Newton og Leibniz

Opgave

I 1671 skrev Newton sin "Methodus Fluxionum et Serierum Infinitarum". Nedenfor gengives uddrag fra den engelske oversættelse fra 1745.

1) Læs afsnittet indtil Problem 1.
   Bemærk, at bevægelserne er lineære bevægelser, og at tiden måles ved et jævnt bevæget punkts bevægelse.
   Hvordan defineres $\dot{x}$, $\dot{y}$ etc. ?
   Hvad er de to basale problemer ?

2) Læs Problem 1. og Eksempel 1 (eksemplet vil afmystifisere Problem 1.)
   Vis, at Newton's regel svarer til:

$$f(x,y) = 0 \Rightarrow \frac{\partial}{\partial x} f(x,y) \frac{dx}{dt} + \frac{\partial}{\partial y} f(x,y) \frac{dy}{dt} = 0 ,$$

og argumenter for, at denne sætning er korrekt.

3) Gengiv Beviset for Problem 1.
   Påpeg problemer heri.
   Hvorfra tror du Newton har hentet bevismetoden ?

4) Vis, at Newton's metode til at finde subtangenten TB = t, svarer til

$$t = y \cdot \frac{dx}{dy} .$$

Bemærk, at

$$TB : BD :: Dc : cd \text{ betyder } \frac{TB}{BD} = \frac{Dc}{cd} .$$

5) Gengiv Problem IX.
   Hvilken hovedsætning formuleres her ?

5-2

Kilde XII

A

TREATISE

OF THE

METHOD of FLUXIONS

AND

INFINITE SERIES,

With its Application to the Geometry of CURVE LINES.

By Sir ISAAC NEWTON, Kt.

Tranflated from the *Latin* Original not yet publiſhed.

Deſigned by the AUTHOR for the Uſe of LEARNERS.

Hac via inſiſtendum eſt.

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MDCCXXXVII.

Tranſition to the Method of FLUXIONS.

And thus much for the Methods of Computation, of which I ſhall make frequent uſe in what follows. Now it remains, that for an illuſtration of the Analytic Art, I ſhould give ſome ſpecimens of Problems, eſpecially ſuch as the nature of Curves will ſupply. Now in order to this, I ſhall obſerve that all the difficulties hereof may be reduced to theſe two Problems only, which I ſhall propoſe, concerning a Space deſcrib'd by local Motion, any how accelerated or retarded.

5-3

Transition to the Method of FLUXIONS.

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I. *The length of the Space deſcrib'd being continually (that is, at all times) given; to find the velocity of the motion at any time propos'd.*

II. *The velocity of the motion being continually given; to find the length of the Space deſcrib'd at any time propos'd.*

Thus in the Equation $xx=y$, if y repreſents the length of the Space at any time deſcrib'd, which (time) another Space x, by increaſing with an uniform celerity $\dot{x}$, meaſures and exhibits as deſcrib'd: then $2x\dot{x}$ will repreſent the celerity, by which the Space y at the ſame moment of time proceeds to be deſcrib'd, and contrariwiſe. And hence it is, that in what follows I conſider things as generated by continual Increaſe, after the manner of a Space, which a thing or point in motion deſcribes.

But ſince we do not conſider the time here, any farther than as it is expounded and meaſured by an equable local motion; and beſides whereas things only of the ſame kind can be compar'd together, and alſo their velocities of increaſe and decreaſe: therefore in what follows I ſhall have no regard to time formally conſider'd, but ſhall ſuppoſe ſome one of the quantities propos'd, being of the ſame kind, to be increas'd by an equable Fluxion, to which the reſt may be refer'd, as it were to time; and therefore by way of analogy it may not improperly receive the name of Time. Whenever therefore the word *Time*, occurs in what follows, (which for the ſake of perſpicuity and diſtinction I have ſometimes uſed,) by that word I would not have it underſtood as if I meant Time in its formal acceptation, but only that other quantity, by the equable increaſe or fluxion whereof, Time is expounded and meaſured.

Now thoſe quantities which I conſider as gradually and indefinitely increaſing, I ſhall hereafter call *Fluents*, or *flowing Quantities*, and ſhall repreſent them by the final letters of the alphabet v, x, y, and z; that I may diſtinguiſh them from other quantities, which in equations may be conſidered as known and determinate, and which therefore are repreſented by the initial letters a, b, c, &c. And the velocities by which every Fluent is increaſed by its generating motion (which I may call *Fluxions*, or ſimply Velocities, or Celerities,) I ſhall repreſent by the ſame letters pointed thus, $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$; that is, for the celerity of the quantity v I ſhall put $\dot{v}$, and ſo for the celerities of the other Quantities x, y, and z, I ſhall put $\dot{x}$, $\dot{y}$, and $\dot{z}$, reſpectively. Theſe things being premis'd, I ſhall now forthwith proceed to the matter in hand; and firſt I ſhall give the ſolution of the two Problems juſt now propos'd.

PROBLEM I.

The Relation of the flowing Quantities to one another being given, to determine the Relation of their Velocities.

SOLUTION. Diſpoſe the equation, by which the given Relation is expreſs'd, according to the dimenſions of ſome one of its flowing Quantities, ſuppoſe x, and multiply its terms by any arithmetical progreſſion, and then by $\frac{\dot{x}}{x}$; and perform this operation ſeparately for every one of the flowing Quantities. Then make the ſum of all the products equal to nothing, and you will have the equation required.

EXAMPLE I. If the relation of the flowing quantities x and y be $x^3 - ax^2 + axy - y^3 = 0$; first dispose the terms according to the dimensions of x, and then according to y, and multiply them in the following manner.

$$\begin{array}{ll|l}
\text{Mult.} & x^3 - ax^2 + axy - y^3 & -y^3 + axy \begin{smallmatrix} -ax^2 \\ +x^3 \end{smallmatrix} \\
\text{by} & \frac{3\dot{x}}{x} \cdot \frac{2\dot{x}}{x} \cdot \frac{\dot{x}}{x} \cdot 0 & \frac{3\dot{y}}{y} \cdot \frac{\dot{y}}{y} \cdot 0 \\
\hline
\text{makes} & 3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y \quad * & -3\dot{y}y^2 + a\dot{y}x \quad *
\end{array}$$

the sum of the products is $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$, which equation gives the relation between the Fluxions $\dot{x}$ and $\dot{y}$. For if you take x at pleasure, the equation $x^3 - ax^2 + axy - y^3 = 0$ will give y; which being determin'd, it will be $\dot{x} : \dot{y} :: 3y^2 - ax : 3x^2 - 2ax + ay$.

$= \dot{x} \times BD$, or $= \dot{x}\sqrt{ax - xx}$, substitute this value instead of it, and there will arise the equation $\overline{2x\dot{x} + a\dot{x}x}\sqrt{ax - xx} + a\dot{x}z - 4\dot{y}y^3 = 0$, which determines the relation of the celerities $\dot{x}$ and $\dot{y}$.

DEMONSTRATION *of the Solution.*

The Moments of flowing quantities (*i. e.* their indefinitely small parts, by the accession of which, in indefinitely small portions of time they are continually increas'd) are as the velocities of their flowing or increasing. Wherefore if the moment of any one, as x, be represented by the product of its celerity $\dot{x}$ into an indefinitely small quantity o, (*i. e.* by $\dot{x}o$,) the moments of the others v, y, and z, will be represented by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$; because $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$, are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$. Now since the moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little accessions of the flowing quantities x and y, by which those quantities are increased through the several indefinitely small intervals of time; it follows that those quantities x and y after any indefinitely small interval of time, become $x + \dot{x}o$ and $y + \dot{y}o$: and therefore the equation which at all times indifferently expresses the relation of the flowing quantities, will as well express the relation between $x + \dot{x}o$ and $y + \dot{y}o$, as between x and y: so that $x + \dot{x}o$ and $y + \dot{y}o$, may be substituted in the same equation for those quantities, instead of x and y.

Therefore let any equation $x^3 - ax^2 + axy - y^3 = 0$ be given, and substitute $x + \dot{x}o$ for x, and $y + \dot{y}o$ for y, and there will arise

x^3

$$\begin{array}{l}
x^3 + 3\dot{x}ox^2 + 3\dot{x}^2oox + \dot{x}^3o^3 \\
-ax^2 - 2a\dot{x}ox - a\dot{x}^2oo \\
+axy + a\dot{x}oy + a\dot{y}ox + a\dot{x}\dot{y}oo \\
-y^3 - 3\dot{y}oy^2 - 3\dot{y}^2ooy - \dot{y}^3o^3 = 0.
\end{array}$$

Now by supposition $x^3 - ax^2 + axy - y^3 = 0$; which therefore being expung'd, and the remaining terms divided by o, there will remain $3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3oo - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + a\dot{y}x + a\dot{x}\dot{y}o - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3oo = 0$. But whereas o is suppos'd to be indefinitely little, that it may represent the moments of quantities, consequently the terms that are multiplied by it, will be nothing in respect of the rest: therefore I reject them, and there remains $3x^2\dot{x} - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$, as above in Example I.

Here it may be observed, that the terms which are not multiplied by o will always vanish; as also those terms that are multiplied by more than one dimension of o; and that the rest of the terms being divided by o, will always acquire the form that they ought to have by the foregoing rule. Q. E. D.

This being done the other things inculcated in the rule will easily follow. As that in the propos'd equation, several flowing quantities may be involv'd; and that the terms may be multiply'd, not only by the number of the dimensions of the flowing quantities, but also by any other arithmetick progression, so that in the operation there may be the same difference of the terms according to any of the flowing quantities, and the progression be dispos'd according to some order of the dimensions of each of them. These things being allow'd, what is taught besides in Examples 3, 4, and 5, will be plain enough of itself.

PROBLEM II.

An Equation being propos'd including the Fluxions of Quantities, to find the Relation of those Quantities to one another.

A particular Solution.

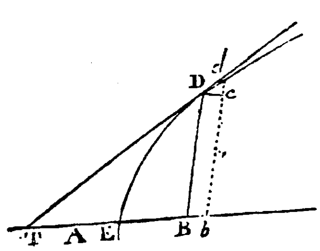
As this problem is the converse of the foregoing, it must be solv'd by proceeding in a contrary manner;

PROBLEM IV.

To draw Tangents to Curves.

The First manner.

Tangents may be variously drawn according to the various relation of curves to right lines: and first, let BD be a right line or ordinate in a given angle to another right line AB, as a base or abscis, and terminated at the curve ED; let this ordinate move thro' an indefinite small space to the place *bd*, so that it may be increased by the moment *cd*, while AB is increased by the moment B*b* to which DC is equal and parallel, let D*d* be produced till it meet with AB in T, and this line will touch the curve in D or *d*, and the triangles *dc*D, DBT



will be similar; so that TB : BD :: D*c*, or B*b* : *cd*. Since therefore the relation of BD to AB is exhibited by the equation by which the nature of the curve is determined, seek for the relation of the Fluxion by PROB. I. Then take TB to BD in the ratio of the Fluxion of AB to the Fluxion of BD, and TD will touch the curve in the point D.

EXAMPLE I. Calling AB$=x$ and BD$=y$, let their relations be $x^3-ax^2+axy-y^3=0$, and the relation of the Fluxion will be $3\dot{x}x^2-2a\dot{x}x+a\dot{x}y-3\dot{y}y^2+a\dot{y}x=0$, so that $\dot{y}:\dot{x}::3xx-2ax+axy+ay:3y^2-ax$ :: BD or (y) : BT. Therefore BT $=\frac{3y^3-axy}{3x^2-2ax+ay}$; therefore the point D being given, and thence DB and AB, or y and x, the length will be given by which the tangent TD is determined.

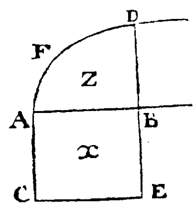
But this method of operation may be thus concinnated: make the terms of the proposed equation equal to nothing, then multiply by the proper number of the dimension of the ordinate, and put the result in the numerator; then multiply the same equation by the proper number of the dimensions of the abscis, and put the product divided by the abscis in the denominator of the value of BT; then take BT towards A if this value be affirmative, but the contrary way if the value be negative.

Thus the equation $\overset{0}{\underset{3}{x^3}}-\overset{0}{\underset{2}{ax^2}}+\overset{1}{\underset{1}{axy}}-\overset{3}{\underset{0}{y^3}}=0$, being multiplied by the upper numbers gives $axy-3y^3$ for the numerator, and multiplied by the lower numbers, and then divided by x, gives $3x^2-2ax+ay$ for the denominator of the value of BT.

PROBLEM IX.

To determine the Area of any Curve propofed.

The Refolution of the Problem depends upon this; that from the relation of the Fluxions being given, the relation of the Fluents may be found, as in PROB. II. Firft if the Right Line BD, by the motion of which the Area required AFDB is defcribed, move upright upon an Abfcifs or Bafe AB given in pofition, conceive (as before) the parallelogram ABEC to be defcribed in the mean time on the other fide BE by a Line



equal

equal to 1, and BE being fuppofed equal to the Fluxion of the Parallelogram, BD will be the Fluxion of the Area required.

Therefore make $AB = x$, then $ABEC \times 1 = 1 \times x = x$, and $BE = \dot{x}$, call $AFDB = z$, and it will be $BD = \dot{z}$, as alfo $\frac{\dot{z}}{\dot{x}}$, becaufe $\dot{x} = 1$; therefore by the equation expreffing BD, at the fame time the ratio of the Fluxion $\frac{\dot{z}}{\dot{x}}$ is expreffed, and thence (by PROB. II. *Cafe* I.) may be found the relation of the Flowing Quantities x and z.

EXAMPLE I. *When* BD *or* $\dot{z}$ *is equivalent to fome fimple Quantity.*

Let there be given $\frac{xx}{a} = \dot{z}$, or $\frac{\dot{z}}{\dot{x}}$, the equation to the Parabola; and (by PROB. II.) there will arife $\frac{x^3}{3a} = z$; therefore $\frac{x^3}{3a}$, or $\frac{1}{3}$ $AB \times BD$ is equal to the Area of the Parabola AFDB.

Opgave

Leibniz udgav aldrig selv en grundig indføring i de basale begreber i sin differentialregning. Ideerne blev dog via Johann Bernoulli videregivet til L'Hospital som udgav dem i "Analyse des Infiniment Petits" i 1696. Her følger begyndelsen af 2. udgaven fra 1715, i en oversættelse af Kirsti Andersen (udgivet af foreningen Videnskabs-historisk Museums Venner, Århus).

1) Hvad er de basale begreber i Leibniz' (L'Hospitals) udgave af differentialregningen ?
2) Find $d\frac{x}{y}$ med Leibniz' metode, dvs. ved at overføre argumenterne i Proposition I og II i sektion I.
3) Find subtangent til $y = x^m$ ved at imitere Proposition I i sektion II.

DE UENDELIGT SMÅ STØRRELSERS ANALYSE

Første del om differensregning

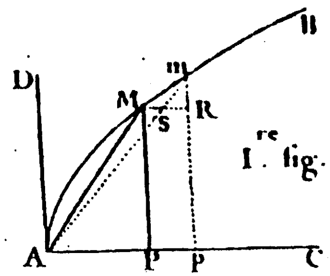
Første afsnit hvor reglerne for denne regning gives

DEFINITION I

Størrelser, der uafbrudt vokser eller aftager, kaldes variable størrelser, hvorimod de, der forbliver de samme, medens de andre ændrer sig, kaldes konstante størrelser. I en parabel er ordinaterne og abscisserne[1] således variable størrelser, medens parameteren er en konstant størrelse.

-----

1. L'Hospital bruger udtrykkene les appliquées et les coupées.



DEFINITION II

Den uendeligt lille del, hvormed en variabel størrelse uafbrudt vokser eller aftager, kaldes <u>differensen</u>.

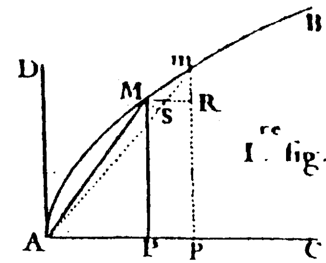
Lad for eksempel AMB være en kurve, der som akse eller diameter har linien AC og som en af sine ordinater PM; lad endvidere pm være en ordinat uendeligt tæt ved den første. Med dette forudsat tegnes MR parallel med AC, korderne AM og Am, den lille cirkelbue MS med centrum A og radius AM, så vil Pp være AP's differens, Rm være PM's, Sm være AM's og Mm være buen AM's. Ligeledes vil den lille trekant MAm, der har buen Mm som grundlinie være differensen af afsnittet AM, og det lille areal MPpm være differensen af arealet begrænset af AP, PM og buen AM. Fig. 1

KOROLLAR

1. Det er klart, at differensen af en konstant størrelse er nul, eller (hvad der er det samme) at konstante størrelser ingen differenser har.

NOTE

I det følgende bruges betegnelsen eller symbolet d til at betegne differensen af en variabel størrelse, som man har udtrykt ved et enkelt bogstav, og for at undgå forvirring gøres der ingen anden brug af d i det følgende. Hvis man for eksempel kalder AP, x; PM, y; AM, z; buen AM, u; det krum- og retlinede areal APM, s; og afsnittet AM, t, så udtrykker dx værdien af Pp, dy værdien af Rm, dz værdien af Sm, du værdien af den lille bue Mm, ds værdien af det lille areal MPpm, og dt værdien af den lille krum- og ret-



I. KRAV ELLER ANTAGELSE

2. Det kræves, at når to størrelser kun adskiller sig med en uendeligt lille størrelse, kan man uden forskel tage den ene for den anden. Eller (hvad der er det samme) at en størrelse, som kun er forøget eller formindsket med en anden størrelse, der er uendeligt mindre end den selv, kan betragtes som forblivende sig selv. Der kræves for eksempel, at man kan tage Ap for AP, pm for PM, arealet Apm for arealet APM, det lille areal MPpm for det lille rektangel MPpR, den lille sektor AMm for den lille trekant AMS, vinklen pAm for vinklen PAM etc.

II. KRAV ELLER ANTAGELSE

3. Det kræves, at en kurve kan betragtes som en uendelig samling af rette linie<stykker>, eller (hvad der er det samme) som en polygon med et uendeligt antal sider - hver uendeligt lille. Ved de vinkler, siderne danner med hinanden, bestemmer de kurvens krumning. Der kræves for eksempel, at kurvestykket Mm og cirkelbuen Ms på grund af deres uendelige lidenhed kan betragtes som rette linier således, at den lille trekant mSM kan opfattes som retlinet.

NOTE

Det antages sædvanligvis i det følgende, at alfabetets sidste bogstaver z, y, x etc. betegner variable størrelser, hvorimod de første a, b, c etc. betegner konstante størrelser således, at x bliver x+dx; y, z etc. bliver y+dy, z+dz etc., og a, b, c, etc. forbliver de samme a, b, c etc.

PROPOSITION I

Problem

4. At tage differensen af flere størrelser, der er lagt til eller trukket fra hinanden.

Lad der være givet $a+x+y-z$, af hvilken differensen skal tages. Hvis det antages, at x forøges med en uendeligt lille del, det vil sige, at den bliver $x+dx$, så bliver y, $y+dy$ og z, $z+dz$, medens konstanten a forbliver den samme, a. Således bliver den givne størrelse $a+x+y-z$, $a+x+dx+y+dy-z-dz$, og dens differens, som man finder ved at trække den fra den sidste størrelse, bliver $dx+dy-dz$. Det er ligesådan med de andre, hvilket giver denne regel.

REGEL I

For adderede eller subtraherede størrelser.

Man tager differensen af hvert led i den givne størrelse, og idet man beholder de samme fortegn, sammensættes en anden størrelse, som er den søgte differens.

PROPOSITION II

Problem

5. At tage differensen af et produkt frembragt af flere størrelser, der er ganget med hinanden.

1°. Differensen af xy er $ydx + xdy$. Thi når x bliver $x+dx$, bliver y, $y+dy$, og xy bliver

$$xy + ydx + xdy + dxdy,$$

som er produktet af $x+dx$ og $y+dy$. Produktets differens er

$$ydx + xdy + dxdy,$$

det vil sige, $ydx + xdy$, fordi $dxdy$ er en uendeligt lille størrelse i forhold til de andre led ydx og xdy: dividerer man for eksempel ydx og $dxdy$ med dx, finder man dels y og dels dy, som er y's differens og derfor uendeligt meget mindre end den. Heraf følger, at differensen af et produkt af to størrelser er lig med produktet af den første størrelses differens og den anden størrelse plus produktet af den

2°. Differensen af xyz er $yzdx + xzdy + xydz$ ...

<Beviset er analogt til det første under 1°>

3°. Differensen af $xyzu$ er

$$uyzdx + uxzdy + uxydz + xyzdu \ldots$$

Det forholder sig således med de andre i det uendelige, hvoraf denne regel dannes.

REGEL II

– For multiplicerede størrelser.

Differensen af et produkt af flere multiplicerede størrelser er lig med summen af produkterne af hver enkelts differens og de øvriges produkt.

Således er differensen af ax, $x0+adx$, det vil sige adx. Differensen af[1] $\overline{a+x}\cdot\overline{b-y}$ er $bdx - ydx - ady - xdy$.

• • •

AFSNIT II

Anvendelse af differensregningen til at bestemme tangenter til alle slags kurver

DEFINITION

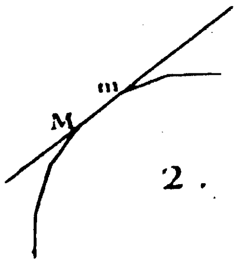
Fig. 2 Hvis man forlænger en af de små sider Mm i den polygon,
Art. 3 som udgør en kurve, så kaldes den således forlængede side <u>tangenten</u> til kurven i punktet M eller m.

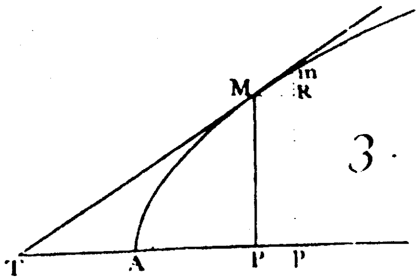
PROPOSITION I

Problem

9. Lad AM være en kurve, hvor relationen mellem abscissen AP og ordinaten PM er udtrykt ved en <algebraisk> ligning.

Fig. 3 I et givet punkt M på denne kurve skal tangenten MT træk-





BEMÆRKNING

10. Når punktet T falder på den modsatte side af punktet A

Fig. 4 <i forhold til P>, x'ernes begyndelsespunkt, er det klart, at når x vokser, aftager y. Følgelig skal der i den givne

Art. 8 lignings differens skiftes fortegn på alle de led, hvor dy findes; med andre ord vil værdien af dx over dy være negativ og derfor også værdien af PT $(y \frac{dx}{dy})$. For ikke at blive forvirret er det imidlertid bedre altid at tage

Afsnit 1 den givne lignings differens efter de foreskrevne regler unden at ændre noget. Thi hvis det ved slutningen af udregningen fremgår, at PT's værdi er positiv, følger det, at man skal tage punktet T på samme side som punktet A, x'ernes begyndelsespunkt, sådan som man antog, da regningen udføres, og modsat, hvis den er negativ, skal det tages til den anden side. De følgende eksempler vil gøre dette klarere.

EKSEMPEL I

11. 1^{o}. Hvis man kræver, at $ax = yy$ skal udtrykke relationen

Fig. 3 mellem AP og PM, er kurven AM en parabel, der som parameter har det givne rette linie<stykke> a. Ved at tage differenserne på begge sider får man

$$adx = 2ydy$$

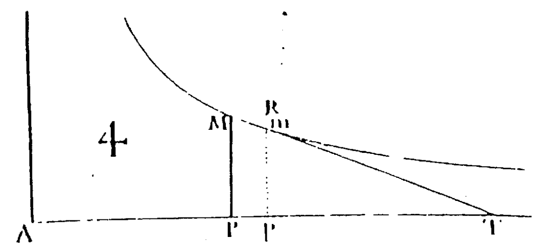
og

$$dx = \frac{2ydy}{a}$$

og

$$PT \left(\frac{ydx}{dy}\right) = \frac{2yy}{a} = 2x ,$$

idet der for yy sættes dens værdi ax. Heraf følger, at når man tager PT som det dobbelte af AP og trækker den rette linie MT, så er den tangenten i punktet M. Det var det, der var forlangt.



2^{o}. Lad ligningen være $aa = xy$, den udtrykker hyperbelens natur mellem asymptoterne. Ved at tage differenserne får man Fig. 4

$$xdy + ydx = 0$$

hvorfor

$$PT \left(\frac{ydx}{dy}\right) = -x .$$

Heraf følger, at hvis man tager PT = PA til den modsatte side af punktet A og trækker den rette linie MT, så er den tangenten i M.

Analysen i 1700–tallet

Opgave 1:

Eulers "Introductio in Analysin Infinitorum" (1748) indeholder de dele af analysen, som går forud for differentialregningen, bl.a. rækkelæren.

1) Læs §1–4 og kommenter Eulers funktionsbegreb

2) Gennemgå Eulers rækkeudvikling af eksponential– og logaritmefunktionen (§114–125). (Der er en del indskud, som du kan gå let hen over.)

3) Gennemgå Eulers rækkeudvikling af cos og sin (§132–134).

4) Gennemgå Eulers udledning af Eulers formler (§138).

Opgave 2:

Eulers "Institutiones Calculi Differentialis" (1755) findes vist desværre ikke oversat til engelsk, så vi må nøjes med en tysk oversættelse.

1) Læs §83–88 og kommenter Euler's brug af uendelig små størrelser.

2) Gennemgå Euler's beregning af differentialet af logaritmefunktionen (§180).

3) Gennemgå Euler's beregning af differentialet af eksponentialfunktionen (§186–188).

Opgave 3 :

I begyndelsen af 1770'erne fandt Lagrange et nyt "algebraisk" grundlag for differentialregningen. Han brugte dette i sin lærebog "Théorie des Fonctions Analytiques" fra 1797. Nedenstående uddrag stammer fra 2. udgaven fra 1813 og er kopieret fra "The History of Mathematics. A Reader".

1. Hvordan definerer Lagrange den afledede ?
2. Hvordan udleder han Taylor rækken ?
3. Påpeg problemer ved Lagrange's grundlag for differentialregningen.

Now let us consider a function $f(x)$ of a variable x. If we replace x by $x + i$, i being any arbitrary quantity, it will become $f(x + i)$ and, by the theory of series, we can expand it in a series of the form

$$f(x) + pi + qi^2 + ri^3 + \cdots,$$

in which the quantities $p, q, r, \ldots$, the coefficients of the powers of i, will be new functions of x, which are derived from the primitive functions of x, and are independent of the quantity i.

But, in order to prove what we claim, we shall examine the actual form of the series representing the expansion of a function $f(x)$ when we substitute $x + i$ for x, which involves only positive integral powers of i.

This assumption is indeed fulfilled in the cases of various known functions; but nobody, to my knowledge, has tried to prove it *a priori*—which seems to me to be all the more necessary since there are particular cases in which it is not satisfied. On the other hand, the differential calculus makes definite use of this assumption, and the exceptional cases are precisely those in which objections have been made to the calculus.

I will first prove that in the series arising by the expansion of the function $f(x + i)$ no fractional power of i can occur except for particular values of x.

[Having accomplished this, Lagrange continues later as follows.]

We have seen that the expansion of $f(x + i)$ generates various other functions $p, q, r, \ldots$, all of them derived from the original function $f(x)$, and we have given the method for finding these functions in particular cases. But in order to establish a theory concerning these kinds of functions we must look for the general law of their derivation.

For this purpose, let us take once more the general formula $f(x + i) = f(x) + pi + qi^2 + ri^3 + \cdots$, and let us suppose that the undetermined quantity x is replaced by $x + o$, o being any arbitrary quantity independent of i. Then $f(x + i)$ will become $f(x + i + o)$, and it is clear that we shall obtain the same result by simply substituting $i + o$ for i in $f(x + i)$. The result must also be the same whether we replace the quantity i by $i + o$ or x by $x + o$ in the expansion $f(x)$.

The first substitution yields $f(x) + p(i + o) + q(i + o)^2 + r(i + o)^3 + \cdots$, or, expanding the powers of $i + o$ and writing out for the sake of simplicity no more than the first two terms of each power (since the comparison of these terms will be sufficient for our purpose):

$$f(x) + pi + qi^2 + ri^3 + si^4 + \cdots + po + 2qio + 3ri^3o + 4si^3o + \cdots.$$

In order to carry out the other substitution, we note that we obtain $f(x) + f'(x)o + \cdots$, $p + p'o + \cdots$, $q + q'o + \cdots$, $r + r'o + \cdots$ when we replace x by $x + o$ in the functions $f(x)$, $p, q, r, \ldots$, respectively; here we retain in the expansion only the terms that include the first power of o. It is clear that the same expression will become $f(x) + pi + qi^2 + ri^3 + si^4 + \cdots + f'(x)o + p'io + q'i^2o + r'i^3o + \cdots$.

Since these two results must be identical whatever the values of i and o may be, comparison of the terms involving $o, io, i^2o, \ldots$, will give: $p = f'(x)$, $2q = p'$, $3r = q'$, $4s = r', \ldots$

Now it is clear that in the same way that $f'(x)$ is the first derived function of $f(x)$, p' is the first derived function of p, q' the first derived function of q, r' the first derived function of r, and so on. Therefore, if, for the sake of greater simplicity and uniformity, we denote by $f'(x)$ the first derived function of $f(x)$, by $f''(x)$ the first derived function of $f'(x)$, by $f'''(x)$ the first derived function of $f''(x)$, and so on, we have $p = f'(x)$, and hence $p' = f''(x)$; consequently $q = \frac{p'}{2} = \frac{f''(x)}{2}$, hence $q' = \frac{f'''(x)}{2}$; consequently $r = \frac{q'}{3} = \frac{f'''(x)}{2\cdot 3}$, hence $r = \frac{f^{iv}(x)}{2\cdot 3}$; consequently $s = \frac{r'}{4} = \frac{f^{iv}(x)}{2\cdot 3\cdot 4}$, hence $s' = \frac{f^{v}(x)}{2\cdot 3\cdot 4}$; and so on.

Then by substituting these values in the expansion of the function $f(x + i)$, we obtain

$$f(x + i) = f(x) + f'(x)i + \frac{f''(x)}{2}i^2 + \frac{f'''(x)}{2\cdot 3}i^3 + \frac{f^{iv}(x)}{2\cdot 3\cdot 4}i^4 + \cdots.$$

This new expression has the advantage of showing how the terms of the series depend on each other and above all how we can form all the derived functions involved in the series provided that we know how to form the first derived function of any primitive function.

We shall call the function $f(x)$ the *primitive function* with respect to the functions $f'(x), f''(x), \ldots$ that are derived from it; these fuctions are called the *derived functions* with respect to the former one. Moreover, we shall call the first derived function $f'(x)$ the *first function*, the second derived function the *second function*, the third derived function the *third function*, and so on. In the same way, if y is supposed to be a function of x, we denote its derived function by $y', y'', y''', \ldots$, respectively, so that, y being the primitive function, y' will be its *first function*, y'' its *second function*, y''' its *third function*, and so on.

Consequently, if x is replaced by $x + i$, y will become

$$y + y'i + \frac{y''i^2}{2} + \frac{y'''i^3}{2\cdot 3} + \cdots.$$

Thus, provided that we have a method of computing the first function of any primitive function, we can obtain, by merely repeating the same operation, all the derived functions, and consequently all the terms of the series that result from expanding the primitive function.

Finally, only a little knowledge of the differential calculus is necessary to recognize that the derived functions $y', y'', y''' \ldots$ of x coincide with the expressions

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \quad \text{respectively.}$$

Euler: Introductio in Analysin Infinitorum 1747. Overs. John D. Blanton, Springer, New York 1988

CHAPTER I

On Functions in General

1. *A constant quantity is a determined quantity which always keeps the same value.*

Quantities of this type are numbers of any sort which keep the same constant value, once they have been assigned; if it is required to represent constant quantities by a symbol, the initial letters of the alphabet are used a, b, c, etc. In algebra, where only fixed quantities are considered, these first letters of the alphabet usually denote known quantities, while the final letters represent unknown quantities, but in analysis this distinction is not so much used, since here it is more a question of considering the former as constants and the latter as variables.

2. *A variable quantity is one which is not determined or is universal, which can take on any value.*

Since all determined values can be expressed as numbers, a variable quantity takes on all possible numbers (all numbers of all types). Just as from the ideas of individuals the ideas of species and genus are formed, so a variable quantity is

a genus in which are contained all determined quantities. Variable quantities of this kind are usually represented by the final letters of the alphabet z, y, x, etc.

3. *A variable quantity is determined when some definite value is assigned to it.*

Hence a variable quantity can be determined in infinitely many ways, since absolutely all numbers can be substituted for it. Nor is the symbol of the variable quantity exhausted until all definite numbers have been assigned to it. Thus a variable quantity encompasses within itself absolutely all numbers, both positive and negative, integers and rationals, irrationals and transcendentals. Even zero and complex numbers are not excluded from the signification of a variable quantity.

4. *A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.*

Hence every analytic expression, in which all component quantities except the variable z are constants, will be a function of that z; thus $a + 3z$; $az - 4z^2$; $az + b\sqrt{a^2 - z^2}$; c^z; etc. are functions of z.

5. *Hence a function itself of a variable quantity will be a variable quantity.*

Since it is permitted to substitute all determined values for the variable quantity, the function takes on innumerable determined values; nor is any determined value excluded from those which the function can take, since the variable quantity includes complex values. Thus, although the function $\sqrt{9 - z^2}$, with real numbers substituted for z, never attains a value greater than 3, nevertheless, by giving z complex values, for instance $5i$, there is no determined value which

6–4

CHAPTER VII

Exponentials and Logarithms Expressed through Series.

114. Since $a^0 = 1$, when the exponent on a increases, the power itself increases, provided a is greater than 1. It follows that if the exponent is infinitely small and positive, then the power also exceeds 1 by an infinitely small number. Let ω be an infinitely small number, or a fraction so small that, although not equal to zero, still $a^\omega = 1 + \psi$, where ψ is also an infinitely small number. From the preceding chapter we know that unless ψ were infinitely small, then neither would ω be infinitely small. It follows that $\psi = \omega$, or $\psi > \omega$, or $\psi < \omega$. Which of these is true depends on the value of a, which is not now known, so we let $\psi = k\omega$. Then we have $a^\omega = 1 + k\omega$, and with a as the base for the logarithms, we have $\omega = \log(1 + k\omega)$.

EXAMPLE

In order that it may be clearer how the number k depends on a, let $a = 10$. From the table of common logarithms, we look for the logarithm of a number which exceeds 1 by the smallest possible amount, for instance, $1 + \frac{1}{1000000}$, so that
$$k\omega = \frac{1}{1000000}.$$
Then $\log\left(1 + \frac{1}{1000000}\right) = \log \frac{1000001}{1000000} = 0.00000043429 = \omega$. Since $k\omega = 0.00000100000$, it follows that $\frac{1}{k} = \frac{43429}{1000000}$ and

$k = \frac{100000}{43429} = 2.30258$. We see that k is a finite number which depends on the value of the base a. If a different base had been chosen, then the logarithm of the same number $1 + k\omega$ will differ from the logarithm already given. It follows that a different value of k will result.

115. Since $a^\omega = 1 + k\omega$, we have $a^{j\omega} = (1 + k\omega)^j$, whatever value we assign to j. It follows that

$$a^{j\omega} = 1 + \frac{j}{1}k\omega + \frac{j(j-1)}{1\cdot 2}k^2\omega^2 + \frac{j(j-1)(j-2)}{1\cdot 2\cdot 3}k^3\omega^3 + \cdots.$$

If now we let $j = \frac{z}{\omega}$, where z denotes any finite number, since ω is infinitely small, then j is infinitely large. Then we have $\omega = \frac{z}{j}$, where ω is represented by a fraction with an infinite denominator, so that ω is infinitely small, as it should be. When we substitute $\frac{z}{j}$ for ω then

$$a^z = (1 + kz/j)^j = 1 + \frac{1}{1}kz + \frac{1(j-1)}{1\cdot 2j}k^2z^2 + \frac{1(j-1)(j-2)}{1\cdot 2j\cdot 3j}k^3z^3 + \frac{1(j-1)(j-2)(j-3)}{1\cdot 2j\cdot 3j\cdot 4j}k^4z^4 + \cdots.$$

This equation is true provided an infinitely large number is substituted for j, but then k is a finite number depending on a, as we have just seen.

116. Since j is infinitely large, $\frac{j-1}{j} = 1$, and the larger the number we substitute for j, the closer the value of the fraction $\frac{j-1}{j}$ comes to 1. Therefore, if j is a number larger than any assignable number, then $\frac{j-1}{j}$ is equal to 1. For the same reason $\frac{j-2}{j} = 1$, $\frac{j-3}{j} = 1$, and so forth. It follows that

$\frac{j-1}{2j} = \frac{1}{2}$, $\frac{j-2}{3j} = \frac{1}{3}$, $\frac{j-3}{4j} = \frac{1}{4}$, and so forth. When we substitute these values, we obtain

$a^z = 1 + \frac{kz}{1} + \frac{k^2z^2}{1\cdot 2} + \frac{k^3z^3}{1\cdot 2\cdot 3} + \frac{k^4z^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$. This equation expresses a relationship between the numbers a and k, since when we let $z = 1$, we have $a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \frac{k^4}{1\cdot\cdot 2\cdot 3\cdot 4} + \cdots$. When $a = 10$, then k is necessarily approximately equal to 2.30258 as we have already seen.

117. Suppose $b = a^n$, and let a be the base for the logarithms, so that $\log b = n$. Since $b^z = a^{nz}$, we have the infinite series $b^z = 1 + \frac{knz}{1} + \frac{k^2n^2z^2}{1\cdot 2} + \frac{k^3n^3z^3}{1\cdot 2\cdot 3} + \frac{k^4n^4z^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$. Now we substitute $\log b$ for n, so that $b^z = 1 + \frac{kz}{1}\log b + \frac{k^2z^2}{1\cdot 2}(\log b)^2 + \frac{k^3z^3}{1\cdot 2\cdot 3}(\log b)^3 + \frac{k^4z^4}{1\cdot 2\cdot 3\cdot 4}(\log b)^4 + \cdots$. Since we know the value of k from the given value of the base a, the general exponential b^z can be expressed in an infinite series whose terms proceed with the powers of z. Having shown this fact, we now go on to show how logarithms can be expressed by infinite series.

118. Since $a^\omega = 1 + k\omega$, where ω is an infinitely small fraction and the relation between a and k is given by $a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \cdots$, if a is taken as the base of the logarithms, then $\omega = \log(1 + k\omega)$ and $j\omega = \log(1 + k\omega)^j$. It is clear that the larger the number chosen for j the more $(1 + k\omega)^j$ will exceed 1. If we let j be an infinite number, the value of the power $(1 + k\omega)^j$ becomes greater than any number greater than 1. Now if we

let $(1 + k\omega)^j = 1 + x$, then $\log(1 + x) = j\omega$. Since $j\omega$ is an finite number, namely the logarithm of $1 + x$, it is clear that j must be an infinitely large number; otherwise, $j\omega$ could not have a finite value.

119. Since we have let $(1 + k\omega)^j = 1 + x$, we have $1 + k\omega = (1 + x)^{\frac{1}{j}}$ and $k\omega = (1 + x)^{\frac{1}{j}} - 1$, so that $j\omega = \frac{j}{k}((1 + x)^{\frac{1}{j}} - 1)$. Since $j\omega = \log(1 + x)$, it follows that $\log(1 + x) = \frac{j}{k}(1 + x)^{\frac{1}{j}} - \frac{j}{k}$ where j is a number infinitely large. But we have

$(1 + x)^{\frac{1}{j}} = 1 + \frac{1}{jx} - \frac{1(j-1)}{j\cdot 2j}x^2 + \frac{1(j-1)(2j-1)}{j\cdot 2j\cdot 3j}x^3 - \frac{1(j-1)(2j-1)(3j-1)}{j\cdot 2j\cdot 3j\cdot 4j}x^4 + \cdots$. Since j is an infinite number, $\frac{j-1}{2j} = \frac{1}{2}$, $\frac{2j-1}{3j} = \frac{2}{3}$, $\frac{3j-1}{4j} = \frac{3}{4}$, etc. Now it follows that $j(1 + x)^{\frac{1}{j}} = j + \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$. As a result we have $\log(1 + x) = \frac{1}{k}\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$, where a is the base of the logarithm and $a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \cdots$.

120. Since we have a series for the logarithm of $1 + x$, we can use this to define the number k when a is the base. If we let $1 + x = a$, since $\log a = 1$, we have

$1 = \frac{1}{k}\left(\frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \cdots\right)$. It follows that $k = \frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \cdots$. If we let

$a = 10$, the value of this infinite series must be approximately equal to 2.30258. We have $2.30258 = \frac{9}{1} - \frac{9^2}{2} + \frac{9^3}{3} - \frac{9^4}{4} + \cdots$, but it is difficult to see how this can be since the terms of this series continually grow larger and the sum of several terms does not seem to approach any limit. We will soon have an answer to this paradox.

121. Since $\log(1 + x) = \frac{1}{k}\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)$, when we substitute $-x$ for x, we obtain

$\log(1 - x) = -\frac{1}{k}\left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$. If we subtract the second series from the first we obtain

$$\log(1 + x) - \log(1 - x) = \log\left(\frac{1 + x}{1 - x}\right)$$

$= \frac{2}{k}\left(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right)$. Now if we let $\frac{1 + x}{1 - x} = a$, so that $x = \frac{a - 1}{a + 1}$, and because $\log a = 1$, we have

$k = 2\left(\frac{a - 1}{a + 1} + \frac{(a - 1)^3}{3(a + 1)^3} + \frac{(a - 1)^5}{5(a + 1)^5} + \cdots\right)$. From this equation we can find the value of k when a is given. For example, if $a = 10$, then

$k = 2\left(\frac{9}{11} + \frac{9^3}{3 \cdot 11^3} + \frac{9^5}{5 \cdot 11^5} + \frac{9^7}{7 \cdot 11^7} + \cdots\right)$ and the terms of this series decrease in a reasonable way so that soon a satisfactory approximation for k can be obtained.

122. Since we are free to choose the base a for the system of logarithms, we

now choose a in such a way that $k = 1$. Suppose now that $k = 1$, then the series found above in section 116,

$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$ is equal to a. If the terms are represented as decimal fractions and summed, we obtain the value for $a = 2.71828182845904523536028 \cdots$. When this base is chosen, the logarithms are called natural or hyperbolic. The latter name is used since the quadrature of a hyperbola can be expressed through these logarithms. For the sake of brevity for this number $2.718281828459 \cdots$ we will use the symbol e, which will denote the base for natural or hyperbolic logarithms, which corresponds to the value $k = 1$, and e represents the sum of the infinite series $1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$.

123. Natural logarithms have the property that the logarithm of $1 + \omega$ is equal to ω, where ω is an infinitely small quantity. From this it follows that $k = 1$ and the natural logarithms of all numbers can be found. Let e stand for the number found above, then

$e^z = 1 + \frac{z}{1} + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$, and the natural logarithms themselves can be found from these series where

$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$, and

$\log\left(\frac{1 + x}{1 - x}\right) = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \cdots$. This last series is strongly convergent if we substitute an extremely small fraction for x. For instance, if $x = \frac{1}{5}$, then

$\log \frac{6}{4} = \log \frac{3}{2} = \frac{2}{1\cdot 5} + \frac{2}{3\cdot 5^3} + \frac{2}{5\cdot 5^5} + \frac{2}{7\cdot 5^7} + \cdots$. If $x = \frac{1}{7}$, then

$\log \frac{4}{3} = \frac{2}{1\cdot 7} + \frac{2}{3\cdot 7^3} + \frac{2}{5\cdot 7^5} + \frac{2}{7\cdot 7^7} + \cdots$, and if $x = \frac{1}{9}$, then

$\log \frac{5}{4} = \frac{2}{1\cdot 9} + \frac{2}{3\cdot 9^3} + \frac{2}{5\cdot 9^5} + \frac{2}{7\cdot 9^7} + \cdots$. From the logarithms of these fractions we can find the logarithms of integers. From the nature of logarithms we have $\log \frac{3}{2} + \log \frac{4}{3} = \log 2$, and $\log \frac{3}{2} + \log 2 = \log 3$, and $2 \log 2 = \log 4$. Further we have

$\log \frac{5}{4} + \log 4 = \log 5$, $\log 2 + \log 3 = \log 6$, $3 \log 2 = \log 8$, $2 \log 3 = \log 9$, $\log 2 + \log 5 = \log 10$.

EXAMPLE

We can now state the values of the natural logarithms of integers from 1 to 10.

log 1	= 0.00000 00000 00000 00000 00000	log 2	= 0.69314 71805 59945 30941 72321
log 3	= 1.09861 22886 68109 69139 52452	log 4	= 1.38629 43611 19890 61883 44642
log 5	= 1.60943 79124 34100 37460 07593	log 6	= 1.79175 94692 28055 00081 24773
log 7	= 1.94591 01490 55313 30510 54639	log 8	= 2.07944 15416 79835 92825 16964
log 9	= 2.19722 45773 36219 38279 04905	log 10	= 2.30258 50929 94045 68401 79914

All of these logarithms are computed from the above three series, with the exception of log 7, which can be found as follows. When in the last series given above, we let $x = \frac{1}{99}$, we obtain

$\log \frac{100}{98} = \log \frac{50}{49} = 0.0202027073175194484078230$. When this is subtracted

from $\log 50 = 2 \log 5 + \log 2 = 3.9120230054281460586187508$ we obtain $\log 49$. But $\log 7 = \frac{1}{2} \log 49$.

124. Let the natural logarithm of $1 + x$ be equal to y, then $y = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$. Now let a be the base of a system of logarithms and let v be the logarithm of $1 + x$ in this system. Then as we have seen, $v = \frac{1}{k}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) = \frac{y}{k}$. It follows that $k = \frac{y}{v}$, and this is the most convenient method of calculating the value of k corresponding to the base a; it is given by the quotient of the natural logarithm of any number divided by the logarithms of that same number with the base a. Suppose the number is a, then $v = 1$ and k is equal to the natural logarithm of a. In the system of common logarithms, where the base is $a = 10$, then k is the natural logarithm of 10. It follows that $k = 2.3025850929940456840179914$, which is the value calculated not far above. If each natural logarithm is divided by this number k, or, which comes to the same thing, multiplied by the decimal fraction $0.4342944819032518276511289$, then the results are the common logarithms, with base $a = 10$.

125. Since $e^z = 1 + \frac{z}{1} + \frac{z^2}{1\cdot 2} + \frac{z^3}{1\cdot 2\cdot 3} + \cdots$, if we let $a^y = e^z$, then after taking natural logarithms, we have $y \log a = z$, since $\log e = 1$. We now substitute this value in the series to obtain

$a^y = 1 + \frac{y \log a}{1} + \frac{y^2(\log a)^2}{1\cdot 2} + \frac{y^3(\log a)^3}{1\cdot 2\cdot 3} + \cdots$. In this way any exponential, with the aid of natural logarithms, can be expressed as an infinite series. Now let j be an infinitely large number, then both exponentials and logarithms can be expressed as powers. That is, $e^z = \left(1 + \frac{z}{j}\right)^j$ and so $a^y = \left(1 + \frac{y \log a}{j}\right)^j$. For natural logarithms, we have $\log(1 + x) = j((1 + x)^{\frac{1}{j}} - 1)$. Other uses of natural logarithms are discussed in integral calculus.

CHAPTER VIII

On Transcendental Quantities Which Arise from the Circle.

126. After having considered logarithms and exponentials, we must now turn to circular arcs with their sines and cosines. This is not only because these are further genera of transcendental quantities, but also since they arise from logarithms and exponentials when complex values are used. This will become clearer in the development to follow.

We let the radius, or total sine, of a circle be equal to 1, then it is clear enough that the circumference of the circle cannot be expressed exactly as a rational number. An approximation of half of the circumference of this circle is

3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651327230664709384 46+.

For the sake of brevity we will use the symbol π for this number. We say, then, that half of the circumference of a unit circle is π, or that the length of an arc of 180 degrees is π.

127. We always assume that the radius of the circle is 1 and let z be an arc of this circle. We are especially interested in the sine and cosine of this arc z. Henceforth we will signify the sine of the arc z by $\sin z$. Likewise, for the cosine of the arc z we will write $\cos z$. Since π is an arc of 180 degrees, $\sin 0\pi = 0$ and $\cos 0\pi = 1$. Also $\sin \frac{\pi}{2} = 1$, $\cos \frac{\pi}{2} = 0$, $\sin \pi = 0$, $\cos \pi = -1$,

$\sin\frac{3}{2}\pi = -1$, $\cos\frac{3}{2}\pi = 0$, $\sin 2\pi = 0$, and $\cos 2\pi = 1$. Every sine and cosine lies between +1 and -1. Further, we have $\cos z = \sin\left(\frac{\pi}{2} - z\right)$ and $\sin z = \cos\left(\frac{\pi}{2} - z\right)$. We also have $(\sin z)^2 + (\cos z)^2 = 1$. Besides these notations we mention also that $\tan z$ indicates the tangent of the arc z, $\cot z$ for the cotangent of the arc z. We agree that $\tan z = \frac{\sin z}{\cos z}$ and $\cot z = \frac{\cos z}{\sin z} = \frac{1}{\tan z}$, all of which is known from trigonometry.

128. We note further that if y and z are two arcs, then

$\sin(y + z) = \sin y \cos z + \cos y \sin z$ and

$\cos(y + z) = \cos y \cos z - \sin y \sin z$. Likewise

$\sin(y - z) = \sin y \cos z - \cos y \sin z$ and

$\cos(y - z) = \cos y \cos z + \sin y \sin z$. Now we substitute the arcs $\frac{\pi}{2}$, π, $\frac{3}{2}\pi$, etc. for y in the previous formulas:

$\sin\left(\frac{\pi}{2} + z\right) = +\cos z$ $\qquad \sin\left(\frac{\pi}{2} - z\right) = +\cos z$

$\cos\left(\frac{\pi}{2} + z\right) = -\sin z$ $\qquad \cos\left(\frac{\pi}{2} - z\right) = +\sin z$

$\sin(\pi + z) = -\sin z$ $\qquad \sin(\pi - z) = +\sin z$

$\cos(\pi + z) = -\cos z$ $\qquad \cos(\pi - z) = -\cos z$

$\sin\left(\frac{3}{2}\pi + z\right) = -\cos z$ $\qquad \sin\left(\frac{3}{2}\pi - z\right) = -\cos z$

$\cos\left(\frac{3}{2}\pi + z\right) = +\sin z$ $\qquad \cos\left(\frac{3}{2}\pi - z\right) = -\sin z$

$\sin(2\pi + z) = +\sin z$ $\qquad \sin(2\pi - z) = -\sin z$

$\cos(2\pi + z) = +\cos z$ $\qquad \cos(2\pi - z) = +\cos z$.

It follows that if n is any integer, then

$\sin\left(\frac{4n+1}{2}\pi + z\right) = +\cos z$ $\qquad \sin\left(\frac{4n+1}{2}\pi - z\right) = +\cos z$

$\cos\left(\frac{4n+1}{2}\pi + z\right) = -\sin z$ $\qquad \cos\left(\frac{4n+1}{2}\pi - z\right) = -\sin z$

$\sin\left(\frac{4n+2}{2}\pi + z\right) = -\sin z$ $\qquad \sin\left(\frac{4n+2}{2}\pi - z\right) = +\sin z$

$\cos\left(\frac{4n+2}{2}\pi + z\right) = -\cos z$ $\qquad \cos\left(\frac{4n+2}{2}\pi - z\right) = -\cos z$

$\sin\left(\frac{4n+3}{2}\pi + z\right) = -\cos z$ $\qquad \sin\left(\frac{4n+3}{2}\pi - z\right) = -\cos z$

$\cos\left(\frac{4n+3}{2}\pi + z\right) = +\sin z$ $\qquad \cos\left(\frac{4n+3}{2}\pi - z\right) = -\sin z$

$\sin\left(\frac{4n+4}{2}\pi + z\right) = +\sin z$ $\qquad \sin\left(\frac{4n+4}{2}\pi - z\right) = -\sin z$

$\cos\left(\frac{4n+4}{2}\pi + z\right) = +\cos z$ $\qquad \cos\left(\frac{4n+4}{2}\pi - z\right) = +\cos z$. These formulas hold whether n is a positive or a negative integer.

129. Let $\sin z = p$ and $\cos z = q$, then $p^2 + q^2 = 1$; if also $\sin y = m$ and $\cos y = n$, then also $m^2 + n^2 = 1$. We have the following identities:

$\sin z = p$ $\qquad \cos z = q$

$\sin(y + z) = mq + np$ $\qquad \cos(y + z) = nq - mp$

6-10

$$\sin(2y + z) = 2mnq + (n^2 - m^2)p$$
$$\cos(2y + z) = (n^2 - m^2)q - 2mnp$$
$$\sin(3y + z) = (3mn^2 - m^3)q + (n^3 - 3m^2n)p$$
$$\cos(3y + z) = (n^3 - 3m^2n)q - (3mn^2 - m^3)p$$

etc. These arcs: z, $y + z$, $2y + z$, $3y + z \cdots$, form an arithmetic progression, however, both their sines and cosines form a recurrent progression which arises from the denominator $1 - 2nx + (m^2 + n^2)x^2$. This is seen from the following: $\sin(2y + z) = 2n \sin(y + z) - (m^2 + n^2)\sin z$, or

$\sin(2y + z) = 2 \cos y \sin(y + z) - \sin z$. In like manner

$\cos(2y + z) = 2 \cos y \cos(y + z) - \cos z$. Furthermore we have

$\sin(3y + z) = 2 \cos y \sin(2y + z) - \sin(y + z)$, and

$\cos(3y + z) = 2 \cos y \cos(2y + z) - \cos(y + z)$. Also

$\sin(4y + z) = 2 \cos y \sin(3y + z) - \sin(2y + z)$, and

$\cos(4y + z) = 2 \cos y \sin(3y + z) - \cos(2y + z)$, etc. The advantage of this law is that when the arcs form an arithmetic progression, then as many of the sines and cosines as may be desired can be expressed with little trouble.

130. Since $\sin(y + z) = \sin y \cos z + \cos y \sin z$, and

$\sin(y - z) = \sin y \cos z - \cos y \sin z$, when we add or subtract these expressions we obtain: $\sin y \cos z = \dfrac{\sin(y + z) + \sin(y - z)}{2}$, and

$\cos y \sin z = \dfrac{\sin(y + z) - \sin(y - z)}{2}$. Furthermore, since

$\cos(y + z) = \cos y \cos z - \sin y \sin z$, and

$\cos(y - z) = \cos y \cos z + \sin y \sin z$, by the same method we obtain:

$\cos y \cos z = \dfrac{\cos(y - z) + \cos(y + z)}{2}$, and

$\sin y \sin z = \dfrac{\cos(y - z) - \cos(y + z)}{2}$. Let $y = z = \dfrac{v}{2}$, then from these last formulas we obtain: $\left(\cos \dfrac{v}{2}\right)^2 = \dfrac{1 + \cos v}{2}$ so that $\cos \dfrac{v}{2} = \sqrt{(1 + \cos v)/2}$,

and $\left(\sin \dfrac{v}{2}\right)^2 = \dfrac{1 - \cos v}{2}$ so that $\sin \dfrac{v}{2} = \sqrt{(1 - \cos v)/2}$. From this we see that if the cosine of an arc is given, then we can find the sine and cosine of the half arc.

131. Let the arcs $y + z = a$ and $y - z = b$, then $y = \dfrac{a + b}{2}$ and $z = \dfrac{a - b}{2}$. When we substitute these values in the formulas above we have the following equations, each of which is, as it were, a theorem:

$$\sin a + \sin b = 2 \sin \frac{a + b}{2} \cos \frac{a - b}{2}$$

$$\sin a - \sin b = 2 \cos \frac{a + b}{2} \sin \frac{a - b}{2}$$

$$\cos a + \cos b = 2 \cos \frac{a + b}{2} \cos \frac{a - b}{2}$$

$$\cos a - \cos b = 2 \sin \frac{a + b}{2} \sin \frac{a - b}{2}.$$

From these results we have, by division, the following theorems:

$$\frac{\sin a + \sin b}{\sin a - \sin b} = \tan \frac{a + b}{2} \cot \frac{a - b}{2} = \frac{\tan \frac{a + b}{2}}{\tan \frac{a - b}{2}},$$

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \tan \frac{a + b}{2}, \qquad \frac{\sin a + \sin b}{\cos b - \cos a} = \cot \frac{a - b}{2},$$

$$\frac{\sin a - \sin b}{\cos a + \cos b} = \tan \frac{a - b}{2}, \qquad \frac{\sin a - \sin b}{\cos b - \cos a} = \cot \frac{a + b}{2},$$

$\dfrac{\cos a + \cos b}{\cos b - \cos a} = \cot \dfrac{a + b}{2} \cot \dfrac{a - b}{2}$. From these we deduce the following theorems:

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{\cos b - \cos a}{\sin a - \sin b},$$

$$\frac{\sin a + \sin b}{\sin a - \sin b} \cdot \frac{\cos a + \cos b}{\cos b - \cos a} = \left(\cot \frac{a - b}{2}\right)^2$$

$$\frac{\sin a + \sin b}{\sin a - \sin b} \cdot \frac{\cos b - \cos a}{\cos a + \cos b} = \left(\tan \frac{a + b}{2}\right)^2.$$

132. Since $(\sin z)^2 + (\cos z)^2 = 1$, we have the factors $(\cos z + i \sin z)(\cos z - i \sin z) = 1$. Although these factors are complex, still they are quite useful in combining and multiplying arcs. Consider the following product: $(\cos z + i \sin z)(\cos y + i \sin y)$, which results in $\cos y \cos z - \sin y \sin z + (\cos y \sin z + \sin y \cos z)i$. Since $\cos y \cos z - \sin y \sin z = \cos(y + z)$ and $\cos y \sin z + \sin y \cos z = \sin(y + z)$ we can express this product as $(\cos y + i \sin y)(\cos z + i \sin z) = \cos(y + z) + i \sin(y + z)$ and likewise

$$(\cos y - i \sin y)(\cos z - i \sin z) = \cos(y + z) - i \sin(y + z)$$

also

$$(\cos x \pm i \sin x)(\cos y \pm i \sin y)(\cos z \pm i \sin z) = \cos(x + y + z) \pm i \sin(x + y + z).$$

133. It now follows that $(\cos z \pm i \sin z)^2 = \cos 2z \pm i \sin 2z$ and $(\cos z \pm i \sin z)^3 = \cos 3z \pm i \sin 3z$. Generally we have $(\cos z \pm i \sin z)^n = \cos nz \pm i \sin nz$. It follows that

$$\cos nz = \frac{(\cos z + i \sin z)^n + (\cos z - i \sin z)^n}{2}$$ and

$$\sin nz = \frac{(\cos z + i \sin z)^n - (\cos z - i \sin z)^n}{2}.$$ Expanding the binomials we

obtain the following series:

$$\cos nz = (\cos z)^n - \frac{n(n-1)}{1\cdot 2}(\cos z)^{n-2}(\sin z)^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}(\cos z)^{n-4}(\sin z)^4 - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}(\cos z)^{n-6}(\sin z)^6 + \cdots$$

and

$$\sin nz = \frac{n}{1}(\cos z)^{n-1}\sin z - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(\cos z)^{n-3}(\sin z)^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}(\cos z)^{n-5}(\sin z)^5 - \cdots.$$

134. Let the arc z be infinitely small, then $\sin z = z$ and $\cos z = 1$. If n is an infinitely large number, so that nz is a finite number, say $nz = v$, then, since $\sin z = z = \frac{v}{n}$, we have

$$\cos v = 1 - \frac{v^2}{1\cdot 2} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4} - \frac{v^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots$$ and

$$\sin v = v - \frac{v^3}{1\cdot 2\cdot 3} + \frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{v^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \cdots.$$ It follows that if v is a given arc, by means of these series, the sine and cosine can be found. In order that the use of these formulas may become clearer, let us take v to be in the same ratio to the quarter circle, or 90 degrees, as m is to n. That is $v = \frac{m}{n}\frac{\pi}{2}$. Since the value of π is known, if we substitute this value we obtain

$$\sin \frac{m}{n}\frac{\pi}{2} = + \frac{m}{n}1.5707963267948966192313216916 - \frac{m^3}{n^3}0.6459640975062462536557565636 + \frac{m^5}{n^5}0.0796926262461670451205055488 - \frac{m^7}{n^7}0.0046817541353186881006854632$$

$$+\frac{m^9}{n^9}0.000160441184787359821872660 5 \qquad -\frac{m^{11}}{n^{11}}0.0000035988432352120853404580$$

$$+\frac{m^{13}}{n^{13}}0.0000000569217292196792681171 \qquad -\frac{m^{15}}{n^{15}}0.0000000006688035109811467224$$

$$+\frac{m^{17}}{n^{17}}0.0000000000060669357311061950 \qquad -\frac{m^{19}}{n^{19}}0.0000000000000437706546731370$$

$$+\frac{m^{21}}{n^{21}}0.0000000000000002571422892856 \qquad -\frac{m^{23}}{n^{23}}0.0000000000000000012538995403$$

$$+\frac{m^{25}}{n^{25}}0.0000000000000000000051564550 \qquad -\frac{m^{27}}{n^{27}}0.0000000000000000000000181239$$

$$+\frac{m^{29}}{n^{29}}0.0000000000000000000000000549$$

and

$$\cos\frac{m}{n}\frac{\pi}{2} = 1.0000000000000000000000000000 \qquad -\frac{m^2}{n^2}1.2337005501361698273543113745$$

$$+\frac{m^4}{n^4}0.2536695079010480136365633659 \qquad -\frac{m^6}{n^6}0.0208634807633529608730516364$$

$$+\frac{m^8}{n^8}0.0009192602748394265802417158 \qquad -\frac{m^{10}}{n^{10}}0.0000252020423730606054810526$$

$$+\frac{m^{12}}{n^{12}}0.0000004710874778818171503665 \qquad -\frac{m^{14}}{n^{14}}0.0000000063866030837918522408$$

$$+\frac{m^{16}}{n^{16}}0.0000000000656596311497947230 \qquad -\frac{m^{18}}{n^{18}}0.0000000000005294400200734620$$

$$+\frac{m^{20}}{n^{20}}0.0000000000000034377391790981 \qquad -\frac{m^{22}}{n^{22}}0.0000000000000000183599165212$$

$$+\frac{m^{24}}{n^{24}}0.0000000000000000000820675327 \qquad -\frac{m^{26}}{n^{26}}0.0000000000000000000003115285$$

$$+\frac{m^{28}}{n^{28}}0.0000000000000000000000010165 \quad -\frac{m^{30}}{n^{30}}0.0000000000000000000000000026.$$

Since it is sufficient to know the sines and cosines of angles only to 45 degrees, the fraction $\frac{m}{n}$ will always be less than ½; because of the powers of the fraction

$\frac{m}{n}$, the series converge quickly. A few terms should be sufficient, especially if the number of decimal places is not so large.

135. Once sines and cosines have been computed, tangents and cotangents can be found in the ordinary way, however, since the multiplication and division of such gigantic numbers is so inconvenient, a different method of expressing these functions is desirable. Since $\tan v = \frac{\sin v}{\cos v}$

$$= \frac{v - \frac{v^3}{1\cdot2\cdot3} + \frac{v^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{v^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} + \cdots}{1 - \frac{v^2}{1\cdot2} + \frac{v^4}{1\cdot2\cdot3\cdot4} - \frac{v^6}{1\cdot2\cdot3\cdot4\cdot5\cdot6} + \cdots}$$

and

$$\cot v = \frac{\cos v}{\sin v}$$

$$= \frac{1 - \frac{v^2}{1\cdot2} + \frac{v^4}{1\cdot2\cdot3\cdot4} - \frac{v^6}{1\cdot2\cdot3\cdot4\cdot5\cdot6} + \cdots}{v - \frac{v^3}{1\cdot2\cdot3} + \frac{v^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{v^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} + \cdots}.$$

If the arc is $v = \frac{m}{n}\frac{\pi}{2}$, then as before

$$\tan v = \frac{2mn}{n^2 - m^2}0.6366197723675$$

$$+\frac{m}{n}0.2975567820597 + \frac{m^3}{n^3}0.0186886502773$$

$$+\frac{m^5}{n^5}0.0018424752034 + \frac{m^7}{n^7}0.0001975800714$$

$$+\frac{m^9}{n^9}0.0000216977245 + \frac{m^{11}}{n^{11}}0.0000024011370$$

$$+\frac{m^{13}}{n^{13}}0.0000002664132 + \frac{m^{15}}{n^{15}}0.0000000295864$$

$$+\frac{m^{17}}{n^{17}}0.0000000032867 + \frac{m^{19}}{n^{19}}0.0000000003651$$

$$+ \frac{m^{21}}{n^{21}}0.0000000000405 + \frac{m^{23}}{n^{23}}0.0000000000045 + \frac{m^{25}}{n^{25}}0.0000000000005$$

$$\cot v = \frac{n}{m}0.6366197723675$$

$$- \frac{4mn}{4n^2 - m^2}0.3183098861837 - \frac{m}{n}0.2052888894145$$

$$- \frac{m^3}{n^3}0.0065510747882 - \frac{m^5}{n^5}0.0003450292554$$

$$- \frac{m^7}{n^7}0.0000202791060 - \frac{m^9}{n^9}0.0000012366527$$

$$- \frac{m^{11}}{n^{11}}0.0000000764959 - \frac{m^{13}}{n^{13}}0.0000000047597$$

$$- \frac{m^{15}}{n^{15}}0.0000000002969 - \frac{m^{17}}{n^{17}}0.0000000000185 - \frac{m^{19}}{n^{19}}0.0000000000011.$$

The basis for these series will be explained at length below in section 197.

136. From what we have seen previously, it is clear that when we know the sines and cosines of angles less than half a right angle, then we also have sines and cosines of greater angles. In fact, if we know the sines and cosines of angles less than only 30 degrees, then from these, by only addition and subtraction, we can find all sines and cosines of larger angles. Since $\sin \frac{\pi}{6} = \frac{1}{2}$, when we let $y = \frac{\pi}{6}$ in the formula from section 130, we have

$\cos z = \sin\left(\frac{\pi}{6} + z\right) + \sin\left(\frac{\pi}{6} - z\right)$ and

$\sin z = \cos\left(\frac{\pi}{6} - z\right) - \cos\left(\frac{\pi}{6} + z\right)$. It follows that from the sines and cosines of angles z and $\frac{\pi}{6} - z$ we obtain $\sin\left(\frac{\pi}{6} + z\right) = \cos z - \sin\left(\frac{\pi}{6} - z\right)$ and

$\cos\left(\frac{\pi}{6} + z\right) = \cos\left(\frac{\pi}{6} - z\right) - \sin z$. In this way we obtain sines and cosines of angles from 30 degrees to 60 degrees, and hence the sines and cosines are defined for all larger angles.

137. A similar strategy can be used to find tangents and cotangents. Since $\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$, we have $\tan 2a = \frac{2 \tan a}{1 - (\tan a)^2}$ and $\cot 2a = \frac{\cot a - \tan a}{2}$. It follows that from tangents and cotangents of arcs less than 30 degrees, we can find tangents and cotangents up to 60 degrees. Let $a = \frac{\pi}{6} - b$, then $2a = \frac{\pi}{3} - 2b$ and $\cot 2a = \tan\left(\frac{\pi}{6} + 2b\right)$. Then we have

$\tan\left(\frac{\pi}{6} + 2b\right) = \frac{\cot\left(\frac{\pi}{6} - b\right) - \tan\left(\frac{\pi}{6} - b\right)}{2}$, which gives tangents of arcs greater than 30 degrees. Secants and cosecants can be found from tangents by means of subtraction. Note that $\csc z = \cot \frac{z}{2} - \cot z$ and $\sec z = \cot\left(\frac{\pi}{4} - \frac{z}{2}\right) - \tan z$. From these remarks it should be very clear how tables of sines can be constructed.

138. Once again we use the formulas in section 133, where we let z be an infinitely small arc and let n be an infinitely large number j, so that jz has a finite value v. Now we have $nz = v$ and $z = \frac{v}{j}$, so that $\sin z = \frac{v}{j}$ and $\cos z = 1$. With these substitutions,

$$\cos v = \frac{\left(1 + \frac{iv}{j}\right)^j + \left(1 - \frac{iv}{j}\right)^j}{2} \quad \text{and} \quad \sin v = \frac{\left(1 + \frac{iv}{j}\right)^j - \left(1 - \frac{iv}{j}\right)^j}{2i}.$$

In the preceding chapter we saw that $(1 + z/j)^j = e^z$ where e is the base of the natural logarithms. When we let $z = iv$ and then $z = -iv$ we obtain $\cos v = \frac{e^{iv} + e^{-iv}}{2}$ and $\sin v = \frac{e^{iv} - e^{-iv}}{2i}$. From these equations we understand how complex exponentials can be expressed by real sines and cosines, since $e^{iv} = \cos v + i \sin v$ and $e^{-iv} = \cos v - i \sin v$.

139. Now let n be an infinitely small number in the formulas of section 130 or let $n = \frac{1}{j}$, where j is an infinitely large number. Then $\cos nz = \cos \frac{z}{j} = 1$ and $\sin nz = \sin \frac{z}{j} = \frac{z}{j}$, since the sine of a vanishing arc $\frac{z}{j}$ is equal to the arc itself, and the cosine of such an arc is equal to 1. With this hypothesis we have

$$1 = \frac{(\cos z + i \sin z)^{\frac{1}{j}} + (\cos z - i \sin z)^{\frac{1}{j}}}{2} \text{ and}$$

$$\frac{z}{j} = \frac{(\cos z + i \sin z)^{\frac{1}{j}} - (\cos z - i \sin z)^{\frac{1}{j}}}{2i}.$$ In section 125 we saw that $\log(1 + x) = j(1 + x)^{\frac{1}{j}} - j$ or, when we substitute y for $1 + x$, we have $y^{\frac{1}{j}} = 1 + \frac{1}{j}\log y$. Now we first substitute $\cos z + i \sin z$ for y, then substitute $\cos z - i \sin z$ for y to obtain

$$1 = \frac{1 + \frac{1}{j}\log(\cos z + i \sin z) + 1 + \frac{1}{j}\log(\cos z - i \sin z)}{2}.$$ Since the terms with logarithms vanish in this equation, nothing follows; however, from the other equation, for the sine, we obtain

$$\frac{z}{j} = \frac{\frac{1}{j}\log(\cos z + i \sin z) - \frac{1}{j}\log(\cos z - i \sin z)}{2i}.$$ From this we obtain

$z = \frac{1}{2i}\log\left(\frac{\cos z + i \sin z}{\cos z - i \sin z}\right)$, so that it becomes clear to what extent logarithms of complex numbers are related to circular arcs.

140. Since $\frac{\sin z}{\cos z} = \tan z$, the arc z can now be expressed through its tangent as follows: $z = \frac{1}{2i}\log\left(\frac{1 + i \tan z}{1 - i \tan z}\right)$. We have seen in section 123 that $\log\left(\frac{1 + x}{1 - x}\right) = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \cdots$. When we substitute $i \tan z$ for x we obtain

$z = \frac{\tan z}{1} - \frac{(\tan z)^3}{3} + \frac{(\tan z)^5}{5} - \frac{(\tan z)^7}{7} + \cdots$. If we let $t = \tan z$ so that z is the arc whose tangent is t, which we will indicate by arctan t, then $z = \arctan t$. When we know the tangent of t, the corresponding arc z is given by $z = \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \cdots$. If the tangent of t is equal to the unit radius, then the arc z is equal to 45 degrees or $z = \frac{\pi}{4}$ and $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. This series, which was first discovered by Leibnitz, can be used to find the value of the circumference of the circle.

141. In order to see the ease with which the length of an arc can be found by means of this series, we should substitute a sufficiently small fraction for the tangent t. For example let us use this series to find the length of the arc z whose tangent is $\frac{1}{10}$. In this case the arc

$z = \frac{1}{10} - \frac{1}{3000} + \frac{1}{500000} - \cdots$, and the approximate value of this series is easily expressed by a decimal fraction. However, from such an arc, we cannot conclude anything about the whole circumference of the circle, since the ratio of the arc whose tangent is $\frac{1}{10}$ to the whole circumference is not given. For this reason, in order to find the circumference, we look for an arc such that not only is it some fractional part of the circumference, but also small and easily expressed. For this purpose it is customary to choose the arc to be 30 degrees, whose tangent is equal to $\frac{1}{\sqrt{3}}$, since smaller arcs have tangents which are extremely irrational. Wherefore, since an arc of 30 degrees has length $\frac{\pi}{6}$, we have $\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3\sqrt{3}} + \frac{1}{5 \cdot 3^2\sqrt{3}} - \cdots$ and

$\pi = \frac{2\sqrt{3}}{1} - \frac{2\sqrt{3}}{3 \cdot 3} + \frac{2\sqrt{3}}{5 \cdot 3^2} - \frac{2\sqrt{3}}{7 \cdot 3^3} + \cdots$. By means of this series the value of π itself, which was previously exhibited, was determined with incredible labor.

142. The labor involved in this calculation is all the more since, first of all, each term is irrational, but also, since each succeeding term is only about one third of the preceding. In order to avoid these inconveniences, let us take the arc to be 45 degrees, that is of length $\frac{\pi}{4}$. Although this arc can be expressed by a series which hardly converges, $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, still we keep this arc and express it by means of two arcs of lengths a and b so that $a + b = \frac{\pi}{4}$, that is 45 degrees. Since $\tan(a + b) = 1 = \frac{\tan a + \tan b}{1 - \tan a \tan b}$, we have $1 - \tan a \tan b = \tan a + \tan b$ and $\tan b = \frac{1 - \tan a}{1 + \tan a}$. If we let

$\tan a = ½$, then $\tan b = \frac{1}{3}$ and both the arcs a and b can be expressed by rational series which converge much more rapidly than the series above. The sum of these two series gives the value of the arc $\frac{\pi}{4}$. It follows that

$$\pi = 4\left(\frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \frac{1}{9 \cdot 2^9} - \cdots\right)$$

$$+ 4\left(\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7} + \frac{1}{9 \cdot 3^9} - \cdots\right).$$ In this way we calculate the length of the semicircle, π, with much more ease than with the series mentioned before.

6-17

Leonhard Euler's

Vollständige Anleitung

zur

Differenzial-Rechnung.

Aus dem Lateinischen übersetzt

und

mit Anmerkungen und Zusätzen begleitet

von

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Erster Theil.

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messen wollen müsse. Aber sie sollten nur erst beweisen, daß diese ihre letzten Theile, davon jeder Körper eine bestimmte Anzahl enthalten soll, keine Ausdehnung haben.

§. 81.

Weil sie also keinen Ausgang aus diesem Labyrinthe finden, und die ihnen gemachten Einwürfe nicht auf die erforderliche Art widerlegen können, so nehmen sie ihre Zuflucht zu Distinktionen, und sagen, daß diese Einwürfe ihren Grund in der sinnlichen Vorstellung und in der Einbildungskraft haben, daß aber bey dergleichen Gegenständen bloß der reine Verstand urtheilen müsse, und daß die Sinnen und alle auf sinnliche Vorstellungen gebaute Schlüsse nicht selten trügen. Also soll der reine Verstand die Möglichkeit davon erkennen können, daß der tausendste Theil eines Cubik-Fußes von Materie gar keine Ausdehnung habe, und eben dies der Einbildungskraft unmöglich scheinen? Daß die Sinne trügen, ist zwar oft der Fall; allein den Mathematikern sollte man diesen Vorwurf am allerwenigsten machen. Denn die Mathematik ist es ja vorzüglich, die uns vor den Täuschungen der Sinne verwahrt, und uns lehrt, daß die sinnlichen Gegenstände ganz anders beschaffen sind, als sie uns erscheinen; ihr danken wir ja die sichersten Vorschriften, deren Befolgung uns wider alle Täuschung der Sinne schützt. Anstatt also durch dergleichen Antworten ihre Behauptungen zu befestigen, machen sie sie nur noch mehr verdächtig.

§. 82.

Doch um zu unserm Endzwecke zurück zu kommen, so sind, wenn auch jemand das Daseyn einer unendlichen Zahl in der Welt leugnen wollte, doch in der theoretischen Mathematik solche Fragen sehr häufig, auf welche nicht anders als mit Annehmung einer unendlichen Zahl geantwortet werden kann. Würde z. B. die Summe aller Zahlen, die diese Reihe, 1 + 2 + 3 + 4 + 5 + ꝛc. ausmachen, verlangt, so kann denn doch diese Summe, da jene Zahlen ohne Ende fortgehen, und beständig wachsen, auf keine Weise eine endliche Zahl seyn, und daraus folgt ihre unendliche Größe nothwendig. Wenn daher eine Größe so groß ist, daß sie größer ist als jede gegebene endliche Größe, so kann sie keine andere als eine unendliche Größe seyn. Um dergleichen Größen anzudeuten bedienen sich die Mathematiker des Zeichens ∞, und zeigen also dadurch eine Größe an, die größer ist als jede endliche Größe, oder größer als jede Größe, die sich angeben läßt. So kann man z. B., da man die Parabel durch eine unendlich lange Ellipse erklären kann, mit Recht behaupten, daß die Axe der Parabel eine unendliche gerade Linie sey.

§. 83.

Es wird aber die Lehre vom Unendlichen durch die Auseinandersetzung des unendlich Kleinen in der Mathematik deutlicher werden. Das leidet keinen Zweifel, daß eine jede Größe so weit vermindert werden kann, daß sie gänzlich verschwindet und zu nichts wird. Eine unendlich kleine Größe aber ist nichts anders als eine verschwindende Größe, und folglich in der That $= 0$. Diese Erklärung des unendlich Kleinen stimmt auch mit der überein, wenn man darunter Größen versteht, die kleiner sind als jede Größe, die sich angeben läßt. Denn wenn eine Größe kleiner ist als jede Größe, die sich angeben läßt, so muß sie nothwendig $= 0$ seyn; weil sich, wenn sie es nicht wäre, eine andere ihr gleiche Größe angeben ließe, welches wider die Voraussetzung streitet. Wir beantworten daher die Frage, was eine unendlich kleine

6-19

Leonhard Euler's

Differentialregning

Oversat til tysk og kommenteret af Johann Andreas Christian Michelsen. Uddrag herfra indlæst på dansk af Torben Rump, 1. april 1995.

Første del **Tredje kapitel** **Om det uendelige og det uendeligt små.**

$ 81

Fordi de altså ikke kan finde nogen udvej af denne labyrint og ikke kan gendrive de indvendinger, man gør mod dem, på den nødvendige måde, så tager de deres tilflugt til destinktioner og siger, at disse indvendinger har deres grund i den sanselige forestilling og i indbildningskraften, men at ved sådanne emner skal udelukkende den rene forstand dømme, og at sanserne og alle på sanselige forestillinger byggende slutninger ikke sjældent bedrager. Altså skal den rene forstand være i stand til at erkende muligheden af, at den tusindte del af en kubikfod af materie slet ikke har nogen udstrækning, og netop dette forekommer indbildningskraften umuligt. At sanserne bedrager er ganske vist ofte tilfældet, men matematikerne er dem, man allermindst skulle gøre denne bebrejdelse, for det er jo fornemmelig matematikken, som værner os mod sansebedragene og lærer os, at de sanselige genstande er ganske anderledes beskafne, end de fremtræder for os. Det er matematikken, vi kan takke for de sikreste forskrifter, som, hvis vi følger dem, beskytter os mod al sansebedrag. I stedet for altså gennem sådanne svar at befæste deres påstande gør de dem kun endnu mere fordægtige.

$ 82

Dog, for nu at komme tilbage til vort endelige formål, så er, selv om også nogle nu ville bestride eksistensen af et uendeligt tal i verden, dog i den teoretiske matematik sådanne spørgsmål meget hyppige, på hvilke der ikke kan svares på anden måde end ved at antage et uendeligt tal. Hvis man f.eks. ville forlange summen af alle tal, som udgør rækken $1 + 2 + 3 + 4 + 5 +$ etc., så kan dog denne sum, da de nævnte tal fortsætter uden ende og bestandigt vokser, på ingen måde være et endeligt tal, og deraf følger nødvendigvis dets uendelige størrelse. Hvis derfor en størrelse er så stor, at den er større end enhver givet endelig atørrelse, så kan den ikke være nogen anden end en uendelig størrelse. For at antyde sådanne størrelser betjener matematikerne sig af tegnet ∞ og angiver altså derved en størrelse, der er større end enhver endelig størrelse eller større end enhver størrelse, som lader sig angive. Således kan man f. eks., da man kan forklare parablen som en uendelig lang ellipse, med rette hævde, at parablens akse er en uendelig lige linie.

$ 83

Men læren om det uendelige vil ved diskussionen af det uendeligt små i matematikken blive tydeligere. Der kan ikke være nogen tvivl om, at enhver størrele kan formindskes så vidt, at den fuldstændig forsvinder og bliver til intet. En uendelig lille størrelse er imidlertid intet andet end en forsvindende størrelse og følgelig faktisk $= 0$. Men denne forklaring af det uendeligt små stemmer også overens med den forklaring, at man derved forstår størrelser, som er mindre end enhver størrelse, som lader sig angive. Thi, hvis en størrelse er mindre end enhver størrelse, som lader sig angive, så må den nødvendigvis være $= 0$; fordi der, hvis den ikke var det, så lod sig angive en anden størrelse, som ville være lig med den, hvilket strider mod forudsætningen. Vi besvarer derfor spørgsmålet, hvad en uendelig lille størrelse er i matematikken, på den måde, at vi siger, at den faktisk er $= 0$. Og dette begreb indeholder ingen af de store hemmeligheder, som man almindeligvis finder i det, og gennem hvilke man lader sig forlede til at fatte mistanke til hele regningen med det uendeligt små. Skulle der imidlertid eksistere tvivl, så vil denne i det følgende, når vi fremfører denne regning, helt blive ellimineret.

$ 84

Da vi altså har vist, at en uendelig lille størrelse virkelig er nul, så må vi frem for alt imødegå den indvending, hvorfor vi ikke bestandigt betegner de uendeligt små størrelser med tegnet 0, men bruger særlige tegn dertil; for da alle nuller er lig hinanden, så forekommer det overflødigt, at man til at betegne dem betjener sig af forskellige tegn, men hvorvel hver to nuller er hinanden lig, sådan at der slet ikke findes nogen forskel imellem dem, så findes der dog to arter af sammenligning af størrelser, af hvilke den ene er den aritmetiske og den anden den geometriske. Ved den førstnævnte iagttager vi forskellen, ved den sidstnævnte ser vi på den kvotient, der udspringer af sammenligningen af størrelserne, og omendskønt det aritmetiske forhold mellem hver to nuller er lige, så er det geometriske det ikke af den grund. Man ser dette meget tydeligt på følgende geometriske proportion $2:1 = 0:0$, hvori det fjerde led lige såvel er nul som det tredje, men på grund af proportionens natur, må, da det første led er dobbelt så stort som det andet, det tredje led også være dobbelt så stort som det fjerde.

$ 85

Dette fremgår også indlysende af den almindelige aritmetik, for da, som enhver ved, nul multipliceret med et hvilket som helst tal atter giver nul, eller $n \cdot 0 = 0$, og altså $n:1 = 0:0$, så er det umiddelbart indlysende, at to nuller, omend de aritmetisk betragtet står i et lighedsforhold, dog har ethvert geometrisk forhold til hinanden. Da altså nullerne kan have ethvert forhold til hinanden, så benytter man sig, for at angive denne forskellighed, med rette af forskellige tegn, især når man skal undersøge det geometriske forhold, som foreligger imellem dem. Men i infinitesimalregningen gør man ikke andet end at beskæftige sig med undersøgelsen af det geometriske forhold mellem forskellige uendeligt små størrelser, og man ville da havne i den største forvirring, dersom man ikke betegnede disse uendeligt små størrelser med forskellige tegn.

$ 86

Når man altså, sådan som det er sædvane i analysen af det uendelige, med dx betegner en uendelig lille størrelse, så er rigtignok både $dx = 0$ og $adx = 0$, hvor a betyder en hvilken som helst endelig størrelse. Men ikke desto mindre er det geometriske forhold $adx:dx$ et endeligt forhold, nemlig $a:1$, og de to uendeligt små størrelser dx og adx må, omend de begge er $= 0$, ikke forveksles med hinanden, når det kommer an på undersøgelsen af deres forhold. På lignende måde forholder det sig, når forskellige uendeligt små størrelser dx og dy forekommer. Thi selv om begge $= 0$, så er deres forhold dog ikke kendt, og i bestemmelsen af forholdet mellem hver to sådanne uendeligt små størrelser består hele differentialregningens anliggende. Hvor ringe end i øvrigt nytten af den slags sammenligninger ved første øjekast synes at være, så stor er den ikke desto mindre, og man lærer dag for dag at indse den mere og mere.

6-21

$ 87

Da altså det uendeligt små virkelig er intet, så er det indlysende, at man hverken forøger eller formindsker en endelig størrelse ved at addere eller subtrahere en uendeligt lille størrelse til henholdsvis fra den. Er altså a en endelig og dx en uendeligt lille størrelse, så er såvel $a + dx$ som $a - dx$ og i det hele taget $a \pm ndx = a$; for forholdet mellem $a \pm ndx$ og a er et lighedsforhold, hvadenten man undersøger dette aritmetisk eller geometrisk. For det aritmetiske forholds vedkommende er dette åbenbart, for da $ndx = 0$, så gælder at $a \pm ndx - a = 0$. Men hvad det geometriske forhold angår, så bliver det åbenbart af det forhold, at $(a \pm ndx):a = 1$. Heraf følger den generelt antagne regel, at de uendeligt små størrelser forsvinder over for de endelige størrelser og altså, når man beskæftiger sig med disse, kan udelades. Herved bortfalder også fuldstændigt den kritik, at analysen af det uendelige skulle fjerne sig fra den geometriske skarphed, da man intet udelader untagen det, som virkelig intet er. Man kan derfor med rette hævde, at man i denne del af den højere matematik iagttager den største geometriske skarphed, således som man finder den i de gamles skrifter.

$ 88

Da den uendeligt lille størrelse dx faktisk er $= 0$, så må også dens kvadrat dx^2, dens kubik dx^3 og enhver anden potens med en positiv eksponent være $= 0$ og altså også forsvinde over for en endelig størrelse, men også den uendeligt lille størrelse dx^2 selv forsvinder over for dx. Thi $dx \pm dx^2$ og dx står i et lighedsforhold, hvad enten man sammenligner dem aritmetisk eller geometrisk. Med hensyn til det førstnævnte er det hævet over enhver tvivl, men hvad det sidstnævnte angår, så er
$(dx \pm dx^2): dx = 1 \pm dx = 1$.
På lignende vis er $dx \pm dx^2 = dx$ og i det hele taget $dx \pm dx^{n+1} = dx$, for så vidt n er et positivt tal. For det geometriske forhold $(dx \pm dx^{n+1}): dx = 1 + dx^n$, og følgeligen, da $dx^n = 0$, et lighedsforhold. Hvis man derfor, som det sker ved potenserne, kalder dx en uendelig lille størrelse af 1. orden, dx^2 en uendelig lille størrelse af 2. orden, dx^3 en uendelig lille størrelse af 3. orden osv., så springer det i øjnene, at de uendeligt små størrelser af de højere ordener forsvinder over for de uendeligt små størrelser af første orden.

$ 89

På lignende vis viser man, at de uendeligt små størrelser af 3. og højere orden forsvinder over for de uendeligt små størrelser af 2. orden, og at i det hele taget de uendeligt små størrelser af enhver højere orden forsvinder over for de uendeligt små størrelser af en lavere orden. Er f. eks. m et mindre tal end n, så er $adx^m + bdx^n = adx^m$, da dx^n er forsvindende i forhold til dx^m.

6-22

$ 178

Af de utallige arter af transcendente eller ikke-algebraiske størrelser, som integralregningen stiller til rådighed, har vi i indledningen til analysen af det uendelige kunnet undersøge nogle hyppigt forekommende arter af disse størrelser, nemlig dem som læren om logaritmerne og cirkelstørrelserne frembyder. Da vi nu har gjort så tydeligt rede for disse størrelsers natur, at man kan betjene sig af dem i regningen lige så let som af de algebraiske størrelser, så vil vi i det nærværende kapitel opsøge disses differentialer, for at deres beskaffenhed og egenskaber kan erkendes endnu tydeligere. På denne måde vil tillige vejen til integralregningen, som er den egentlige kilde til de transcendente størrelser, blive banet.

$ 179

I første række kommer altså de logaritmiske størrelser eller sådanne funktioner af x, som foruden algebraiske udtryk også indeholder en logaritme til x eller en funktion af samme, i betragtning. Da nu de algebraiske størrelser i den forbindelse ikke kan frembyde nogen vanskelighed, så kommer det udelukkende an på påvisningen af differentialet af logaritmen til enhver funktion af x. Men omend der gives såre mange arter af logaritmer, så vil vi dog her for nemheds skyld kun betragte de hyperbolske logaritmer, og vi kan begrænse os dertil, fordi logaritmerne i forskellige systemer har et konstant forhold til hinanden, og man altså af den hyperbolske logaritme meget let kan udlede en hvilken som helst anden logaritme. Er nemlig den hyperbolske logaritme til en funktion p = ln(p)[*)], så er logaritmen til just denne funktion i et andet system = mln(p), hvor m betyder det tal, som udtrykker forholdet mellem logaritmerne i dette system og de hyperbolske. Udtrykket ln(p) skal derfor i nærværende undersøgelse stedse betyde den hyperbolske logaritme til p.

*) *Euler bruger betegnelsen* l p *i stedet for* ln(p)

$ 180

Vi vil altså opsøge differentialet til den hyperbolske logaritme til x og i den forbindelse sætte y = ln(x), således at værdien af differentialet dy skal bestemmes. Sætter man nu x + dx i stedet for x, så går y over i y + dy, og derfor bliver

$$y + dy = \ln(x + dx) \text{ og } dy = \ln(x+dx) - \ln(x) = \ln(1 + dx/x).$$

Fra indledningen til analysen af det uendelige er imidlertid kendt, at

$$\ln(1+ z) = z - z^2/2 + z^3/3 - z^4/4 + \text{osv.}$$

og sætter man derfor dx/x i stedet for z, så bliver

$$dy = dx/x - dx^2/2x^2 + dx^3/3x^3 - dx^4/4x^4 + \text{osv.}$$

og da alle følgende led i denne række forsvinder over for det første, så bliver

$$d\ln(x) = dy = dx/x$$

og følgelig differentialet til enhver anden logaritme, som forholder sig til den hyperbolske som n:1, = ndx/x.

$ 181

Hvis altså der er givet logaritmen til en eller anden funktion af x, som vi vil kalde p, altså ln(p), så finder man ad just denne vej, at differentialet til samme = dp/p, og således fås til bestemmelsen af differentialerne til logaritmerne følgende regel: Man søge differentialet til størrelsen p, hvis logaritme er givet, og dividerer samme med denne størrelse p. Denne regel fremgår også af udtrykket $(p^o - 1^o)/o$, til hvilken vi ovenfor har reduceret logaritmen til p. Lad $\omega = 0$, så bliver, fordi $\ln(p) = (p^\omega - 1)/\omega$, $d\ln(p) = d(1/\omega\, p^\omega) = p^{\omega-1}dp = dp/p$, idet $\omega = 0$. I den forbindelse skal bemærkes at dp/p er differentialet til den hyperbolske logaritme til p, således at man altså, hvis den almindelige logaritme til p er givet, skal multiplicere differentialet dp/p med tallet 0,43429448.. etc.

$ 182

Takket være denne regel kan man meget let finde differentialet til logaritmen til enhver funktion af x, hvilket følgende eksempler vil bekræfte

I Er $\mathbf{y = ln(x)}$, så er $\mathbf{dy = dx/x}$

II Er $\mathbf{y = ln(x^n)}$, så sætte man $x^n = p$, således at altså $y = ln(p)$. Følgelig har man da $dy = dp/p$, og da $dp = nx^{n-1}dx$, så bliver

$$\mathbf{dy = ndx/x}$$

Just dette fremgår af logaritmernes natur. Thi da $ln(x^n) = nln(x)$, så er også $dln(x^n) = ndln(x) = ndx/x$

III Er $\mathbf{y = ln(1+xx)}$, så er $\mathbf{dy = 2xdx/(1+xx)}$

IV Er $\mathbf{y = ln(1/\sqrt{(1 - xx)})}$, så er da $y = -ln(\sqrt{(1 - xx)}) = -½ln(1 - xx)$, $\mathbf{dy = xdx/(1 - xx)}$

V Er $\mathbf{y = ln(x/\sqrt{(1 + xx)})}$, så er da $y = ln(x) - ½ln(1 + xx)$,

$$dy = dx/x - xdx/(1 + xx) = \mathbf{dx/(x(1+xx))}$$

VI Er $\mathbf{y = ln(x + \sqrt{(1 + xx)})}$, så bliver

$dy = (dx + xdx/\sqrt{(1 + xx)})/((x + \sqrt{(1 + xx)})$

$= (xdx + dx\sqrt{(1 + xx)})/((x + \sqrt{(1 + xx)})\sqrt{(1 + xx)})$

og da tælleren og nævneren er delelige med $x + \sqrt{(1 + xx)}$, så bliver

$$\mathbf{dy = dx/\sqrt{(1 + xx)}}$$

VII Er $\mathbf{y = (1/\sqrt{-1})ln(x\sqrt{-1} + \sqrt{(1 - xx)})}$, så sætte man $x\sqrt{-1} = z$. Da altså nu $y = (1/\sqrt{-1})ln(z + \sqrt{(1 + zz)})$, så erp.gr.a. det foregående at $dy = (1/\sqrt{-1})dz/\sqrt{(1 + zz)}$ og da $dz = dx\sqrt{-1}$ så bliver

$$\mathbf{dy = dx/\sqrt{(1 - xx)}}$$

Omend altså den givne logaritme indbefatter imaginære størrelser i sig, bliver dog desuagtet differentialet til samme reelt.

$ 183

Hvis den størrelse, hvis logaritme er givet, består af faktorer, så lader logaritmen selv sig dele i flere andre. Er f. eks. $y = ln(pqrs)$, så bliver, da $y = ln(p) + ln(q) + ln(r) + ln(s)$, $dy = dp/p + dq/q + dr/r + ds/s$. Opsplitningen finder ligeledes sted, hvis den størrelse, hvis logaritme skal differentieres, er en brøk. Er nemlig $y = ln((pq)/(rs))$, så bliver, da $y = ln(p) + ln(q) - ln(r) - ln(s)$, $dy = dp/p + dq/q - dr/r - ds/s$. Ej heller potenserne frembyder nogen vanskelighed. Thi er $y = ln((p^m q^n)/(r^\mu s^\rho))$, så bliver, da $y = mln(p) + nln(q) - \mu ln(r) - \rho ln(s)$, $dy = mdp/p + ndq/q - dr/r - \rho ds/s$.

I Er $y = ln((a + x)(b + x)(c + x))$, så bliver, fordi da $y = ln(a + x) + ln(b + x) + ln(c + x)$, så bliver

$$dy = dx/(a + x) + dx/(b + x) + dx/(c + x)$$

II Er $\mathbf{y = ½ln((1 + x)/(1 - x))}$, så bliver $y = ½ln((1 + x) - ½ln((1 - x)$ og altså

$$\mathbf{dy} = ½dx/(1 + x) - ½dx/(1- x) = \mathbf{dx/(1 - xx)}$$

III Er $\mathbf{y = ½ln((\sqrt{(1 + xx)} + x)/(\sqrt{(1 + xx)} - x))}$, så er $y = ½ln(\sqrt{(1 + xx)} + x) - ½ln(\sqrt{(1 + xx)} - x)$, og altså

$$\mathbf{dy} = ½dx/\sqrt{(1 + xx)} + ½dx/\sqrt{(1 + xx)} = \mathbf{dx/\sqrt{(1 + xx)}}$$

Just dette differential finder man endnu lettere, hvis man bortskaffer den irrationale nævner i brøken ved multiplikation i tæller og nævner med $\sqrt{(1 + xx)} + x$. Så får man nemlig

$$y = ½ln(\sqrt{(1 + xx)} + x)^2 = ln(\sqrt{(1 + xx)} + x)$$

og deraf er på basis af det foregående kendt, at

$$dy = dx/\sqrt{(1 + xx)}$$

IV Er $\mathbf{y = \ln[(\sqrt{(1+x)} + \sqrt{(1-x)})/(\sqrt{(1+x)} - \sqrt{(1-x)})]}$, så sætte man tælleren i denne brøk $\sqrt{(1+x)} + \sqrt{(1-x)} = p$, og nævneren $\sqrt{(1+x)} - \sqrt{(1-x)} = q$, hvorved man får
$y = \ln(p/q) = \ln(p) - \ln(q)$ og $dy = dp/p - dq/q$. Da nu
$dp = dx/(2\sqrt{(1+x)}) - dx/(2\sqrt{(1-x)}) = -dx(\sqrt{(1+x)} - \sqrt{(1-x)})/(2\sqrt{(1-xx)}) =$
$-qdx/(2\sqrt{(1-xx)})$ og
$dq = dx/(2\sqrt{(1+x)}) + dx/(2\sqrt{(1-x)}) = pdx/(2\sqrt{(1-xx)})$, så bliver
$dp/p - dq/q = -qdx/(2p\sqrt{(1-xx)}) - pdx/(2q\sqrt{(1-xx)}) = (pp+qq)dx/(2pq\sqrt{(1-xx)})$, og
da $pp + qq = 4$ og $pq = 2x$, så bliver

$$\mathbf{dy = -dx/(x\sqrt{(1-xx)})}$$

Dette differentiale finder man imidlertid på en lettere måde, hvis man forvandler den givne logaritme på følgende måde

$$y = \ln[(1+\sqrt{(1-xx)})/x] = \ln[1/x + \sqrt{(1/xx - 1)}],$$

idet man nemlig multiplicerer tæller og nævner med $\sqrt{(1+x)} + \sqrt{(1-x)}$. For sætter man i den forbindelse $1/x + \sqrt{(1/xx - 1)} = p$, så bliver

$$dp = -dx/(xx) - dx/(x^3\sqrt{(1/xx - 1)}) = -dx/(xx) - dx/(xx\sqrt{(1-xx)})$$
$$= -dx(1 + \sqrt{(1-xx)})/(xx\sqrt{(1-xx)}$$ og altså fås, eftersom $p = (1+\sqrt{(1-xx)})/x$,
$$dy = dp/p = -dx/(x\sqrt{(1-xx)})$$ som før.

$ 184

Da nu de første differentialer, dersom man dividerer dem med dx, bliver algebraiske størrelser, så lader de næstfølgende differentialer sig let udlede efter forskrifterne i det foregående kapitel, forudsat at differentialet dx betragtes som en konstant størrelse. Således er, hvis man sætter $y = \ln(x)$,

$dy = dx/x$ og $dy/dx = 1/dx$
$ddy = -dx^2/x^2$ og $ddy/dx^2 = -1/x^2$
$d^3y = 2dx^3/x^3$ og $d^3y/dx^3 = 2/x^3$
$d^4y = -6dx^4/x^4$ og $d^4y/dx^4 = -6/x^4$, etc.

og hvis p er en algebraisk størrelse, og $y = \ln(p)$, så er også, omend y ikke er en algebraisk størrelse, dog dy/dx, ddy/dx^2, d^3y/dx^3 etc. algebraiske funktioner af x.

$ 185

Efter således at differentiationen af logaritmerne er blevet forklaret, så lader de funktioner, som består af algebraiske og logaritmiske størrelser, sig let differentiere, og lige så ringe vanskelighed frembyder de størrelser, som alene er sammensat af logaritmerne. Følgende eksempler vil gøre dette tydeligt.

I Er $\mathbf{y = \ln(x)^2}$, så sætte man $\ln(x) = p$. Da nu $y = p^2$, så bliver $dy = 2pdp$, og da $dp = dx/x$, så er

$$\mathbf{dy = (2dx/x)\ln(x)}$$

II På samme måde bliver, hvis $\mathbf{y = \ln(x)^n}$, $\mathbf{dy = (ndx/x)\ln(x)^{n-1}}$ og hvis altså $\mathbf{y = \sqrt{\ln(x)}}$, så er, da $n = ½$, $\mathbf{dy = dx/(2x\sqrt{\ln(x)})}$

III Er endvidere p en eller anden funktion af x og $\mathbf{y = \ln^n(p)}$, så bliver $\mathbf{dy = (ndp/p)\ln^{n-1}(p)}$. Da nu differentialet af p kan findes af det foregående, så er derved også differentialet til y kendt.

IV Er $y = \ln(p)\ln(q)$, og er p og q funktioner af x, så er i henhold til den ovenstående om faktorer givne regel

$$dy = (dp/p)\ln(q) + (dq/q)\ln(p)$$

V Hvis $\mathbf{y = x\ln(x)}$, så finder man efter den selvsamme regel

$$\mathbf{dy} = dx\ln(x) + xdx/x = \mathbf{dx\ln(x) + dx}$$

VI Hvis $\mathbf{y = x^m\ln(x) - (1/m)x^m}$, så finder man, når man opsøger differentialerne til delene, at $d[x^m\ln(x)] = mx^{m-1}dx\ln(x) + x^{m-1}dx$ og $d[(1/m)x^m] = x^{m-1}dx$. Det giver altså

$$\mathbf{dy = mx^{m-1}dx\ln(x)}$$

VII Er $\mathbf{y = x^m\ln^n(x)}$, så bliver $\mathbf{dy = mx^{m-1}dx\ln^n(x) + nx^{m-1}dx\ln^{n-1}(x)}$

6-25

VIII Forekommer logaritmer til logaritmer, f. eks. hvis **$y = \ln(\ln(x))$**, så sætte man $\ln(x) = p$, hvorved $y = \ln(p)$ og $dy = dp/p$. Men $dp = dx/x$ og altså bliver **$dy = dx/(x\ln(x))$**.

IX Er **$y = \ln(\ln(\ln(x)))$**, så bliver, hvis man sætter $\ln(x) = p$, $y = \ln(\ln(p))$ og altså ifølge det foregående eksempel $dy = dp/(p\ln(p))$. Da nu $dp = dx/x$, så bliver **$dy = dx/[x\ln(x)\ln(\ln(x))]$**

$ 186

Efter denne forklaring af logaritmernes differentiering vil vi skride til de eksponetielle størrelser, eller til de potenser, hvis eksponent er en foranderlig størrelse. Af den slags funktioner af x, lader differentialerne sig finde ved differentiation af logaritmerne på følgende måde. Skal differentialet af a^x findes, så sætte man $y = a^x$, hvorved, når man tager logaritmerne, $\ln(y) = x\ln(a)$. Differentierer man nu, så bliver $dy/y = dx\ln(a)$, og altså $dy = ydx\ln(a)$; og da $y = a^x$, så bliver endvidere $dy = a^x dx\ln(a)$, og dette er det søgte differentiale af a^x. På lignende måde finder man, hvis p er en funktion af x, at differentialet af den eksponentielle størrelse a^p er $a^p dp\ln(a)$.

$ 187

Just dette differentiale kan imidlertid også umiddelbart afledes af det, der blev sagt i indledningen om de eksponentielle størrelsers natur. Lad der være givet udtrykket a^p, hvori p skal betyde en funktion af x, således at, hvis man sætter $x + dx$ i stedet for x, p overgår til $p + dp$. Sætter man derfor $y = a^p$, så bliver, hvis x går over i $x + dx$, $y + dy = a^{p+dp}$ og altså $dy = a^{p+dp} - a^p = a^p(a^{dp} - 1)$. Men nu er det vist, at man kan udtrykke enhver eksponentiel størrelse a^z ved følgende række $1 + z\ln(a) + z^2\ln^2(a)/2 + z^3\ln^3(a)/6 +$ etc. Følgelig bliver $a^{dp} = 1 + dp\ln(a) + dp^2\ln^2(a)/2 +$ etc. og $a^{dp} - 1 = dp\ln(a)$, fordi samtlige de følgende led forsvinder over for $dp\ln(a)$; og der gælder følgelig at $dy = d(a^p) = a^p dp\ln(a)$.

Differentialet af en eksponetiel størrelse er altså et produkt af den eksponentielle størrelse, af eksponentens differentiale dp og af logaritmen til den konstante størrelse a i den tænkte eksponents variable potens.

$ 188

Er altså e tallet, hvis hyperbolske logaritme er $=1$, så er differentialet til $e^x = e^x dx$. Betragter man derfor dx som en konstant størrelse, så bliver differentialet til $e^x dx = e^x dx^2$, og dette er det andet differentiale af e^x. Ligeledes bliver det trdje differentiale $= e^x dx^3$. Er derfor $y = e^{nx}$, så bliver $dy/dx = ne^{nx}$; $ddy/dx^2 = n^2e^{nx}$; $d^3y/dx^3 = n^3e^{nx}$; $d^4y/dx^4 = n^4e^{nx}$ osv. Man ser heraf, at det første, det andet og de følgende differentialer af e^{nx} udgør en geometrisk progression, og at $d^my/dx^m = n^me^{nx}$, og følgelig at $d^my/(ydx^m)$ er en konstant størrelse $= n^m$.

$ 189

Hvis den til en variabel potens opløftede størrelse selv er en variabel størrelse, så finder man differentialet deraf på en lignende måde. Lad p og q være funktioner af x og $y = p^q$. Tager man nu logaritmerne, så bliver $\ln(y) = q\ln(p)$, og differentierer man, så bliver $dy/y = dq\ln(p) + qdp/p$, hvoraf der fås at $dy = ydq\ln(p) + yqdp/p = p^q dq\ln(p) + qp^{q-1}dp$, fordi $y = p^q$.

OPGAVE 1.
Chauchy on the derivative as a limit

Diskutér Chauchy's begreber, især "differentialet", og sammenlign med forgængernes og vores måde at gøre det på

OPGAVE 2.
Chauchy on Maclaurin's Theorem.

Diskutér igen Chauchy's begreber, herunder hans modeksempel (70), (71). Hvilken definition af f'(x) gøres suspekt med dette eksempel ?

Kilde XVIII fra A Source Book in Classical Analysis (ed. G. Birkhoff), Harvard University Press, Cambridge Mass. 1973, pp. 1-11.

2. *Cauchy on the Derivative as a Limit.*[1]

THIRD LESSON

Derivatives of Functions of One Variable

When the function $y = f(x)$ is continuous between two given limits of the variable x, and one assigns a value between these limits to the variable, an infinitesimal increment Δx of the variable produces an infinitesimal increment in the function

[6] "Imaginaire" has been translated "complex" throughout this book. Cauchy uses x for z, z for r, and $\sqrt{-1}$ for i.

[7] In modern notation, let $A = \lim\sup_{n\to\infty} |a_n|^{1/n}$.

[1] A. L. Cauchy, *Résumé des leçons . . . sur le calcul infinitésimal* (Paris, 1823); *Oeuvres* (2), IV, 22ff, 27ff. Our translation has been adapted from the translation by Evelyn Walker (E. W.) in Smith, *Source Book.*

itself. Consequently, if we then set $\Delta x = h$,[2] the two terms of the *difference quotient* ["rapport aux différences"]

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h} \tag{1}$$

will be infinitesimals. But whereas these terms tend to zero simultaneously, the ratio itself may converge to another limit, either positive or negative. This limit, when it exists, has a definite value for each particular value of x; but it varies with x. Thus, for example, if we take $f(x) = x^m$, m being a [positive] integer, the ratio of the infinitesimal differences will be

$$\frac{(x+h)^m - x^m}{h} = mx^{m-1} + \frac{m(m-1)}{1 \cdot 2} x^{m-2}h + \cdots + h^{m-1},$$

and it will have for [its] limit the quantity mx^{m-1}, that is to say, a new function of the variable x. The same will hold generally; only the form of the new function which serves as the limit of the ratio $[f(x+h) - f(x)]/h$ will depend upon the form of the given function $y = f(x)$. In order to indicate this dependence, we give to the new function the name derivative ["fonction dérivée"] and we designate it, using a prime, by the notation y' or $f'(x)$.[3]

FOURTH LESSON

Differentials of Functions of a Single Variable

Let $y' = f(x)$ remain a function of the independent variable x; let h be an infinitesimal and k a finite quantity. If we set $h = \alpha k$, α will also be an infinitesimal quantity, and we will have identically

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x+\alpha k) - f(x)}{\alpha k},$$

whence one concludes that

$$\frac{f(x+\alpha k) - f(x)}{\alpha} = \frac{f(x+h) - f(x)}{h} k. \tag{1}$$

The limit toward which the left side of equation (1) converges as the variable α tends to zero, the quantity k remaining constant, is called the *differential* of the function $y = f(x)$. We indicate this differential by the symbol d, as follows:

$$dy \quad \text{or} \quad df(x).$$

It is easy to obtain its value when we know that of the derivative y' or $f'(x)$. Indeed, taking the limits of the two sides of equation (1), we shall find generally

$$df(x) = kf'(x). \tag{2}$$

In the special case where $f(x) = x$, equation (2) reduces to

$$dx = k. \tag{3}$$

[2] Cauchy uses i for h and h for k.
[3] The phrase "fonction dérivée" and the notation $f'(x)$ were due to Lagrange.

Thus the differential of the independent variable x is just the finite constant k. Granting this, equation (2) becomes

$$df(x) = f'(x)\,dx \tag{4}$$

or, what amounts to the same thing,

$$dy = y'\,dx. \tag{5}$$

It follows from these last [equations] that the derivative $y' = f'(x)$ of any function $y = f(x)$ is precisely equal to dy/dx, that is, to the ratio of the differential of the function to that of the variable, or, if one wishes, to the coefficient by which the second differential must be multiplied in order to obtain the first. It is for this reason that we sometimes give to the derivative the name of *differential coefficient*.[4]

3. *Cauchy on Maclaurin's Theorem*[1]

NINTH LESSON

Theorems of Maclaurin and Taylor

When, in some interval and for values of θ satisfying $0 \leqq \theta \leqq 1$ ["inférieurs à l'unité"], one of the [two] expressions

$$x^n f^{(n)}(\theta x)/n!, \tag{1}$$

$$x^n(1-\theta)^{n-1} f^{(n)}(\theta x)/(n-1)! \tag{2}$$

tends to zero ["décroit indéfiniment"] as n increases, then, setting $n = \infty$ in equation (8) or (61) of the Eighth Lesson, one finds that

$$f(x) = f(0) + xf'(0) + x^2 f''(0)/2! + x^3 f'''(0)/3! + \cdots. \tag{3}$$

Therefore . . . the [infinite] series whose general term is the product $x^n f^{(n)}(0)/n!$ is convergent in the given interval, and its sum is $f(x)$. This is Maclaurin's Theorem.

[Cauchy then illustrates Maclaurin's Theorem by the examples e^x, $\cos x$, $\sin x$, $\arctan x$, and extends it to Taylor's Theorem. He remarks: "It is essential to observe that the formulas of Maclaurin and Taylor are valid not only for real but also for imaginary values of the function $f(x)$," and illustrates with the power series expansions for $\cos x + i \sin x$ and $e^{ax}(\cos bx + i \sin bx)$.]

TENTH LESSON

Rules for the Convergence of Series.
Application to Maclaurin and Taylor series.

[Cauchy first establishes, not too rigorously, some general tests for convergence and divergence. He then uses these tests to justify the following result.]

THEOREM 3. Let $f(x)$ be any real or complex function of the variable x, and let

[4] After this Cauchy gives the rules for differentiating various elementary functions: algebraic, exponential, trigonometric, and inverse trigonometric. (E. W.)

[1] A. L. Cauchy, *Leçons de calcul différentiel* (Paris, 1829); *Oeuvres* (2), IV, 364, 384–385, and 391–395.

$\beta_n = |f^{(n)}(0)/(n!)|$. Let moreover $\Phi = \limsup \beta_n^{1/n}$, or also the limit (if it exists) of the limit β_{n+1}/β_n. Then the Maclaurin series [whose terms are]

(31) $$f(0),\ xf'(0),\ x^2f''(0),\ x^3f'''(0), \ldots$$

will converge for all [real or complex] x with $|x| < 1/\Phi$, and divergent whenever $|x| > 1/\Phi$.

[Theorem 4 presents the analogous result for Taylor series.]

. . . . .

Arguments similar to those which we have just used to establish Theorems 3 and 4, when applied to Taylor series instead of Maclaurin series, lead immediately to two other theorems which we are going to state.

THEOREM 5. Let $f(x)$ be a real or complex function of the real variable x, h a real or imaginary constant, and γ_n the modulus of the expression $f^n(x)/(n!)$. Let moreover[2] $\gamma = \limsup \sqrt[n]{\gamma_n}$—or, if it exists, the limit of the ratio γ_{n+1}/γ_n. Then the Taylor series

(64) $$f(x) + hf'(x) + h^2f''(x)/2! + h^3f'''(x)/3! + \cdots$$

will converge if $|h| < \gamma^{-1}$, and diverge whenever $|h| > \gamma^{-1}$.

Examples. If we take for $f(x)$ any of the three functions e^x, $\cos x$, or $\sin x$, then $\gamma = 0$ and $\gamma^{-1} = \infty$. Therefore the series (64) will converge for all finite h, real or complex, as we have already remarked.

If we take for $f(x)$ one of the functions $\log x$ or x^μ, then $\gamma = 1/|x|$ and $\gamma^{-1} = |x|$. Therefore the (Taylor) series (64) will converge as long as $|h| < |x|$, as is known.

. . . . .

One might think that the Maclaurin series [of $f(x)$] always had $f(x)$ for sum, when it was convergent, and that when all its successive terms vanish $f(x)$ itself vanishes. But to become convinced of the contrary, it suffices to observe that the second condition holds if one sets

(70) $$f(x) = e^{-1/x^2},$$

and the first, if one sets

(71) $$g(x) = e^{-x^2} + e^{-1/x^2}.$$

However, the function e^{-1/x^2} does not vanish identically, and the series derived from the first supposition has for sum not the binomial $e^{-x^2} + e^{-1/x^2}$, but its first summand.

As to the rest, one easily recognizes that a real or complex function $f(x)$ cannot be the sum of a convergent (power) series, ordered by ascending powers of x, unless this series coincides with the Maclaurin series [of $f(x)$]. Indeed, let

(72) $$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

Differentiating this equation repeatedly with respect to x, and observing[3] that Eq.

[2] In Cauchy's words: "the limit toward which, as n increases without limit, the greatest values of $\gamma_n^{1/n}$ tend."

[3] Note that Cauchy casually assumes here that a convergent power series can be differentiated termwise.

(12) of Lesson 3 holds even in the case where the polynomial has an infinite number of terms, we obtain

$$(73)\qquad \begin{aligned} f'(x) &= a_1 + 2a_2 x + 3a_2 x^2 + \cdots, \\ f''(x) &= 1 \cdot 2a_2 + 2 \cdot 3a_3 x + \cdots, \\ f'''(x) &= 1 \cdot 2 \cdot 3a_3 + \cdots. \end{aligned}$$

Hence, setting $x = 0$, we obtain from Eqs. (72) and (73)

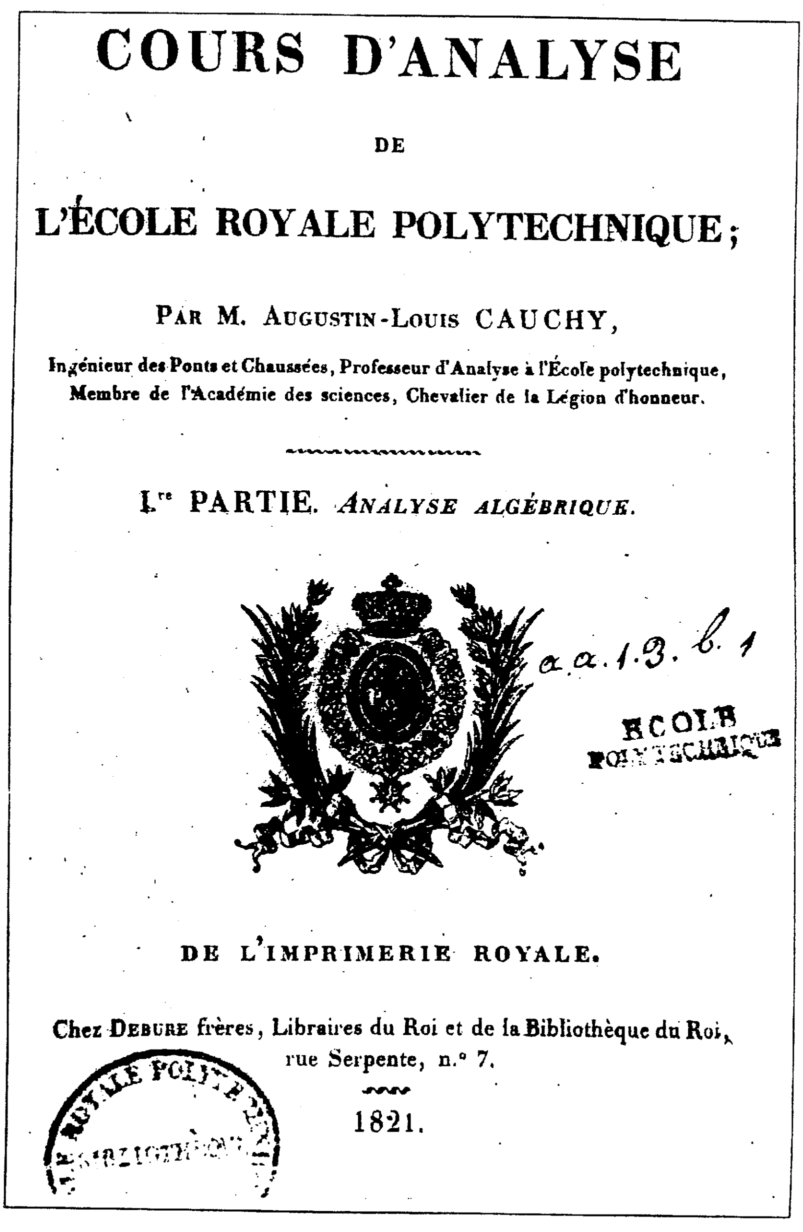
$$(74)\qquad f(0) = a_0, \quad f'(0) = a_1, \quad f''(0) = 1 \cdot 2a_2, \quad f'''(0) = 1 \cdot 2 \cdot 3a_3, \quad \text{etc.},$$

and consequently

$$(75)\qquad a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = f''(0)/2!, \quad a_3 = f'''(0)/3!,$$

etc. Now, substituting into formula (72) the preceding values of $a_0, a_1, a_2, \ldots,$ we are evidently led to the Maclaurin series for x.

One can prove in the same way that the function $f(x+h)$ cannot be the sum of a power series arranged in order of increasing exponents, unless this series is the Taylor series of $f(x)$.



COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

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1821.

Cauchy as young academician. Half-length portrait by Boilly, 1821. Published by permission of the École Polytechnique.

The Montagne Sainte-Geneviève in 1840. Lithograph. In the foreground, the École Polytechnique, in the middle ground, left, the Collège Royal Henri IV, formerly École Centrale du Panthéon, which the young Cauchy attended in 1802–1804, and, in the background, the Panthéon. Published by permission of the École Polytechnique.

Gauss in 1828 (lithograph by S. Bendixen).

OPGAVE 3.
Fra : Mat 3MH noter (Københavns Universitet) ved Jesper Lützen. Opgave 27.

I sin "Resumé des leçons données à l'école polytechnique sur le calcul infinitésimal" fra 1823 gav Cauchy en ny definition af integralet, der i århundredret forud var blevet defineret som det omvendte af differentialet.
Nedenstående oversættelse stammer fra Judith V. Grabiner "The Origins of Cauchy's Rigorous Calculus", MIT Press, Cambridge 1981, s- 171-174.

1. Hvordan defineres integralet ?
2. Gennemgå Cauchy's bevis for at integralet af en kontinuert funktion over et begrænset interval eksisterer
3. Påpeg to basale sætninger som bruges uden at det bemærkes.
4. Hvilken fordel har Cauchy's definition frem for den tidligere definition af integralet som det omvendte af differentialoperatoren.

Suppose that the function $y = f(x)$ is continuous with respect to the variable x between the two finite limits $x = x_0, x = X$. We designate by $x_1, x_2, \ldots, x_{n-1}$ new values of x placed between these limits and suppose that they either always increase or always decrease between the first limit and the second. We can use these values to divide the difference $X - x_0$ into elements

(1) $x_1 - x_0, x_2 - x_1, x_3 - x_2, \ldots, X - x_{n-1}$,

which all have the same sign. Once this has been done, let us multiply each element by the value of $f(x)$ corresponding to the left-hand end point [*origine*] of that element: that is, the element $x_1 - x_0$ will be multiplied by $f(x_0)$ [*Calcul infinitésimal* has the misprint $f(x)$], the element $x_2 - x_1$ by $f(x_1), \ldots$ and finally the element $X - x_{n-1}$ by $f(x_{n-1})$; and let

(2) $$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \cdots + (X - x_{n-1})f(x_{n-1})$$

be the sum of the products so obtained. The quantity S clearly will depend upon

1st: the number n of elements into which we have divided the difference $X - x_0$;

2nd: the values of these elements and therefore the mode of division adopted.

It is important to observe that if the numerical values of these elements become very small and the number n very large, the mode of division will have only an insignificant effect on the value of S. This in fact can be proved as follows.

If we were to suppose all the elements of the difference $X - x_0$ reduced to a single one, which would just be that difference, we would have simply that

(3) $$S = (X - x_0)f(x_0).$$

When instead we take the expressions (1) for the elements of the difference $X - x_0$, the value of S, determined in this case by equation (2), will be equal to the sum of the elements multiplied by a mean of the coefficients $f(x_0)$, $f(x_1), f(x_2), \ldots, f(x_{n-1})$.*

Moreover, since these coefficients [the $f(x_k)$] are particular values of the expression $f[x_0 + \theta(X - x_0)]$ for values of θ between zero and one, we can prove by arguments similar to those used in the seventh lecture [in proving the theorem about bounds on the differential quotient: see the appendix, p. 169] that the mean in question is another value of the same expression, corresponding to a value of θ between the same limits. We can then substitute the following for equation (2):

$$(4) \quad S = (X - x_0)\, f[x_0 + \theta(X - x_0)],$$

where θ will be a [nonnegative] number less than one.

To go from the mode of division we have just considered to another in which the numerical values of the elements of $X - x_0$ are still smaller, it suffices to divide each of the expressions (1) [that is, the $(x_k - x_{k-1})$] into new elements. We must then replace in the right-hand side of equation (2) the product $(x_1 - x_0)f(x_0)$ by a sum of similar products, for which sum we may substitute an expression of the form $(x_1 - x_0)f[x_0 + \theta_0(x_1 - x_0)]$, where θ_0 is a [nonnegative] number less than one: note that we will have a relation between this sum and the product $(x_1 - x_0)\, f(x_0)$, which is similar to the relation that exists between the values of S given by equations (4) and (3). Similarly, we must substitute for the product $(x_2 - x_1) \cdot f(x_1)$ a sum of terms that can be written in the form $(x_2 - x_1)f[x_1 + \theta_1(x_2 - x_1)]$, where θ_1 again designates a [nonnegative] number less than one. Continuing in this way, we finally conclude that in the new mode of division the value of S will be of the form

* Cauchy had defined a *mean* of a set of elements $\{a_1, \ldots, a_n\}$, which he designated by $M(a_1 \cdots a_n)$, to be a quantity included between the minimum and maximum of the elements of the set. At this point in the discussion of the integral, to justify the conclusion quoted above, Cauchy referred to the *Cours d'analyse*, Theorem III, Corollary (see the edition of Cauchy's *Oeuvres*, series 2, vol. 3, p. 28). This corollary, restated in modern index notation for clarity, is as follows: Suppose we are given a set of n quantities all having the same sign: $\{y_1, y_2, \ldots, y_n\}$. Consider another set $\{a_1, \ldots, a_n\}$ of n quantities, and recall that their mean $M(a_1 \cdots a_n)$ is included between the minimum and maximum of the a_k. Then the corollary states $a_1 y_1 + a_2 y_2 + \cdots + a_n y_n = (y_1 + \cdots + y_n) M(a_1 \cdots a_n)$.

Applying this to the problem in the text, let $a_k = f(x_{k-1}), k = 1, 2, \ldots, n$, and let $y_k = x_k - x_{k-1}, k = 1, 2, \ldots, n-1$, $y_n = X - x_{n-1}$. Using the corollary, Cauchy's conclusion is $f(x_0)(x_1 - x_0) + \cdots + f(x_{n-1})(X - x_{n-1}) = (X - x_0)\, M[f(x_0) \cdots f(x_{n-1})]$.

$$(5)\quad S = (x_1 - x_0) f[x_0 \theta_0 (x_1 - x_0)] + (x_2 - x_1) f[x_1 + \theta_1 (x_2 - x_1)] + \cdots + (X - x_{n-1}) f[x_{n-1} + \theta_{n-1}(X - x_{n-1})].$$

If in equation (5) we set

$$f[x_0 + \theta_0 (x_1 - x_0)] = f(x_0) \pm \varepsilon_0,$$

$$f[x_1 + \theta_1 (x_2 - x_1)] = f(x_1) \pm \varepsilon_1,$$

$$\vdots$$

$$f[x_{n-1} + \theta_{n-1}(X - x_{n-1})] = f(x_{n-1}) \pm \varepsilon_{n-1},$$

we can derive

$$(6)\quad S = (x_1 - x_0)[f(x_0) \pm \varepsilon_0] + (x_2 - x_1)[f(x_1) \pm \varepsilon_1] + \cdots + (X - x_{n-1})[f(x_{n-1}) \pm \varepsilon_{n-1}].$$

Then, working out these products,

$$(7)\quad S \doteq (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) + \cdots + (X - x_{n-1}) f(x_{n-1}) \pm \varepsilon_0 (x_1 - x_0) \pm \varepsilon_1 (x_2 - x_1) \pm \cdots \pm \varepsilon_{n-1}(X - x_{n-1}).$$

We may add that if the elements $x_1 - x_0, x_2 - x_1, \ldots, X - x_{n-1}$ have very small numerical values, each of the quantities $\pm \varepsilon_0, \pm \varepsilon_1, \ldots, \pm \varepsilon_{n-1}$ will be very close to zero, and therefore the same will be true for the sum $\pm \varepsilon_0 (x_1 - x_0) \pm \varepsilon_1 (x_2 - x_1) \pm \cdots \pm \varepsilon_{n-1}(X - x_{n-1})$, which is equivalent to the product of $X - x_0$ by a mean between these quantities [the ε_k; again, Cauchy has used the corollary about means]. Granting this, when we compare equations (2) and (7) we see that we would not change perceptibly the value of S that was calculated by a mode of division in which the elements of the difference $X - x_0$ have very small numerical values if we went to a second mode of division in which each of those elements was further subdivided into others.

Now suppose that we consider two separate modes of division of the difference $X - x_0$, in both of which the elements of the difference have very small numerical values. We can compare these two modes with a third mode, chosen so that each element, from either the first or second mode, is formed by bringing together several elements of the third mode. To satisfy this condition, it suffices for each of the values of x placed between the limits x_0 and X in the first two modes to be used in the third; and we can prove that we change the value of S very little in going from the first or the second mode to the third—and therefore, in going from the first to the second. Thus, when the elements of the difference $X - x_0$ become infinitely small, the mode of division has only an imperceptible effect on the value of S; and, if we let the numerical values of these elements decrease while their number increases, the value of S ultimately becomes, for all practical purposes [*sensiblement*], constant. Or, in other words, it ultimately reaches a certain limit that depends uniquely on the form of the function $f(x)$ and on the bounding values x_0, X of the variable x. This limit is what is called a *definite integral.*

Victor J. Katz: A History of Mathematics

Side Bar 16.1 What Is a Limit?

Leibniz (1684): If any continuous transition is proposed terminating in a certain limit, then it is possible to form a general reasoning, which covers also the final limit.

Newton (1687): The ultimate ratio of evanescent quantities . . . [are] limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*.

Maclaurin (1742): The ratio of $2x + o$ to a continually decreases while o decreases and is always greater than the ratio of $2x$ to a while o is any real increment, but it is manifest that it continually approaches to the ratio of $2x$ to a as its limit.

D'Alembert (1754): This ratio $[a : 2y + z]$ is always smaller than $a : 2y$, but the smaller z is, the greater the ratio will be and, since one may choose z as small as one pleases, the ratio $a : 2y + z$ can be brought as close to the ratio $a : 2y$ as we like. Consequently, $a : 2y$ is the limit of the ratio $a : 2y + z$.

Lacroix (1806): The limit of the ratio $(u_1 - u)/h$ . . . is the value towards which this ratio tends in proportion as the quantity h diminishes, and to which it may approach as near as we choose to make it.

Cauchy (1821): If the successive values attributed to the same variable approach indefinitely a fixed value, such that they finally differ from it by as little as one wishes, this latter is called the limit of all the others.

Side Bar 16.2 Definitions of Continuity

Euler (1748): A continuous curve is one such that its nature can be expressed by a single function of x. If a curve is of such a nature that for its various parts . . . different functions of x are required for its expression, . . . , then we call such a curve discontinuous.

Bolzano (1817): A function $f(x)$ varies according to the law of continuity for all values of x inside or outside certain limits . . . if [when] x is some such value, the difference $f(x + \omega) - f(x)$ can be made smaller than any given quantity provided ω can be taken as small as we please.

Cauchy (1821): The function $f(x)$ will be, between two assigned values of the variable x, a continuous function of this variable if for each value of x between these limits, the [absolute] value of the difference $f(x + \alpha) - f(x)$ decreases indefinitely with α.

Dirichlet (1837): One thinks of a and b as two fixed values and of x as a variable quantity that can progressively take all values lying between a and b. Now if to every x there corresponds a single, finite y in such a way that, as x continuously passes through the interval from a to b, $y = f(x)$ also gradually changes, then y is called a continuous function of x in this interval.

Heine (1872): A function $f(x)$ is continuous at the particular value $x = X$ if for every given quantity ϵ, however small, there exists a positive number η_0 with the property that for no positive quantity η which is smaller than η_0 does the absolute value of $f(X \pm \eta) - f(X)$ exceed ϵ. A function $f(x)$ is continuous from $x = a$ to $x = b$ if for every single value $x = X$ between $x = a$ and $x = b$, including $x = a$ and $x = b$, it is continuous.

Mittag-Leffler om Weierstrass i Acta Mathematica, 21, 1897

To shape a good function theory is not a task for a beginner, no matter how gifted. First, one must gain a command of all that is known about Analysis, and work through all relationships, even the most highly developed ones, which are known or have been stated. Only then can one think of shaping a theory of functions which governs and clarifies all. By its nature, such an undertaking can only be the crown of a mathematical lifework.[4]

Die Elemente der Functionenlehre af E. Heine

Journal für die Reine und Angewandte Mathematik, 74 (1872)

Werth dort nicht existire, wenn die Abscisse, durch n getheilt, eine irrationale Zahl ist, konnte nur so lange als berechtigt gelten, als man den Irrationalitäten nicht eine selbständige Existenz beilegte. (Durch die numerische Berechnung der Summe wird man sich übrigens bei Berücksichtigung einer geringeren Anzahl n von Gliedern dem Mittelwerthe, bei einer beliebig grossen dem Werthe vor oder nach dem Sprunge nähern. Die Annäherung an den Mittelwerth kann man durch ein grösseres n nur dann vergrössern, wenn man für die kritische irrationale Abscisse einen solchen rationalen Werth gesetzt hat, der dem wahren Werthe derselben hinlänglich nahe kommt).

§. 2. Bedingungen der Continuität.

1. *Definition.* Eine Function $f(x)$ heisst *bei einem bestimmten einzelnen Werthe $x = X$ continuirlich*, wenn, für jede noch so klein gegebene Grösse ε, eine andere positive Zahl η_0 von solcher Beschaffenheit existirt, dass für keine positive Grösse η, die kleiner als η_0 ist, der Zahlwerth von $f(X \pm \eta) - f(X)$ das ε überschreitet.

1. *Folgerung.* Zwei Functionswerthe für Argumente x, welche zwischen $X - \eta$ und $X + \eta$ liegen, können sich höchstens um 2ε unterscheiden.

Erläuterung. Eine Function ist nur ein Aggregat von einzelnen Werthen (A, §. 1, Def.); ein Zusammenhang zwischen denselben, so dass ein Werth sich aus den Werthen in der Umgebung ergiebt, wird erst durch die Continuität hergestellt.

1. *Lehrsatz.* *) Ist eine Function $f(x)$ bei $x = X$ continuirlich, so bilden für jede Zahlenreihe x_1, x_2, etc., die das Zeichen X besitzt, auch $f(x_1)$, $f(x_2)$, etc. eine Zahlenreihe mit dem Zahlzeichen $f(X)$; *und umgekehrt*, wenn für jede Zahlenreihe x_1, x_2, etc., die das Zeichen X besitzt, auch $f(x_1)$, $f(x_2)$, etc. eine Zahlenreihe mit dem Zeichen $f(X)$ bilden, so ist $f(x)$ bei $x = X$ continuirlich.

*) Den Satz, dass die Function nur und immer continuirlich ist, wenn $f(X) - f(x_n)$, für jede Zahlenreihe von X beliebig klein wird, mit seinem Beweise, entlehne ich dem Herrn *Cantor*. Während ich mich hier auf Functionen mit einer Veränderlichen beschränke, hat Herr *Cantor* allgemein Functionen mehrerer Veränderlichen behandelt; er wird zeigen, dass diese Functionen die Continuität, welche ich an einer anderen Stelle (Dieses Journal, Bd. 71, S. 361) eine gleichmässige nannte, besitzen, wenn sie *in jedem einzelnen Punkte* gewisse Bedingungen erfüllen. Den allgemeinen Gang des Beweises einiger Sätze im §. 3 nach den Principien des Herrn ***Weierstrass*** kenne ich durch mündliche Mittheilungen von ihm selbst, von Herrn *Schwarz* und *Cantor*, so dass bei diesen Beweisen nur die Durchführung im Einzelnen von mir herrührt.

7-10

Beweis. Erstens. Jede Zahlenreihe x_1, x_2, etc. lässt sich mit Hülfe einer Elementarreihe als $X+\eta_1$, $X+\eta_2$, etc. darstellen. Ist nun die Function continuirlich, so werden für jede gegebene Grösse ε (B, §. 2, Def. 1) die Glieder der Reihe η_1, η_2, etc. unter η_0 herabsinken, so dass, von einem gewissen Werthe von n an, $f(X+\eta_n)-f(X)$, d. h. $f(x_n)-f(X)$ nicht mehr ε überschreitet. Diese Differenz ist, da man ε beliebig klein nehmen kann, das allgemeine Glied einer Elementarreihe, $f(x_1)-f(X)$, $f(x_2)-f(X)$, etc., deren Zahlzeichen daher Null wird. Andererseits ist es auch (A, §. 3, Def. 1) gleich

$$[f(x_1),\ f(x_2),\ \text{etc.}]-f(X),$$

wodurch der erste Theil bewiesen ist, nämlich die Gleichheit

$$f(X) = [f(x_1),\ f(x_2),\ \text{etc.}].$$

Zweitens. Erfüllt nun die Function die vorstehende Bedingung, welche besagt, dass für jede Zahlenreihe x_1, x_2, etc. ohne irgend eine Ausnahme, deren Zahlzeichen X ist, $f(x_1)-f(X)$, $f(x_2)-f(X)$, etc. beliebig klein werden, so folgt daraus ihre Continuität. Würde nämlich, wenn man eine bestimmte Zahl ε festhält (B, §. 2, Def. 1), wie klein man auch eine Zahl η_0 nimmt, niemals die Bedingung der Continuität erfüllt sein, würden also noch immer Werthe η unter η_0 existiren, für welche $f(X+\eta)-f(X)$ über ε bleibt, so sei für irgend eine Grösse von η_0 ein solcher Werth von η (unter diesem η_0), für welchen obige Differenz nicht kleiner als ε ist, gleich η'. Für einen halb so grossen Werth von η_0 möge die Differenz bei $\eta=\eta''$ nicht kleiner als ε sein; für ein η_0 gleich der Hälfte des früheren (dem Viertel des ersten) möge dies bei $\eta=\eta'''$ geschehen, u. s. f. Da die Werthe von η_0 eine Elementarreihe bilden, so ist dasselbe mit (den kleineren) η', η'', η''', etc. der Fall; es würden also $X+\eta'$, $X+\eta''$, etc. eine Zahlenreihe x_1, x_2, etc. mit dem Zeichen X vorstellen, ohne dass doch $f(x_1)-f(X)$, $f(x_2)-f(X)$, etc. unter ε sinken — gegen die Voraussetzung.

2. *Lehrsatz.* Eine continuirliche Function $f(x)$ ist für jedes x bekannt, wenn sie für jeden rationalen Werth dieser Veränderlichen gegeben ist.

Beweis. Es sei X eine irrationale, durch die Reihe x_1, x_2, x_3, etc. gegebene Grösse; ferner mögen y_1, y_2, y_3, etc. rationale Zahlen vorstellen, die sich von x_1, x_2, x_3, etc. um weniger als 1, $\frac{1}{2}$, $\frac{1}{3}$, etc. unterscheiden. Da die x von den gleichnamigen y nur um Glieder einer Elementarreihe verschieden sind, so ist auch (A, §. 2, Def. 2) X gleich $[y_1, y_2, \text{etc.}]$, also (B, §. 2, Lehrs. 1)

$$f(X) = [f(y_1),\ f(y_2),\ \text{etc.}],$$

folglich bekannt.

3. *Lehrsatz.* Jede ganze Potenz x^m ist bei jedem einzelnen Werthe $x=X$ continuirlich.

Beweis. Es sei wiederum $X=[x_1, x_2, \text{etc.}]$, woraus folgt (A, §. 3, Def. 1) dass

$$X^m = [x_1^m,\ x_2^m,\ \ldots]$$

sei. Dies ist aber (B, §. 2, Lehrs. 1) die Bedingung der Continuität für eine Function $f(x)=x^m$ bei X.

2. *Folgerung.* Jede ganze Function ist bei jedem einzelnen Werthe der Veränderlichen continuirlich.

4. *Lehrsatz.* Es ist $\sin x$ bei jedem einzelnen Werthe $x=X$ continuirlich.

Beweis. Man hat nachzuweisen, dass $\sin x_1$, $\sin x_2$, etc. eine Zahlenreihe bilden, und zweitens, dass das Zeichen derselben $\sin X$ ist. Beides folgt, wenn man gezeigt hat, dass die Reihe $\sin X-\sin x_1$, $\sin X-\sin x_2$, etc. eine elementare ist. In der That wird aber $\sin X-\sin x_n$ oder

$$\left[X-x_n,\ X-x_n-\frac{X^3-x_n^3}{6},\ \text{etc.}\right]$$

mit wachsendem n beliebig klein.

§. 3. Eigenschaften continuirlicher Functionen.

1. *Definition.* Eine Function $f(x)$ heisst ***continuirlich*** von $x=a$ bis $x=b$, wenn sie bei jedem einzelnen Werthe $x=X$ zwischen $x=a$ und $x=b$, mit Einschluss der Werthe a und b, continuirlich ist (B, §. 2, Def. 1); sie heisst ***gleichmässig continuirlich*** von $x=a$ bis $x=b$, wenn für jede noch so kleine gegebene Grösse ε eine solche positive Grösse η_0 existirt, dass für alle positiven Werthe η, die kleiner als η_0 sind, $f(x\pm\eta)-f(x)$ unter ε bleibt. Welchen Werth man auch x geben möge, nur vorausgesetzt, dass x und $x\pm\eta$ dem Gebiete von a bis b angehören, muss *dasselbe* η_0 das Geforderte leisten.

1. *Lehrsatz.* Jede ganze Potenz von x ist zwischen irgend welchen gegebenen Grenzen gleichmässig continuirlich.

Beweis. Da $(x\pm\eta)^m-x^m$, jedenfalls unter dem Producte aus η und einer in den gegebenen Grenzen festen Grösse bleibt, so lässt sich diese Differenz offenbar für alle x durch denselben Werth von η beliebig klein machen.

1. *Folgerung.* Jede ganze Function ist zwischen beliebig gegebenen Grenzen gleichmässig continuirlich.

2. *Lehrsatz.* Besitzt eine (für jedes einzelne x) von a bis b continuirliche Function $f(x)$ für zwei zwischen a und b liegende Zahlen $x = x_1$ und $x = x_2$ entgegengesetzte Vorzeichen, so verschwindet sie für einen dazwischen liegenden Werth von x.

Beweis. *) Es mögen $x_2 - x_1 = \delta$ und $f(x_1)$ positiv sein. Man bilde nun die Zahlen

$$x_3 = x_2 - \frac{\delta}{2}, \quad x_4 = x_3 \pm \frac{\delta}{4}, \quad x_5 = x_4 \pm \frac{\delta}{8},$$

allgemein

$$x_{n+1} = x_n \pm \frac{\delta}{2^{n-1}},$$

und zwar gilt, bei der Bildung von x_{n+1} aus x_n das positive oder negative Vorzeichen, je nachdem $f(x_n)$ das positive oder negative Vorzeichen besitzt; — ist der Functionswerth $f(x_n)$ für irgend ein n Null, so bedarf der Satz keines weiteren Beweises, weshalb dieser Fall ausgeschlossen bleibt.

Die Zahlen x_1, x_2, etc. bilden eine Zahlenreihe, da (A, §. 1, Def. 1) $x_{n+\nu} - x_n$, wie aus den vorstehenden Gleichungen durch eine ganz elementare Rechnung hervorgeht, selbst im ungünstigsten Falle, wenn nämlich die Functionswerthe für x_{n-1}, x_n, etc., $x_{n+\nu-1}$ sämmtlich dasselbe Vorzeichen besitzen, mit wachsendem n beliebig klein wird. Das Zahlzeichen dieser Zahlenreihe sei X; ich zeige, dass $f(X)$ Null ist.

Wäre dies nicht Null, so ist es eine bestimmte Zahl, die 4ε heisse. Man bilde nun eine solche Grösse η_0, dass $f(X \pm \eta) - f(X) < \varepsilon$ (B, §. 2, Def. 1), und nehme n so gross, dass x_n, x_{n+1}, etc. sich von X um weniger als η_0 unterscheiden, wodurch $f(X)$ sich von $f(x_n)$, $f(x_{n+1})$, etc. um weniger als ε unterscheidet. Dann ist die Differenz $f(x_n) - f(x_{n+\nu})$ kleiner als 2ε. Nimmt man nun ν so gross, dass $f(x_n)$ und $f(x_{n+\nu})$ entgegengesetzte Vorzeichen haben (dass dies immer erreicht werden kann, wird unten gezeigt), so leuchtet ein, dass $f(x_n)$ selbst kleiner als 2ε, mithin $f(X)$ kleiner als 3ε, also nicht gleich 4ε ist.

Würde aber, wie gross man auch für ein bestimmtes n die Zahl ν nimmt, $f(x_{n+\nu})$ immer dasselbe Vorzeichen wie x_n behalten, so sei x_m diejenige Zahl der Reihe mit dem niedrigsten Index, von welcher an die Vorzeichen

*) Es schien zweckmässig, selbst auf Kosten der Kürze, beim Beweise geometrische Anschauungen auszuschliessen.

der Function $f(x)$ nicht mehr wechseln, so dass sie also für x_m, x_{m+1}, etc. dieselben bleiben. Da $f(x_1)$ und $f(x_2)$ entgegengesetzte Vorzeichen besitzen, so ist m wenigstens gleich 2; folglich haben sicher $f(x_{m-1})$ und $f(x_m)$ entgegengesetzte Vorzeichen. Bezeichnet α die positive oder negative Einheit, je nachdem $f(x_{m-1})$ positiv oder negativ ist, so wird, nach diesen Voraussetzungen,

$$x_m = x_{m-1} + \alpha\delta 2^{2-m}, \quad x_{m+1} = x_m - \alpha\delta 2^{1-m}, \quad x_{m+2} = x_{m+1} - \alpha\delta 2^{-m}, \text{ etc.},$$

folglich

$$x_{m+\mu} - x_{m-1} = -\alpha\delta 2^{2-m}\left(-1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots + \frac{1}{2^\mu}\right).$$

Mit wachsendem μ sinkt die rechte Seite, welche das Zeichen von α besitzt, unter jeden Grad der Kleinheit, so dass das Zahlzeichen X der oben gebildeten Zahlenreihe genau x_{m-1} wäre. Es würde das allgemeine Glied x_n dieser Zahlenreihe sich also x_{m-1} beliebig nähern, und dabei hätte die ganz bestimmte Grösse $f(x_{m-1})$ das Zeichen von α, $f(x_n)$ das entgegengesetzte. Dies ist, wegen der Continuität von $f(x)$, nicht möglich.

2. *Folgerung.* Sobald a eine positive ganze nicht quadratische Zahl bezeichnet, besitzt die Gleichung $x^2 - a = 0$ keine ganze, folglich keine rationale Wurzel. Es hat aber die linke Seite für gewisse verschiedene Werthe von x entgegengesetzte Vorzeichen, also die Gleichung eine irrationale Wurzel. Hierdurch ist bewiesen, dass nicht alle Zahlzeichen sich auf rationale Zahlen reduciren, sondern *dass es auch irrationale Zahlen giebt.* (A, §. 5).

3. *Lehrsatz.* Eine Function $f(x)$, die von $x = a$ bis $x = b$ so beschaffen ist, dass zwischen je zwei Zahlen x_1 und x_2, wie nahe sie auch gewählt werden, noch andere liegen, für welche $f(x)$ verschiedene Zeichen besitzt, ist discontinuirlich.

Beweis. Wäre sie continuirlich, so sei sie für einen bestimmten Werth ξ von x gleich 2ε. Es lässt sich dann eine Grösse η_0 so bestimmen, dass

$$f(\xi \pm \eta) - f(\xi) < \varepsilon,$$

für jeden Werth η unter η_0. Zwischen $x = \xi$ und $x = \xi + \eta_0$ ändert $f(x)$ das Zeichen, muss demnach dazwischen, für einen Werth $x = \xi + \eta$, verschwinden (B, §. 3, Lehrs. 2), so dass $f(\xi)$ sich von Null höchstens um ε unterscheidet, also nicht 2ε sein kann.

4. *Lehrsatz.* Wenn die (für jedes einzelne x) von $x = a$ bis $x = b$ continuirliche Function $f(x)$ von $x = a$ bis $x = b$ nie negativ, aber zwischen diesen Grenzen kleiner wird als jede angebbare Grösse, so *erreicht* sie auch den Werth Null.

Beweis. Da $f(x)$ für jedes bestimmte x auch einen bestimmten Werth besitzt, so kann es für ein solches x nur dann kleiner als jede angebbare

Grösse sein, wenn es dort verschwindet. Es seien nun x_1 und x_2 zwei derartige Zahlen, dass zwischen ihnen andere liegen, für welche $f(x)$ beliebig klein wird; behält man die Bezeichnung im Beweise des zweiten Lehrsatzes bei, bildet also Zahlen x_3, x_4, etc. durch die dort angeführten Recursionsformeln, in denen über die Wahl des unbestimmt gelassenen Vorzeichens noch das Nähere angegeben werden soll, so könnten zunächst $x = x_3$, oder $x = x_4$, etc., $x = x_n$, solche Stellen sein, an denen $f(x)$ beliebig klein wird. Dann verschwindet es aber an diesen Stellen, wie aus dem Eingange dieses Beweises ersichtlich ist, und der Satz wäre bewiesen. Es handelt sich also nur noch um den Beweis, wenn die Function weder für x_3, noch für x_4, etc. verschwindet, wie viele von diesen Zahlen man auch bilden möge.

Die Zahlen x, für welche $f(x)$ beliebig klein wird, sind entweder grösser als x_3, oder kleiner als x_3, oder zum Theil grösser, zum Theil kleiner. Im ersten Falle bilde man x_4 aus x_3 mit Hülfe des positiven Vorzeichens, im zweiten mit dem negativen, im dritten, wie willkürlich festgesetzt wird, mit dem positiven. In ähnlicher Art wird x_5 aus x_4 gebildet, u. s. f., so dass eine Zahlenreihe x_1, x_2, etc. mit dem Zahlzeichen X entsteht; ich zeige, dass $f(X)$ Null ist.

Wäre es nicht Null sondern 3ε, so bilde man, wie im zweiten Lehrsatze, η_0 und nehme n so gross wie dort, d. h. so gross, dass x_n, x_{n+1}, etc. sich von X um weniger als η_0 unterscheiden. Sind nun x_n und $x_{n+\nu}$ Werthe, zwischen denen Zahlen x liegen, für welche $f(x)$ beliebig klein, z. B. $< \varepsilon$, wird, so kann $f(X)$, welches sich von allen Zahlen $f(x)$, wo $X-\eta_0 < x < X+\eta_0$, um weniger als ε unterscheidet, höchstens 2ε und nicht 3ε sein. Würde es aber kein ν geben, würde man also von x_n an immer zu grösseren oder immer zu kleineren Werthen x_{n+1}, x_{n+2}, etc. gelangen, so sei x_m das letzte von den zu bildenden x, welches resp. einen kleineren oder grösseren Werth besitzt, als das vorhergehende; x_{m+1}, x_{m+2}, etc. sind dann sämmtlich resp. grösser oder kleiner als x_m und bilden eine wachsende oder abnehmende Reihe von Gliedern, welche aber immer unter oder über x_{m-1} bleiben. Man findet durch dieselbe Rechnung wie bei dem zweiten Lehrsatze in dem entsprechenden Falle, $X = x_{m-1}$. Während $f(X) = f(x_{m-1})$ einen bestimmten Werth 3ε besitzt, würde daher $f(x)$ beliebig klein sein müssen für Werthe x, die beliebig nahe an x_{m-1} liegen, nämlich zwischen $X = x_{m-1}$ und x_n, wie gross man auch n nimmt. Dieses ist aber, wegen der Continuität von $f(x)$, nicht möglich.

3. *Folgerung.* Wenn eine (für alle einzelnen Werthe) von $x = a$ bis $x = b$ continuirliche Function nicht überall gleiche Werthe besitzt, so erreicht sie für einen bestimmten Werth von x ein Maximum und ebenso ein Minimum.

5. *Lehrsatz.* Wenn die von $x = a$ bis $x = b$ (für alle einzelnen Werthe) continuirliche Function $f(x)$ für jeden einzelnen Werth, der zwischen a und einer rationalen oder irrationalen Zahl X liegt, wo $a < X < b$, wie nahe man auch X kommt, nicht positiv, über X hinaus aber positiv wird, so ist $f(X) = 0$.

Beweis. Es sei x_1, x_2, etc. eine Zahlenreihe für X, deren Glieder sämmtlich unter X liegen sollen. Dann wird

$$f(X) = [f(x_1),\ f(x_2),\ \text{etc.}]$$

nicht positiv; negativ kann es wegen der Continuität von $f(x)$ unmöglich sein, weil es dann einen bestimmt angebbaren, von Null verschiedenen negativen Werth besässe, während $f(x)$ selbst für den kleinsten Werth, der x grösser als X macht, nach der Voraussetzung, positiv ist. Es bleibt daher nur übrig, dass $f(X)$ Null ist.

6. *Lehrsatz.* Eine von $x = a$ bis $x = b$ (für alle einzelnen Werthe) continuirliche Function $f(x)$ ist auch gleichmässig continuirlich. (B, §. 3, Def. 1).

Beweis. Bezeichnet 3ε eine beliebige Grösse, so existirt eine solche Zahl, dass von $x = a$ bis zu ihr hin $f(x) - f(a)$ absolut $\leqq 3\varepsilon$ ist. Ein Werth, der dies leistet, ist der grösste und macht zugleich $f(x) - f(a) - 3\varepsilon = 0$. (B, §. 3, Lehrs. 5). Dieser Werth sei x_1. In ähnlicher Art findet man eine Zahl x_2 als die grösste, welche bewirkt, dass von $x = x_1$ bis $x = x_2$ immer $f(x) - f(x_1) \leqq 3\varepsilon$ bleibt. So fährt man fort; kommt man nach einer endlichen Anzahl n von Operationen zu $x_n = b$ oder findet, dass $f(x) - f(x_{n-1})$, von $x = x_{n-1}$ bis $x = b$, noch nicht 3ε überschreitet, so ist der Satz bewiesen.

Es bliebe noch der Fall übrig, dass kein n existirt, dass also die Grössen x_1, x_2, etc. eine unendliche Reihe von wachsenden Grössen bilden, die unter b liegen. Diese Reihe wäre dann eine Zahlenreihe, deren Zahlzeichen X sei; hervorzuheben ist ihre Eigenschaft, nach der für jedes n die Gleichung besteht: $f(x_{n+1}) - f(x_n) = 3\varepsilon$. Nun sei η_0 von der Beschaffenheit, dass $f(X)$ sich von $f(X - \eta)$ um weniger als ε unterscheidet, so lange $\eta \leqq \eta_0$. Zwischen die Zahlen $X - \eta_0$ und X mögen von der obigen Zahlenreihe x_n, x_{n+1}, etc. fallen, so dass (B, §. 2, Folg. 1) $f(x_{n+1}) - f(x_n)$ kleiner als 2ε wäre, während es andererseits 3ε sein müsste. Die zu Grunde liegende Annahme ist daher unmöglich, und die Function eine gleichmässig continuirliche.

Halle, im October 1871.