Written Exam
Introduction to Linear and Integer Programming (DM515)

Institut for Matematik & Datalogi
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Thursday, June 24, 2010, 9:00–13:00, U26

All usual helping tools (textbooks, lecture notes, etc.) together with pocket calculators are allowed. It is not allowed to use computers, smartphones and personal digital assistants.

The exam consists of 6 tasks and relative subtasks distributed on 10 pages.

The weight in the evaluation of each task is given in percent. Higher weights are set on tasks that constitute necessary minimal knowledge to pass the exam. Weigths are not representative of the difficulty of the task.

Remember to justify all your statements. You may refer to results from the textbooks or the lecture notes in the syllabus. In particular, it is possible to justify a statement by saying that it derives trivially from a result in the textbook (if this is true!). You may use all methods or extensions that have been used in the assignment sheets, published during the course. However, it is not allowed to answer a subtask exclusively by reference to an exercise seen during the course. Reference to other books (outside the course material) is not accepted as answer to a task!

You may write your answers in Danish or in English.

Task 1 (25%)

Consider the following IP problem:

$$\begin{aligned} \max \quad & 4x_1 + 7x_2 \\ s.t. \quad & x_1 + 3x_2 \leq 12 \\ & 4x_1 + 6x_2 \leq 27 \\ & 4x_1 + 2x_2 \leq 20 \\ & x_1, x_2 \geq 0, x_1, x_2 \in \mathbb{Z} \end{aligned} \tag{1}$$

Subtask a

Give a heuristic primal bound and describe how you determined it.

Subtask b

Write the LP relaxation (1lp) of (1) to obtain a dual bound. Explain the relation between the optimal solution of (1lp) and the optimal solution of (1).

Subtask c

Write the first simplex tableau of (1lp) and indicate which variables constitute a basic solution. Call s_1, s_2, s_3 the slack variables.

Subtask d

Explain which variable leaves the basis and which variable enters the basis in the first iteration of the simplex algorithm with largest coefficient pivot rule. Show that the answer would be the same if, instead, the largest increase pivot rule was used.

Subtask e

After a number of iterations the tableau is the following:

x1	x2	s1	s2	s3	-z	b
0	1	2/3	-1/6	0	0	7/2
1	0	-1	1/2	0	0	3/2
0	0	8/3	-5/3	1	0	7
0	0	-2/3	-5/6	0	1	-61/2

Argue that an optimal solution for (1lp) has been found and give for it the value of x_1 and x_2 together with its objective function value. Report the optimality gap for (1) at this stage.

Subtask f

From the second row of the last tableau derive a Gomory cut and write it in the space of the original variables.

Argue shortly that the cut is a valid inequality for (1) and that it will make the current optimal solution of (1lp) infeasible.

Subtask g

Introduce the cut in the tableau and explain how the solution algorithm will continue. Indicate the new pivot and explain how you found it. (You do not need to carry out the simplex iteration.)

Subtask h

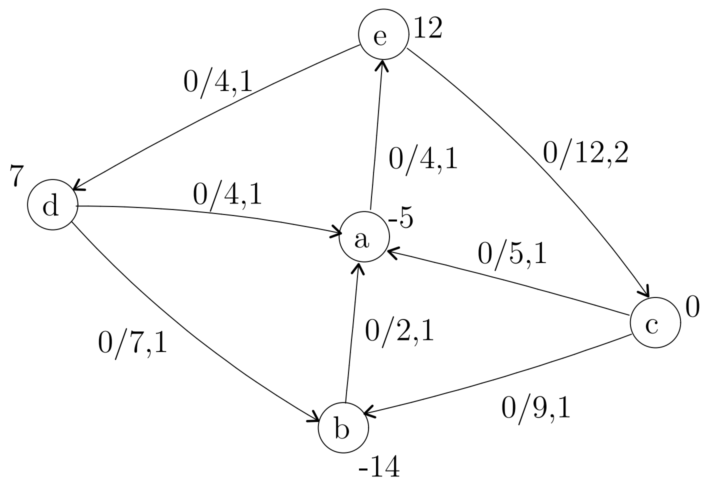
After the introduction of the cut the tableau of the optimal solution to the new LP problem is the following.

x1	x2	s1	s2	s3	s4	-z	b
0	1	2/3	0	0	-1/3	0	11/3
0	0	0	1	0	-2	0	1
0	0	8/3	0	1	-10/3	0	26/3
1	0	-1	0	0	1	0	1
0	0	-2/3	0	0	-5/3	1	-89/3

Explain how the solution process would continue from this stage by branch and bound. Define the next branching and indicate what can be done in each open node.

Task 2 (25%)

Let $\mathcal{N} = (V, A, l, u, c, b)$ be the network depicted below. Numbers on vertices correspond to the balance values, numbers on each arc ij correspond to $l_{ij}/u_{ij}, c_{ij}$.



Subtask a

Find a feasible flow. Explain how you found it and give evidence of its feasiblity. (Hint: solve a network flow problem after an opportune network transformation).

Subtask b

Let x be the flow found at the previous point (or, in case you did not solved the previous point, let x be any feasible flow such as the one made by $x_{ed} = 3, x_{da} = 4, x_{db} = 6, x_{ae} = 0, x_{ba} = 1, x_{ca} = 0, x_{cb} = 9, x_{ec} = 9$). Draw the residual network $\mathcal{N}(x)$ and evaluate the cost of the flow.

Subtask c

Starting from x find the minimum cost flow x^* in $\mathcal{N}$. Draw x^* in $\mathcal{N}$ and give its value.

Task 3 (20%)

The matrix description of the simplex method is at the basis of the *revised simplex method* that is implemented in computer programs for solving LP problems. In the revised simplex method the iterations are not carried out on the tableau but, instead, faster updates are performed by keeping the basis matrix and its inverse.

For the LP problem

$$\begin{aligned} \max \quad & x_1 + 2x_2 + x_3 + x_4 \\ & 2x_1 + x_2 + 5x_3 + x_4 \le 8 \\ & 2x_1 + 2x_2 + 4x_4 \le 12 \\ & 3x_1 + x_2 + 2x_3 \le 18 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned} \tag{2}$$

the inverse of the basis matrix for the basic variables $\mathbf{x}_B = (x_3, x_2, s_3)$ and non-basic variables $\mathbf{x}_N = (x_1, x_4, s_1, s_2)$, where s_1, s_2, s_3 are slack variables, is

$$A_B^{-1} = \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0 & 0.5 & 0 \\ -0.4 & -0.3 & 1 \end{bmatrix}$$

[Added: January 31, 2011. Due to an error, the one given is in fact the transpose of the inverse.]

Subtask a

Using only matrix operations, like in the revised simplex method, show that the given basis structure corresponds to an optimal solution and give the value of its variables.

Subtask b

What is the value of the simplex multipliers and of the optimal solution of the dual problem?

Subtask c

Are all constraints "binding", or is there slack capacity in some of them? Answer this question in the light of the complementary slackness theorem.

Subtask d

Explain how you determine whether the basis (x_3, x_2, s_3) remains optimal if the second constraint's right-hand side is changed to 26. In case it remains optimal, determine the change in the optimal objective value.

Subtask e

What will be the new basis structure if the objective function is changed to

$$3x_1 + 2x_2 + x_3 + x_4?$$

Explain how you determine the entering and the leaving variables.

Subtask f

A new variable is introduced with coefficients of 3, 5, 6 in the first, second and third constraint, respectively. Determine what cost coefficient should the new variable have in order to have a change in the structure of the optimal basis.

Task 4 (15%)

Subtask a

Prove that the polyhedron $P = \{(x_1, \ldots, x_m, y) \in \mathbb{R}^{m+1} : y \leq 1, x_i \leq y \text{ for } i = 1, \ldots, m\}$ has integer vertices. [Hint: start by writing the constraint matrix.]

Subtask b

Consider the following (integer) linear programming problem:

$$\begin{aligned}
\min \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_3x_4 \\
& x_3 + x_4 \geq 10 \\
& x_2 + x_3 + x_4 \geq 20 \\
& x_1 + x_2 + x_3 + x_4 \geq 30 \\
& x_2 + x_3 \geq 15 \\
& x_1, x_2, x_3, x_4 \in \mathbb{Z}_0^+
\end{aligned} \tag{3}$$

The constraint matrix has consecutive 1's in each column. Matrices with consecutive 1's property for each column are totally unimodular. Show that this fact holds for the specific numerical example (3). [Hint: rewrite the problem in standard form (that is, in equation form) and add a redundant row $\mathbf{0} \cdot \mathbf{x} = 0$ to the set of constraints. Then perform elementary row operations to bring the matrix to a known form.] What algorithm beside the simplex could be used to solve the problem?

Subtask c

Use one of the two previous results to show that the problem in the next page can be solved efficiently when formulated as a mathematical programming problem. (You do not need to find numerical results.)

Shift scheduling. The administrators of a department of a urban hospital have to organize the working shifts of nurses maintaining sufficient staffing to provide satisfactory levels of health care. Staffing requirements at the hospital during the whole day vary from hour to hour and are reported in the table:

Hour	Staffing requirement
0 am to 6 am	2
6 am to 8 am	8
8 am to 11 am	5
11 am to 2 pm	7
2 pm to 4 pm	3
4 pm to 6 pm	4
6 pm to 8 pm	6
8 pm to 10 pm	3
10 pm to 12 pm	1

According to union agreements, nurses can work according to one of the seven shift patterns below each with its own cost

pattern	Hours of work	total hours	cost
1	0 am to 6 am	6	720 Dkk
2	0 am to 8 am	6	800 Dkk
3	6 am to 2 pm	8	740 Dkk
4	8 am to 4 pm	8	680 Dkk
5	2 pm to 10 pm	8	720 Dkk
6	4 pm to 12 pm	6	780 Dkk
7	6 pm to 12 pm	6	640 Dkk

The department administrators would like to identify the assignment of nurses to working shifts that meets the staffing requirements and minimizes the total cost.

Task 5 (10%)

Given the following LP problem:

$$
\begin{array}{rl}
\max & z = x_1 - 2x_2 \\
\text{s.t.} & x_1 + 2x_2 - x_3 \geq 0 \\
& 4x_1 + 3x_2 + 4x_3 \leq 3 \\
& 2x_1 - x_2 + 2x_3 = 1 \\
& x_2, x_3 \geq 0
\end{array}
$$

Subtask a

Derive the dual problem using the Lagrange multipliers approach and report the steps of the derivation.

Task 6 (5%)

A car rental company at the beginning of each month wants to have a certain number of cars in each of the towns in which it operates. For the towns $A, B, C, \ldots, G$ the number of cars desired is $30, 40, 55, 60, 80, 40, 55$, respectively. At the end of the current month there are instead in the stations in these towns $65, 90, 95, 15, 60, 10, 25$ cars, respectively. To move one car from one station to the other causes a cost that we may assume proportional to the distance between the two stations. The table indicates the distances (in hundreds of kilometers) between every city pair of stations.

from .. to..	D	E	F	G
A	5	6	10	9
B	9	11	9	15
C	12	10	14	15

Formulate the problem of deciding the cars to move while minimizing the costs in mathematical programming terms. Which algorithm could you use to solve the problem beside the simplex?