DM551/MM851

Algorithms and Probability

October 7, 2020

Derangements

Example: Peter likes betting. He hears:

- Team A is expected to win.
- Team B is expected to be 2nd.
- Team C is expected to be 3rd.
- Team D is expected to be 4th.
- Team E is expected to be 5th.

Peters bets on these 5 events.

In how many ways can he lose all bets?

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Property P_i — place i was correct.

Want $D = N(P_1' P_2' P_3' P_4' P_5')$.

Derangements

$$D = N - \sum_{i=1}^{5} N(P_i) + \sum_{1 \le i < j \le 5} N(P_i P_j) - \sum_{1 \le i < j < k \le 5} N(P_i P_j P_k) + \sum_{1 \le i < j < k < l \le 5} N(P_i P_j P_k P_l) - N(P_1 P_2 P_3 P_4 P_5).$$

$N = 5!$.

$N(P_i) = (5-1)! \quad \forall i$.

$N(P_i P_j) = (5-2)! \quad \forall i, j$.

$N(P_i P_j P_k) = (5-3)! \quad \forall i, j, k$.

$N(P_i P_j P_k P_l) = (5-4)! \quad \forall i, j, k, l$.

$N(P_1 P_2 P_3 P_4 P_5) = (5-5)!$.

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$N(P_1 P_2 P_3 P_4 P_5) = (5-5)!$.

How many terms are the in sum where there are s properties?

a. $P(5, s)$

b. $C(5, s)$

c. 5^s

d. $s!$

Derangements

Suppose all permutations were equally likely.

What is the probability of a derangement?

$$D_n/n! = 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^n \frac{1}{n!}$$

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The infinte sum

$$1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^n \frac{1}{n!} + \ldots$$

gives $1/e \approx 0.368$.

$\frac{1}{e} - \frac{1}{(n+1)!} \leq \frac{D_n}{n!} \leq \frac{1}{e} + \frac{1}{(n+1)!}$.

The probability of at least one object being fixed is approximately $1 - 1/e \approx 0.632$.