Institut for Matematik og Datalogi
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February 23, 2011
JFB

Cryptology – F11 – Week 5

Lecture, February 18

After discussing the problems, we figured out why the Legendre symbol of an element, x, of the multiplicative group modulo a prime p can be computed using $x^{(p-1)/2} \pmod{p}$.

Lecture, February 22

We continued with chapter 5 in the textbook, covering more on quadratic residuosity and covering primality testing. (We didn't cover the Solovay-Strassen primality test, but you should read about that in the textbook.)

Class cancelled on February 25

Lecture, March 1

We will finish chapter 5 and possibly begin on chapter 6.

Problem sessions March 2 and March 8

1. With RSA, there are often recommendations to use a public exponent $e = 3$.

   a. What would the advantage to this be?

   b. If $e = 3$, the two prime factors dividing the modulus, p and q, must be such that $p \equiv q \equiv 2 \pmod{3}$. Why is it impossible to have one or both of p and q congruent to 0 or 1 modulo 3?

   c. Suppose that $e = 3$, $p = 3r + 2$ and $q = 3s + 2$. What would the decryption exponent d be?

2. Suppose the modulus used in RSA has 1024 bits. What is the unicity distance of this RSA cryptosystem? Why?

3. Suppose that user A wants to send a message $s \in \{s_1, s_2, ..., s_k\}$ to user B, where $s_i < 1024$ for $1 \leq i \leq k$. Assume that RSA is secure (when the modulus is large enough and is the product of two equal length prime factors). **a** Why would you still advise user A not to use RSA directly?

   b What would you recommend instead, if you still wanted to use RSA?

4. Show all steps in the calculations of the Jacobi symbol $\left(\frac{29}{35}\right)$, using the standard algorithm (using the four properties of the Jacobi symbol given in the textbook).

5. Show all steps of the execution of one call to the Solovay-Strassen Primality Test, checking if 35 is prime. Assume that the random integer a chosen is 19.

6. A *Carmichael number* is a composite integer n such that for all $x \in Z_n^*$, $x^{n-1} \equiv 1 \pmod{n}$.

   a Explain why the existence of Carmichael numbers (there are in fact infinitely many of them) make primality testing more difficult.

   b Explain why Carmichael numbers are easy to factor using intermediate calculations from the Miller–Rabin primality test.

   c Show that 561 is a Carmichael number. Try to do it without explicitly checking all elements of $\mathbb{Z}_{561}^*$.

7. Do problems 5.9, 5.13, 5.16 and 5.17 in the textbook.

8. Suppose $n = 11,820,859$ is an RSA modulus. Suppose you know $\phi(n) = 11,813,904$. Find the factors of n. Show your work. (You may use Maple to solve the quadratic equation, but explain how you used it.)

9. Find all square roots of 64 modulo 105.

10. Find a primitive element (generator) in the multiplicative group modulo 103 ($\mathbb{Z}_{103}^*$) and show that it is a primitive element.

11. Consider the following proposal for a primality test for an integer n: Check if $2^n - 2$ is divisible by n. Answer "prime" if it is and "composite" if it is not.

    a. Give an odd prime for which this test works correctly and an odd composite for which it also works correctly.

    b. Prove that the test answers "prime" for all primes.

    c. Show that it answers "prime" incorrectly for $n = 341$. Use Fermat's Little Theorem to compute $2^{341} \pmod{p}$ for each prime factor of 341. Then use the Chinese Remainder Theorem, to compute $2^{341} \pmod{341}$.

12. Suppose the Solovay-Strassen Primality Test is used to find the primes p and q for use in the RSA cryptosystem. (Assume that random integers of the required length are chosen and tested for primality until two are found where the test does not discover that they are composite.) Even assuming that the primality test is executed several times, there is still a small probability of choosing a number which is not prime. Suppose the p chosen is prime, but q is not.

    a. Suppose q is a Carmichael number. (Recall that a *Carmichael number* is a composite integer n such that for all $x \in Z_n^*$, $x^{n-1} \equiv 1 \pmod{n}$.) Would encryption and decryption still work properly? Prove your answer. (Hint: try using the Chinese Remainder Theorem.)

    b. Suppose q is a not a Carmichael number. Would encryption and decryption still work properly? Prove your answer. (Hint: try using the Chinese Remainder Theorem.)

    c. What other problem could exist if q is a composite number?