Expectations

Example: What is the expected value of the first successful Bernoulli trial?

Answer:

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\begin{aligned}
\sum_{i=1}^{\infty} i q^{i-1} p &= \sum_{i=1}^{\infty} i q^{i-1} - \sum_{i=1}^{\infty} i q^{i} \\
&= \sum_{j=0}^{\infty} (j+1) q^{j} - \sum_{i=1}^{\infty} i q^{i} \qquad (j = i-1) \\
&= 1 + \sum_{j=1}^{\infty} (j+1-j) q^{j} \\
&= 1 + \sum_{j=1}^{\infty} q^{j} \;=\; 1 + (\sum_{j=0}^{\infty} q^{j}) - 1 \\
&= \frac{1}{1-q} \\
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With a fair die, the expected number of throws before a 1 is 6.

Linearity of Expectations

A *linear function* has the form

$$f(X_1, X_2, ..., X_n) = a_0 + a_1X_1 + a_2X_2 + ... + a_nX_n$$

where $a_i \in \mathbb{R}$ for $0 \leq i \leq n$.

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Thm. Let f be a linear function, S be a sample space, and $X_1, X_2, ..., X_n$ be random variables defined on S. Then, $E[f(X_1, X_2, ..., X_n)] = f(E[X_1], E[X_2], ..., E[X_n])$.

Linearity of Expectations

Pf. Let $f(X_1, ..., X_n) = a_0 + a_1 X_1 + ... + a_n X_n$ where $a_i \in \mathbb{R}$ for $0 \leq i \leq n$. Then,

$$\begin{aligned}
E[f(X_1, ..., X_n)] &= \sum_{s\in S} p(s) f(X_1(s), ..., X_n(s)) \\
&= \sum_{s\in S} p(s)(a_0 + a_1 X_1(s) + ... + a_n X_n(s)) \\
&= \sum_{s\in S} \left(p(s) a_0 + \sum_{i=1}^{n} p(s) a_i X_i(s) \right) \\
&= a_0 + \sum_{i=1}^{n} \left(\sum_{s\in S} p(s) a_i X_i(s) \right) \\
&= a_0 + \sum_{i=1}^{n} \left(a_i \sum_{s\in S} p(s) X_i(s) \right) \\
&= f(E[X_1], E[X_2], ..., E[X_n]). \qquad \square
\end{aligned}$$

Linear Search Algorithm

procedure linear_search($x, a_1, a_2, ..., a_n$)
$i \leftarrow 1$
while ($i \leq n$ and $x \neq a_i$) $i \leftarrow i + 1$
{ Either $i = n + 1$ or $x = a_i$}

if $i \leq n$ **then** $location \leftarrow i$
else $location \leftarrow 0$
return $location$

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Worst case: n comparisons of elements ($2n + 1$ comparisions).

Insertion Sort Algorithm

procedure InsertionSort(List):
{ Input: List is a list }
{ Output: List, with same entries, but in nondecreasing order }

$N := 2$
while ($N \leq$ length(List))
  Pivot := Nth entry
  $j := N - 1$
  while ($j > 0$ and jth entry $>$ Pivot)
    move jth entry to loc. $j + 1$
    $j := j - 1$
  place Pivot in $j + 1$st loc.
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            move jth entry to loc. j + 1
            j := j − 1
        place Pivot in j + 1st loc.
        N := N + 1
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Worst case: $\sum_{i=2}^{n}(i-1) = \frac{n(n-1)}{2}$ comparisons of elements

Expectation, Variance, Standard Deviation

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The **variance** of a random variable is $V[X] = E[(X - E[X])^2]$.

$V[X] = E[X^2 - 2XE[X] + E^2[X]]$.

By the linearity of expectations, this is $E[X^2] - 2E[XE[X]] + E[E^2[X]]$.

Since $E[X]$ is a real number, this is $E[X^2] - 2E^2[X] + E^2[X]$.

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Thus, $V[X]$ is also $E[X^2] - E^2[X]$.

If X and Y are independent random variables, then $V[X + Y] = V[X] + V[Y]$. If $X_1, X_2, ..., X_n$ are pairwise independent random variables, then $V[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} V[X_i]$.

The **standard deviation** of a random variable is the positive square root of the variance.

Variance

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How many successes in the ith trial?

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The variance of the binomial distribution —

$V[X] = V[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} V[X_i]$

since the X_i are pairwise independent.

$V[X_i] = E[X_i^2] - E^2[X_i] = p - p^2 = pq$.

Thus, for the binomial distribution, $V[X] = \sum_{i=1}^{n} pq = npq$.

Chebyshev's Inequaltiy

Thm. [Chebyshev's Inequality] Let X be a random variable on sample space S, with probability function p, and $r > 0$. Then

$$p(|X(s) - E[X]| \geq r) \leq V[X]/r^2$$

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- $V(X) \geq \sum_{s \in A} r^2 p(s)$.
- $p(A) = \sum_{s \in A} p(s) \leq V(X)/r^2$. $\square$