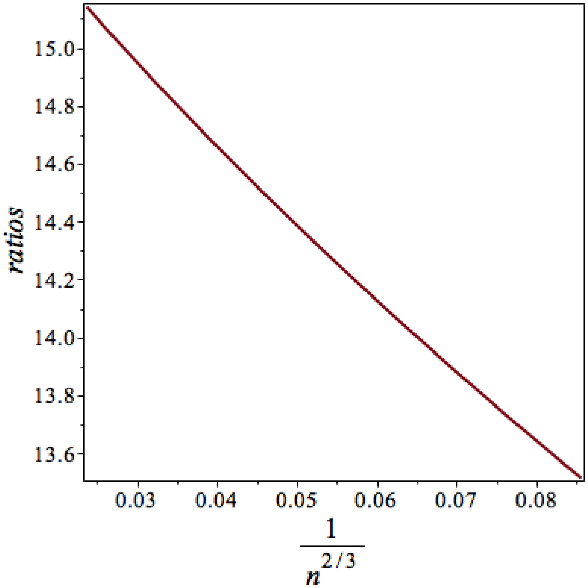

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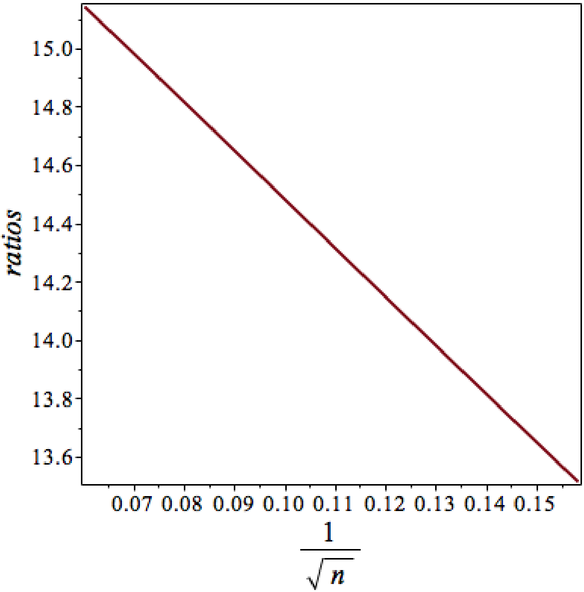

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Since the graph of the ratios plotted against $1/\sqrt{n}$ is convex and the graph against $1/n^{2/3}$ is concave, we might guess that

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$$r_n^{(1)} = 1 - \frac{(\sigma - 1)\log \kappa}{n^{2-\sigma}} + O(1/n^2).$$

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Now we construct modified ratios of ratios to eliminate the $O(1/n^2)$ term:

$$r_n^{(2)} = \frac{n^2 r_n^{(1)} - (n-1)^2 r_{n-1}^{(1)}}{2n-1} = 1 + \frac{(\sigma - 1)\log \kappa}{n^{2-\sigma}} + o(1/n^2).$$

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Then the plot of $\log(r_n^{(2)} - 1)$ against $\log(n)$ should be linear, with gradient $\sigma - 2$.
Taking the local gradients of this plot gives us an estimate of $\sigma - 2$ for each value of n, so we plot these estimates against $1/n$.

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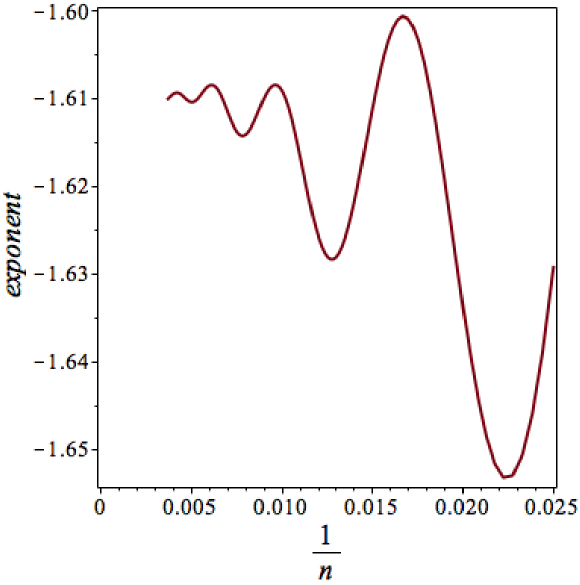

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Including this in our analysis, we get

$$r_n^{(2)} = 1 + \frac{c_1 (\log n)^{2/3}}{n^{2-\sigma}} + O\left(\frac{1}{n^{2-\sigma}(\log n)^{1/3}}\right).$$

Including the $(\log n)^{2/3}$ term in our estimates of $\sigma - 2$, we get a new plot:

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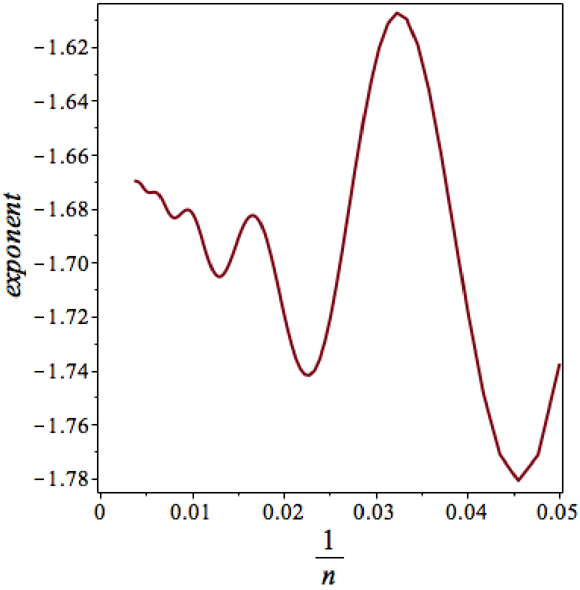

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Now, assuming that

$$a_n \sim \mu^n \kappa^{n^{1/3}(\log n)^{2/3}} n^g,$$

we get

$$r_n = \mu \left(1 + \frac{\log \kappa (\log n)^{2/3}}{3n^{2/3}} + \frac{2 \log \kappa}{3n^{2/3}(\log n)^{1/3}} + \frac{g}{n} + o(1/n) \right).$$

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We show the plots of the esimates for each value of n:

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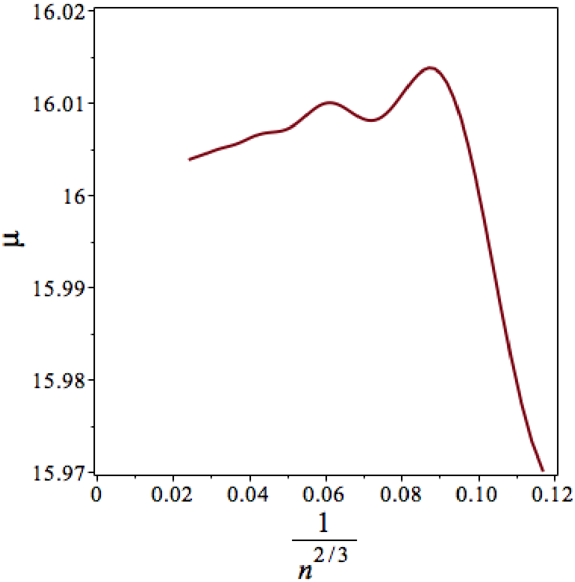

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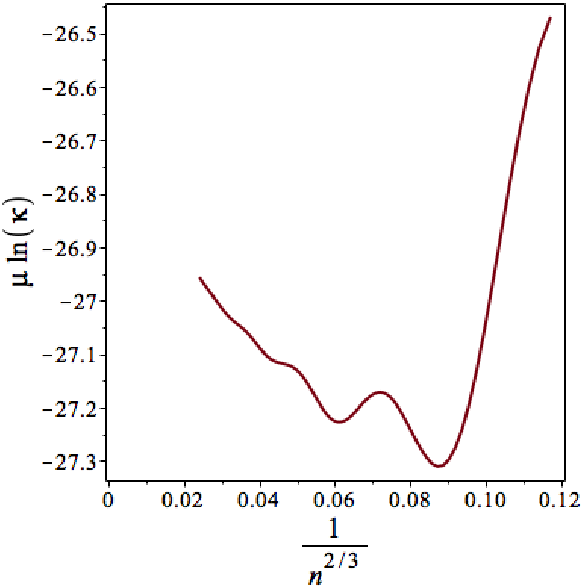

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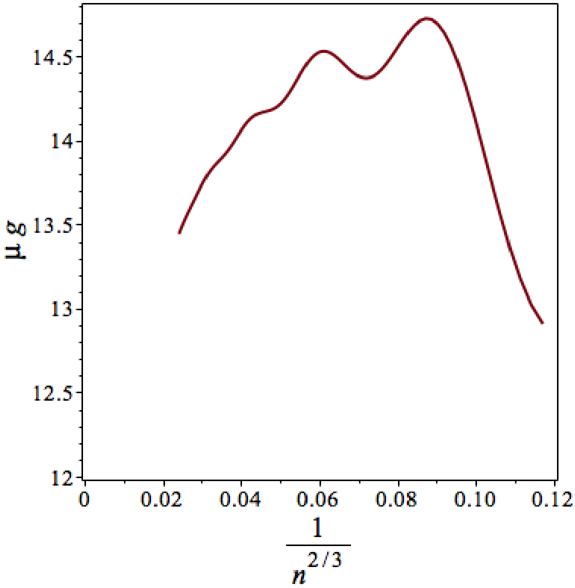

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We also get $\mu \log \kappa \approx -26.7$ and $\mu g \approx 10$, so $\kappa \approx 0.19$ and $g \approx \frac{5}{8}$.

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We use the method of differential approximants to estimate the next 100 terms before analysing the sequence further.

Differential approximants

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- Calculate the unique polynomials $P, Q_0, Q_1, \ldots, Q_M$ (up to scaling) of degrees $L, M, d_0, \ldots, d_M$ such that the first 32 coefficients of

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- Approximate F by the solution $\tilde{F}$ of

$$\sum_{k=0}^{M} Q_k(x) \left(x \frac{d}{dx} \right)^k \tilde{F}(x) = P(x)$$

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Surprisingly, these estimates generally seem to be very accurate. We give the equivalent result for $\mathbb{Z} \wr \mathbb{Z}$ to justify this method:

Differential approximant results for $\mathbb{Z} \wr \mathbb{Z}$.

Using only the terms for $n \leq 31$, we approximate the next 80 ratios. The left column gives the actual error of this approximation, and the right column give the estimated error.

n	Actual error	1 standard deviation
1	2.69×10^{-17}	2.02×10^{-17}
5	1.14×10^{-13}	7.85×10^{-14}
10	3.37×10^{-11}	2.08×10^{-11}
20	2.22×10^{-8}	1.23×10^{-8}
30	9.63×10^{-7}	5.39×10^{-7}
40	1.22×10^{-5}	6.88×10^{-6}
50	7.59×10^{-5}	4.73×10^{-5}
60	3.13×10^{-4}	2.23×10^{-4}
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The actual error is consistently less than twice the estimate error.

Differential approximant results for Thompson's group F.

Using only the terms for $n \leq 31$, we approximate the next 100 ratios. The left column gives the estimated value and right column give the estimated error.

n	Estimated term	1 standard deviation
1	12.1393	4.47×10^{-20}
10	12.3773	3.76×10^{-14}
20	12.5722	2.43×10^{-9}
30	12.7224	1.25×10^{-8}
40	12.8433	2.02×10^{-7}
50	12.9437	1.85×10^{-6}
60	13.02893	1.11×10^{-6}
80	13.16718	2.19×10^{-5}
100	13.2756	2.17×10^{-4}

Analysis of Thompson's group F

As we did for $\mathbb{Z}_2 \wr \mathbb{Z}$ and $\mathbb{Z} \wr \mathbb{Z}$, we plot all 132 ratios against $\frac{1}{n^{1-\sigma}}$ for different values of σ:

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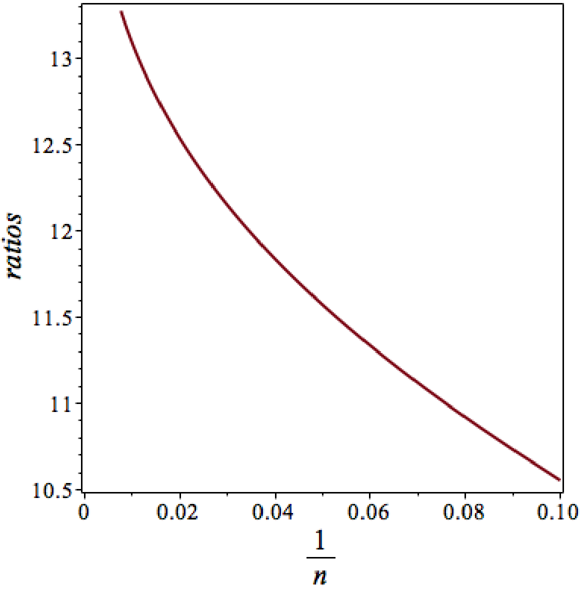

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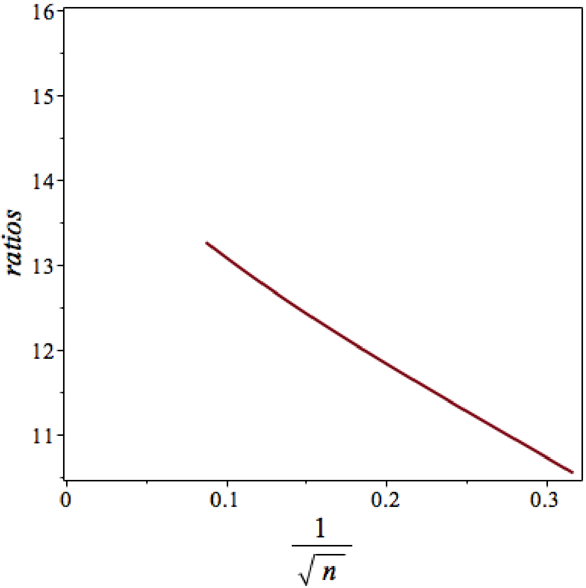

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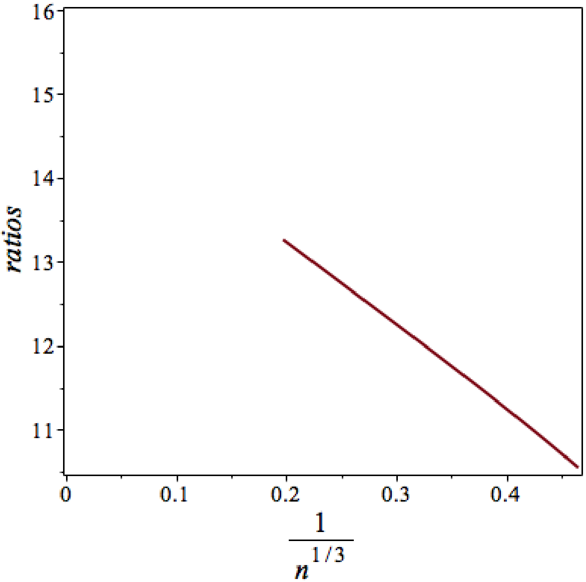

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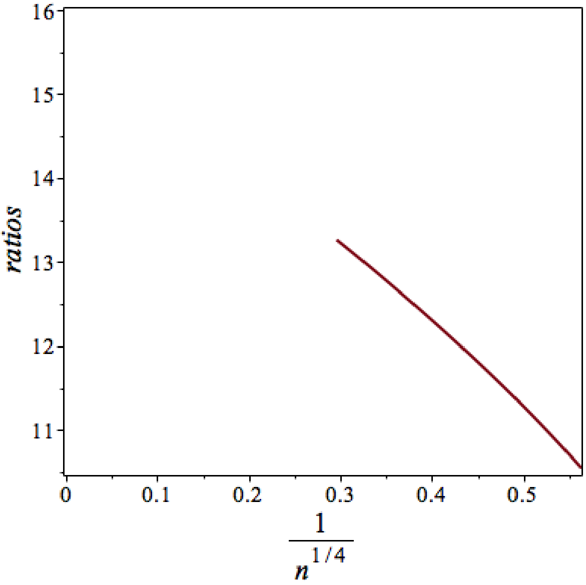

Analysis of Thompson's group F

Again, none of these graphs are linear, so if

$$a_n \sim \mu^n \kappa^{n^\sigma} n^g,$$

then we do not have a clear value for σ.
To estimate the value of σ, we take modified ratios of ratios $r_n^{(2)}$, as we did for $\mathbb{Z} \wr \mathbb{Z}$, then take the local gradients of the graph of $\log(r_n^{(2)} - 1)$ against $\log(n)$ to estimate $\sigma - 2$.
This analysis amplifies any inaccuracy in the terms, so we only use $n \leq 75$.

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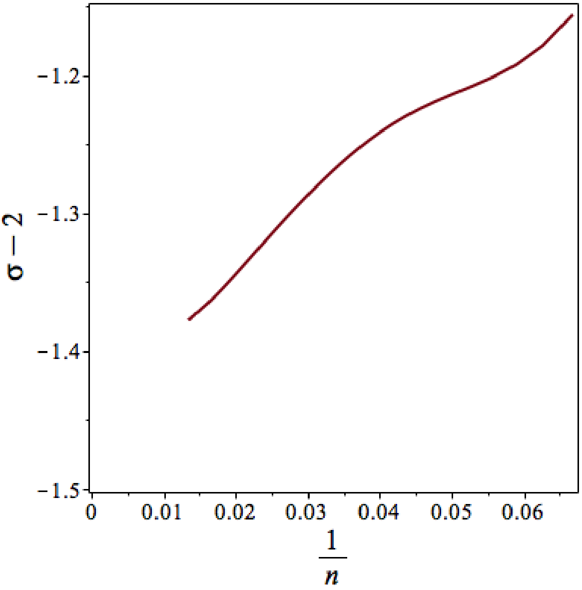

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If we assume instead that the terms behave like

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then we get the following estimates of σ:

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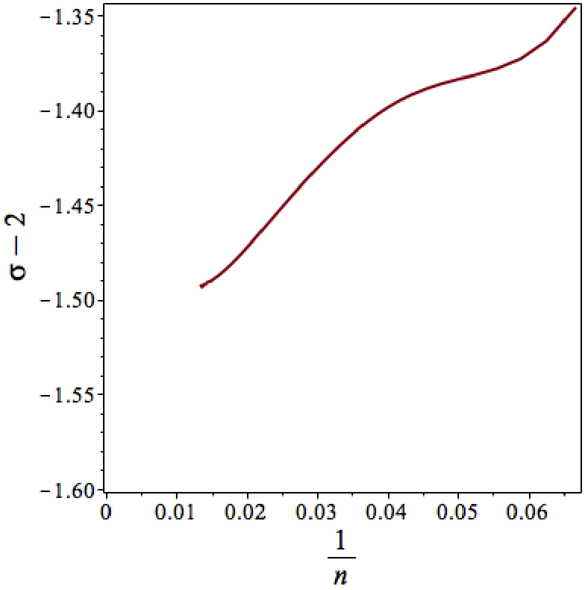

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we get

$$r_n = \mu \left(1 + \frac{\log \kappa (\log n)^c}{2n^{1/2}} + \frac{c \log \kappa}{n^{1/2}(\log n)^{1-c}} + \frac{g}{n} + o(1/n) \right).$$

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We show the plots of the esimates for $c = 0$, and varying n:

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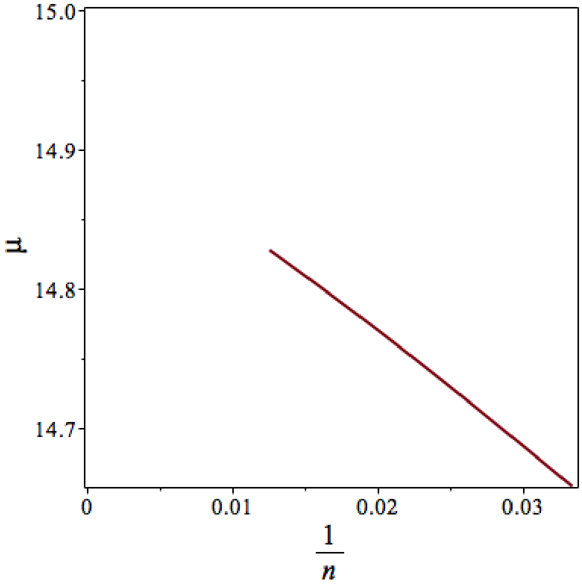

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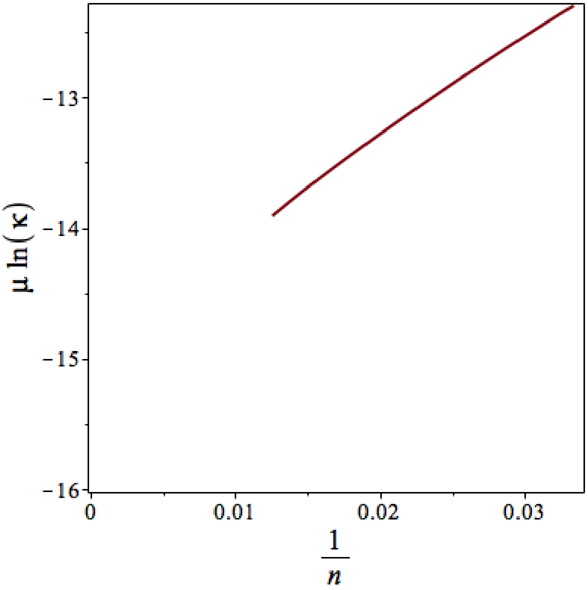

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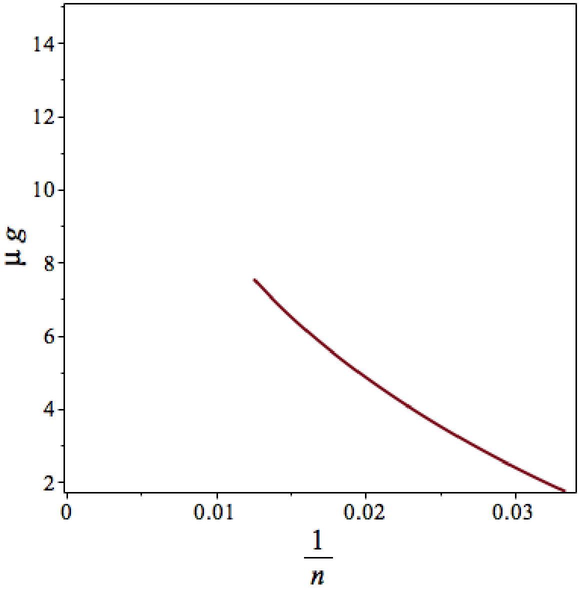

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For other values of c we get similar values of μ, all well below 16, which would be needed for amenability.

Some rigourous analysis

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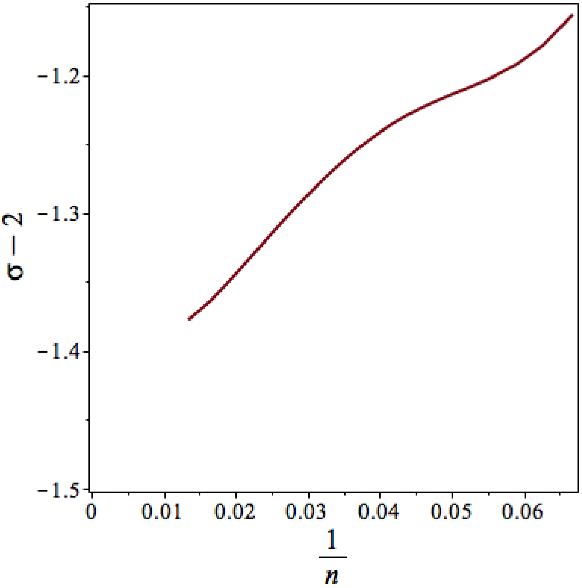
In fact, it follows that if Thompson's group is amenable, then our earlier estimates of $\sigma - 2$ for Thompson's group must converge to -1, if they converge to anything.

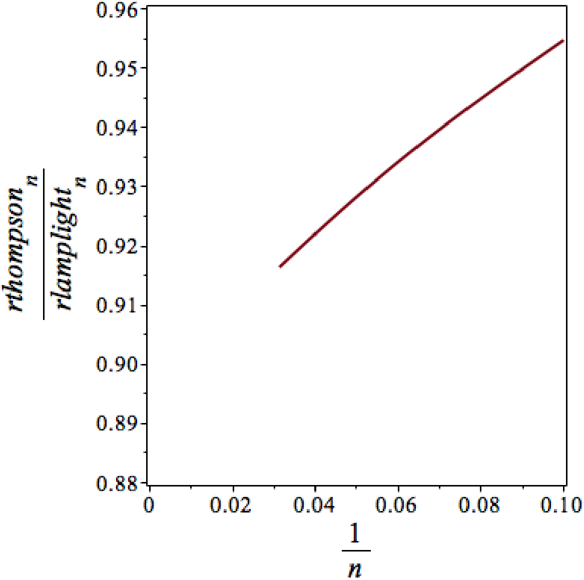
Estimates of σ for Thompson's group F



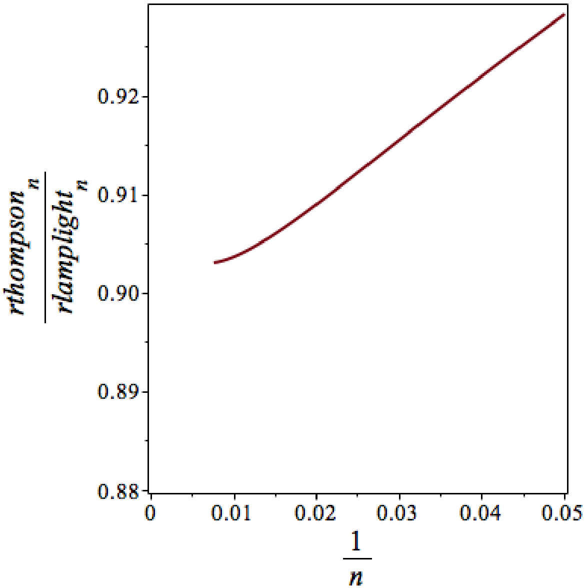
Analysis of Thompson's group F

For one final test, we look at the the ratio of the nth ratio r_n for Thompson's group over the nth ratio s_n for the group $\mathbb{Z} \wr \mathbb{Z}$. Since the ratios s_n converge to 16, Thompson's group F is amenable if and only if this ratio r_n/s_n converges to 1. We plot these ratios against 1/n:

Comparison of Thompson's group F with $\mathbb{Z} \wr \mathbb{Z}$ using the 32 known terms



Comparison of Thompson's group F with $\mathbb{Z} \wr \mathbb{Z}$ using all 132 terms



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Further questions:

- How does the cogrowth sequence for Thompson's group really behave? The conclusion that Thompson's group is not amenable would be somewhat more convincing if we could confidently say exactly how the cogrowth sequence really is behaving.
- Can we get more confident about this by using our methods in conjuction with methods for approximating a lot more of the coefficients?

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Each loop of length $2n$ can be seen as two paths of length n to the same vertex, so the number of loops of length $2n$ is equal to the sum of the squares $p_n(v)^2$ over all vertices v in Γ.

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So, during the algorithm we calculate the number $p_n(v)$ of paths to each vertex v within the ball of radius n.

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One way to compute each $p_n(v)$ is to first compute $p_{n-1}(u)$ for each vertex u in the ball of radius u, then calculate each $p_n(v)$ using those terms. Unfortunately, this becomes essentially impossible for $n > 24$ due to memory usage.

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Then to compute $p_n(v)$ we sum the values of $p_k(u)p_{n-k}(u^{-1}v)$ for each vertex u which is in the ball of radius k and such that $u^{-1}v$ is in the ball of radius $n-k$.

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For large n, there are on average only about 8 such vertices u, so the algorithm will be reasonably fast as long as we can quickly find all of these vertices u for each v.

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This algorithm parallelises very easily, so we ran it on the University of Melbourne's new high performance computer, Spartan. to calculate t_{31}, it ran for about two weeks on about 150 cores.

Thank you