Quasi-hamiltonian paths in semicomplete multipartite digraphs

Jørgen Bang-Jensen\textsuperscript{a}, Alessandro Maddaloni\textsuperscript{a} and Sven Simonsen\textsuperscript{a} 1

\textsuperscript{a} Department of Mathematics and Computer Science, University of Southern Denmark, Odense DK-5230, Denmark.

June 13, 2013

Abstract

A quasi-hamiltonian path in a semicomplete multipartite digraph $D$ is a path which visits each maximal independent set (also called a partite set) of $D$ at least once. This is a generalization of a hamiltonian path in a tournament.

In this paper we investigate the complexity of finding a quasi-hamiltonian path, in a given semicomplete multipartite digraph, from a prescribed vertex $x$ to a prescribed vertex $y$ as well as the complexity of finding a quasi-hamiltonian path whose end vertices belong to a given set of two vertices $\{x,y\}$. While both of these problems are polynomially solvable for semicomplete digraphs (here all maximal independent sets have size one), we prove that the first problem is NP-complete and describe a polynomial algorithm for the latter problem.

Keywords: Polynomial algorithm , NP-complete, semicomplete multipartite digraph, extended semicomplete digraph, multipartite tournament, quasi-hamiltonian path.

1. Introduction

Notation not introduced below or in the next section follows [? ].

We always denote by $V$ ($A$) the set of vertices (arcs) of a digraph and by $n$ ($m$) its cardinality.

A digraph $D = (V,A)$ is semicomplete multipartite if whenever there is an arc between $u$ and $v$ but no arc between $u$ and $w$, there is an arc between $v$ and $w$. Equivalently, there is a partition $V_1, V_2, \ldots, V_c$ of $V$ into independent sets so that every vertex in $V_i$ shares an arc with every vertex in $V_j$ for $1 \leq i < j \leq c$. We call $V_1, V_2, \ldots, V_c$ the partite sets of $D$. A multipartite tournament, or MT in short, is a semicomplete multipartite digraph, or SMD in short, without cycles of length two. A semicomplete digraph (tournament) is a SMD (MT) where $\alpha(D) = 1$, where $\alpha(D)$ is the size of a largest independent set in $D$.

Let $\{v_1, \ldots, v_a\}$ be the vertex set of a digraph $D$ and let $M_1, \ldots, M_a$ be digraphs which are pairwise vertex disjoint. The composition $D[M_1, \ldots, M_a]$ is the digraph which has arc set $\cup_{i=1}^{a} A(M_i) \cup \{h_i h_j | h_i \in V(M_i), h_j \in V(M_j), v_i v_j \in A(D)\}$.

\footnote{1E-mail addresses: jbj@imada.sdu.dk (J. Bang-Jensen), maddaloni@imada.sdu.dk (A. Maddaloni) and svsim@imada.sdu.dk (S. Simonsen)}
A digraph $D$ is **extended semicomplete** if $D = S[M_1,\ldots,M_s]$ where $S$ is a semicomplete digraph and $M_1,\ldots,M_s$ are digraphs with no arcs, we sometimes call these sets modules.

An $(x,y)$-**path** $(x,y)$-path in a digraph $D$ is a path which starts in $x$ and ends in $y$ (has end vertices in $\{x,y\}$).

A digraph $D$ is **k-strong** if it has at least $k + 1$ vertices and $D - X$ is strongly connected for every $X \subseteq V$ with $|X| \leq k - 1$.

A digraph is **hamiltonian-connected** (weakly hamiltonian-connected) if it contains an $(x,y)$-hamiltonian ($(x,y)$-hamiltonian) path for every choice of distinct vertices.

Thomassen [?] proved that every 4-strong semicomplete digraph $D$ is hamiltonian connected and gave an infinite class of 3-strong tournaments which are not hamiltonian-connected.

In [?] a polynomial algorithm for testing whether a given semicomplete digraph contains an $(x,y)$-hamiltonian path was given. The algorithm relies on the results in [?] as well as additional technical results. Interestingly, the algorithm cannot be modified to solve the problem of finding the longest $(x,y)$-path in a semicomplete digraph and the complexity of this problem is still open.

**Conjecture 1.1.** [?] There exists a polynomial algorithm for finding the longest $(x,y)$-path in a semicomplete digraph.

In [?] Thomassen also gave a complete characterization of weakly hamiltonian connected tournaments. That characterization implies a polynomial algorithm for deciding whether a given tournament contains an $(x,y)$-hamiltonian path. From this we can also derive a polynomial algorithm for finding the longest $(x,y)$-path in a tournament (and in fact, in a semicomplete digraph).

For multipartite tournaments it is an open problem whether the existence of an $(x,y)$-hamiltonian path can be decided in polynomial time and the problem is most likely quite difficult\(^2\) (see [?] for an algorithm for this problem in the case of extended tournaments). This makes it interesting to study problems for multipartite tournaments which contain the above mentioned hamiltonian path problems for tournaments as special cases.

Let $D = (V,A)$ be a semicomplete multipartite digraph with partite sets $V_1,V_2,\ldots,V_c$. A **quasi-hamiltonian** path in $D$ is a path $P$ such that $V(P) \cap V_i \neq \emptyset$ for $1 \leq i \leq c$ (so for semicomplete digraphs a quasi-hamiltonian path is a hamiltonian path). A digraph is **quasi-hamiltonian-connected** (weakly quasi-hamiltonian-connected) if it contains an $(x,y)$-quasi-hamiltonian ($(x,y)$-quasi-hamiltonian) path for every choice of distinct vertices. The following result generalizes Thomassen’s well known result [?] that every 4-strong tournament is hamiltonian-connected

**Theorem 1.2.** [?] Every 4-strong multipartite tournament $D$ is quasi-hamiltonian-connected, that is, it contains a quasi-hamiltonian path starting in $x$ and ending in $y$ for every choice of distinct vertices of $D$.

In [?] Guo et al. introduced the following notion, which generalizes weak hamiltonian-connectivity in semicomplete digraphs. A semicomplete multipartite digraph $D$ with partite sets $V_1,V_2,\ldots,V_c$ is **weakly quasi-hamiltonian set-connected** if every pair of distinct partite sets $V_i,V_j$, $1 \leq i < j \leq c$ contains vertices $x \in V_i$, $y \in V_j$ such that $D$ has an $(x,y)$-quasi-hamiltonian path. The authors of [?] gave a complete characterization of weakly quasi-hamiltonian set-connected multipartite tournaments which generalizes Thomassen’s work on weakly hamiltonian connected tournaments in [?]. An easy corollary of their work is the following.

**Theorem 1.3.** [?] There is a polynomial algorithm for deciding whether a multipartite tournament $D$ with given distinct partite sets $V_i,V_j$ contains an $(x,y)$-quasi-hamiltonian path for some $x \in V_i, y \in V_j$.

It is mentioned in [?] that they chose that way of generalizing weak hamiltonian-connectedness of tournaments to multipartite tournaments, because the straightforward generalization to weak quasi-hamiltonian paths would not lead to a nice generalization of Thomassen’s characterization for the existence $(x,y)$-hamiltonian paths in tournaments.

\(^2\)For instance the hamiltonian cycle problem is known to be polynomially solvable for semicomplete multipartite digraphs [?] but the algorithm is highly non-trivial.
In this paper we consider the complexity of finding quasi-hamiltonian \((x,y)\)- and \([x,y]\)-paths in semi-complete multipartite digraphs. We prove that, somewhat surprisingly, it is NP-complete to decide whether a semicomplete multipartite digraph contains an \((x,y)\)-quasi-hamiltonian path. In doing so, we also identify a problem which is NP-complete for tournaments and thus add to the short list\(^3\) of (natural) problems which are NP-complete already for tournaments. We prove that for the subclass of extended semicomplete digraphs a polynomial algorithm does exist by reducing the problem to the \((x,y)\)-hamiltonian path problem for semicomplete digraphs.

Our main contribution is a polynomial algorithm for weak quasi-hamiltonicity, namely we prove that there exists a polynomial algorithm for deciding whether there exists an \([x,y]\)-quasi-hamiltonian path in a given semicomplete multipartite digraph. Since our main objective was to prove that this problem is indeed polynomial and in order to keep the presentation reasonably simple, we have chosen to present an algorithm whose correctness is fairly easy to verify rather than going for an algorithm with a much better complexity.

We conclude the paper with a number of open problems and some remarks.

2. Further terminology

Given a digraph \(D = (V,A)\) and a vertex set \(X \subseteq V\) we denote by \(D(X)\) the subgraph of \(D\) induced by \(X\).

Given a subdigraph \(F\) of a digraph \(D\) and a set of vertices \(U \subseteq V(F)\), we denote by \(N_F^+(U)\) (\(N_F^-(U)\)) the out-neighborhood (in-neighborhood) of \(U\) restricted to \(F\), we denote by \(d_F^+(U)\) (\(d_F^-(U)\)) the out-degree (in-degree) of \(U\) restricted to \(F\), sometimes, when \(F = D\), no subscript is used.

Given two disjoint vertex sets \(X,Y \subseteq V\) we write \(X \Rightarrow Y\) to indicate that there is no arc from \(Y\) to \(X\), we write \(X \rightarrow Y\) to indicate that all the possible arcs from \(X\) to \(Y\) are present.

Given a path, or a cycle, \(P\) and two vertices \(a \neq b\) such that \(P\) contains an \((a,b)\)-path \(Q\), we denote by \(P[a,b]\) the subpath \(Q\), we denote by \(P[a,b],P[a,b],[P]a,b\) the paths \(Q - a, Q - b, Q - \{a,b\}\) respectively.

Given vertex sets \(X,Y \subseteq V\) an \((X,Y)\)-path is a path which starts in \(X\) ends in \(Y\) and has no internal vertex in \(X \cup Y\).

Given a semicomplete multipartite digraph \(D = (V,A)\) and a vertex set \(X \subseteq V\), we denote by \(H(X)\) the union of all partite sets of \(D\) that intersect \(X\).

We note that finding an \((x,y)\)-quasi-hamiltonian path (an \([x,y]\)-quasi-hamiltonian path) in a semicomplete multipartite digraph \(D\) is simply a matter of finding an \((x,y)\)-path (an \([x,y]\)-path) if \(D\) has at most two partite sets. Given this we assume from here on that all semicomplete multipartite digraphs considered have at least three partite sets.

Note also that if we create a copy of \(y\), \(y'\), and add it to the original digraph, then there is a one-to-one correspondence between \((x,y)\)-paths (\([x,y]\)-paths) in the original digraph and \((x,y')\)-paths (\((x,y')\)-paths) in the new digraph. Therefore we will always implicitly assume that \(x \neq y\).

3. Strong quasi-hamiltonian connectedness is NP-complete

A polynomial algorithm for finding an \((x,y)\)-hamiltonian path in a semicomplete digraph was given in [? ]. We prove that the related quasi-hamiltonian problem on SMDs is NP-complete, but first we investigate a related problem on tournaments.

**Problem 1.** Given a strong tournament \(D = (V,A)\), two vertices \(x,y \in V\) and a partition \(C\) of \(V\). Decide whether \(D\) contains an \((x,y)\)-path intersecting all vertex sets in \(C\).

---

\(^3\)These include Feedback vertex set [? ? ] and cycle through a prescribed set of arcs [? ] as well as Feedback arc set [? ? ? ].
Let $F$ respectively.

\section{Theorem 3.1. Problem 1 is NP-complete.}

\begin{proof}
We prove NP-completeness by reducing from 3-SAT.

Let $W[s, t, p, q]$ be the digraph (the variable gadget shown in Fig 1) with vertices
\{s, t, a, b, v_1, \ldots, v_p, u_1, \ldots, u_q\} where we define vertex sets $T = \{v_1, \ldots, v_p\}$ and $F = \{u_1, \ldots, u_q\}$. Let $W[s, t, p, q]$ have the arcs
\[ \{sa, sb, ab, at, ts, a \rightarrow T \rightarrow b, b \rightarrow F \rightarrow a, t \rightarrow T \rightarrow s, t \rightarrow F \rightarrow s, F \rightarrow T\} \]
and add arcs such that $W(T)$ and $W(F)$ are acyclic tournaments with acyclic ordering $v_1, \ldots, v_p$ and $u_1, \ldots, u_q$ respectively.

![Figure 1: The variable gadget $W$.](image)

We observe that an instance of the gadget $W = W[s, t, p, q]$ contains no $(s, t)$-path that intersects both $T$ and $F$, since $N_{W-s}^-(F) = \{b\} = N_{W-s}^+(T)$ and $N_{W-t}^-(T) = \{a\} \cup F$ while $N_{W-s}^+(F) = \{a\} \cup T$ which does not permit such a path. But $W$ contains an $(s, t)$-path that contains all vertices of $\{a, b\} \cup T$ and an $(s, t)$-path that contains all vertices of $\{a, b\} \cup F$, we call these the \textbf{true} and the \textbf{false} path respectively.

Let $F$ be an instance of 3-SAT with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$. The ordering of the clauses $C_1, \ldots, C_m$ induces an ordering of the occurrences of a variable $x$ and its negation $\bar{x}$ in these.

We construct a digraph $D$ from $F$ by associating, with each variable $x_i$, a copy $W_i = W[s_i, t_i, p_i, q_i]$, where $T_i$ has vertices $v_{i,1}, \ldots, v_{i,p_i}$ and $F_i$ has vertices $u_{i,1}, \ldots, u_{i,q_i}$, where $p_i$ is the number of non-negated occurrences of $x_i$ over all clauses while $q_i$ is the number of negated occurrences. Then we identify $x = s_1, t_i = s_{i+1}$ for all $1 \leq i < n$ and $t_n = y$. Finally we turn $D$ into a strong tournament by adding all arcs $W_i \rightarrow W_{i-1}$ for all $1 < i \leq n$.

Let $C$ be the partition of $V(D)$ given by
\[ \{\{s_1\}, \ldots, \{s_n\}, \{y\}, \{a_1\}, \ldots, \{a_n\}, \{b_1\}, \ldots, \{b_n\}, B_1, \ldots, B_m\} \]
where $B_j$ is a vertex set on 3 vertices representing the clause $C_j$ defined such that $v_{i,k}$ is in $B_j$ if the $k$‘th occurrence of $x_i$ is in $C_j$ while $u_{i,k}$ is in $B_j$ if the $k$‘th occurrence of $\bar{x}_i$ is in $C_j$.

We claim that $D$ contains an $(x, y)$-path intersecting all vertex sets in $C$ if and only if $F$ has a satisfying truth assignment.

Given a satisfying truth assignment for $F$ we can construct an $(x, y)$-path $P$ that intersects all vertex sets of $C$ as follows: For all $1 \leq i \leq n$, if $x_i$ is true add the true path in $W_i$ to $P$ if $x_i$ is false add the false path in $W_i$ to $P$. Now $P$ obviously contains all singleton vertex sets $\{s_i\}, \{a_i\}, \{b_i\}$ and $\{y\}$. But $P$ also contains at least one vertex of every $B_j$ since one literal of $C_j$ was true say for the variable $x_i$ and the corresponding path through $W_i$ was chosen.

Conversely given an $(x, y)$-path $P$ in $D$ such that $P$ intersects all vertex sets in $C$ it is easy to see that $P = P[s_1, t_1]P[s_2, t_2] \ldots P[s_n, t_n]$ where $V(P[s, t]) \subset W_i$, since $s_j$ is an $(s_i, s_h)$-separator for all
\( i \leq j \leq h \). So for all \( 1 \leq i \leq n \) we have \( V(P) \cap T_i = \emptyset \) or \( V(P) \cap F_i = \emptyset \). We can thus assign \( x_i \) the value true if \( V(P) \cap F_i = \emptyset \) and false otherwise. This truth assignment satisfies every clause \( C_j \) since \( P \) intersects \( B_j \); say \( v_{i,k} \in V(P) \cap B_j \) then \( V(P) \cap T_i \neq \emptyset \) implies \( V(P) \cap F_i = \emptyset \) and \( x_i \) is set to true satisfying \( C_j \). A symmetric argument applies if \( u_{i,k} \in V(P) \cap B_j \).

Changing the proof slightly we get.

**Corollary 3.2.** It is NP-complete to decide if a strong semicomplete multipartite digraph \( D \) contains an \( (x,y) \)-quasi-hamiltonian path.

**Proof.** In the proof of Theorem 3.1 remove all internal arcs in \( D(B_j) \) for all \( 1 \leq j \leq m \). Given that these arcs could not be part of any \( (x,y) \)-path the rest of the proof remains the same with \( D \) now a strong semicomplete multipartite digraph with partite sets exactly the sets of \( C \).

### 4. Strong quasi-hamiltonian connectedness in extended semicomplete digraphs

We show that for a subclass of semicomplete multipartite digraphs, namely the extended semicomplete digraphs, the \((x,y)\)-quasi-hamiltonian path problem is polynomial.

To establish the claimed result we use the following theorem.

**Theorem 4.1.** \(|?|\) There exists an \( O(n^7) \) algorithm that, given a semicomplete digraph \( D = (V,A) \) and two vertices \( x,y \in V(D) \), either outputs an \((x,y)\)-hamiltonian path or decides that no such path exists.

To connect this result to the problem of finding an \((x,y)\)-quasi-hamiltonian path in an extended semicomplete digraph we establish the following.

**Lemma 4.2.** Let \( D = S[M_1, \ldots , M_s] \) be an extended semicomplete digraph and let \( x,y \in V(D) \). Let \( D' \) be the semicomplete digraph obtained from \( D \) by adding all possible arcs inside each \( M_i \). There exists an \((x,y)\)-quasi-hamiltonian path in \( D \) if and only if there exists an \((x,y)\)-hamiltonian path in \( D' \).

**Proof.** Let \( Q \) be an \((x,y)\)-quasi-hamiltonian path in \( D \). Fix \( v_1, \ldots, v_s \) such that \( v_i \in V(Q) \cap V(M_i) \) for \( 1 \leq i \leq s \) and let \( v_i^+, \ldots, v_s^+ \) be their respective successors on \( Q \). We can assume, modulo a reenumeration, that \( Q \) visits \( v_1, \ldots, v_s \) in that order. Note that we can possibly have \( v_s = y \), in which case we define \( v_s^+ \) to be \( y \). Let \( V_i = V(M_i) \cap V(Q) \) for \( i = 1, \ldots, s \) and let \( H_1, \ldots, H_s \) be hamiltonian paths of the complete digraphs \( D'(V_1), \ldots, D'(V_s) \) respectively. For all \( 1 \leq i \leq s \) we have that \( D'(M_i) \) is complete and so: \( v_i \) has an arc to the first vertex of \( H_i \), the last vertex of \( H_i \), which has the same out-neighbors of \( v_i \), has an arc to \( v_i^+ \notin M_i \) and moreover these arcs are not used by \( Q \) or any of the \( H_i \). We conclude that the path

\[
Q[x,v_1]H_1Q[v_1^+,v_2]H_2 \ldots Q[v_{s-1}^+,v_s]H_sQ[v_s^+,y]
\]

is an \((x,y)\)-hamiltonian path in \( D' \). Note that if \( V(M_i) \subset V(Q) \), then \( H_i = \emptyset \), in which case our path just continues along \( Q \) to \( v_{i+1} \).

Vice versa assume there exists an \((x,y)\)-hamiltonian path in \( D' \). Then, in particular, the set of \((x,y)\)-paths in \( D' \) containing at least one vertex for each \( M_i \) is non empty. Among these paths, let \( Q \) be one that minimizes the total number of arcs. \( Q \) corresponds to an \((x,y)\)-quasi-hamiltonian path in \( D \), because it uses no arc in \( \bigcup_i A(M_i) \). To see the last assertion assume, by contradiction, that \( Q \) contains an arc \( st \in A(M_i) \), for some \( i \). Clearly \( Q \) must visit a module different from \( M_i \) before visiting \( s \), or after visiting \( t \). Assume, without loss of generality, the latter holds and let \( u \) be the first vertex on \( Q \) after \( t \) which is not in \( M_i \). We have that \( su \in A(D) - A(Q) \), hence the path \( Q[x,s]Q[u,y] \) would have less arcs than \( Q \) and still contain at least one vertex for each \( M_i \), contradiction. \(%}
From the proof of Lemma 4.2 it is easy to see how, given an \((x,y)\)-hamiltonian path \(Q\) in \(D'\) one can construct an \((x,y)\)-quasi-hamiltonian path in \(D\) in time \(O(n)\): for each arc \(st \in Q \cap M_i\), if \(t \neq y\), find the first vertex \(u\) of \(Q\) after \(t\) which is not in \(M_i\) and replace the path \(Q[s,u]\) with the arc \(su\). If \(t = y\), find the last vertex \(u\) of \(Q\) before \(s\) which is not in \(M_i\) and replace the path \(Q[u,t]\) with the arc \(ut\).

Thus we have the following

**Theorem 4.3.** There exists an \(O(n^2)\) algorithm that, given an extended semicomplete multipartite digraph \(D\) and two vertices \(x, y \in V(D)\), either outputs an \((x,y)\)-quasi-hamiltonian path or decides that no such path exists.

**Proof.** Given \(D\), construct \(D'\) defined as in Lemma 4.2 in time \(O(n^2)\). Run the algorithm from Theorem 4.1 on input \((D', x, y)\): if the output is an \((x,y)\)-hamiltonian path in \(D'\), then produce an \((x,y)\)-quasi-hamiltonian path in \(D\) in time \(O(n)\). If the algorithm outputs that no \((x,y)\)-hamiltonian path exists in \(D'\), then output (correctly, by Lemma 4.2) that no \((x,y)\)-quasi-hamiltonian path exists in \(D\). 

\[\square\]

5. Weak quasi-hamiltonian connectedness in semicomplete multipartite digraphs

We construct our polynomial algorithm for finding an \([x,y]\)-quasi-hamiltonian path in a SMD. To that end we start by introducing some lemmas, the first of which is a generalization of the merging lemma from [?].

**Lemma 5.1.** Let \(D = (V, A)\) be a semicomplete multipartite digraph, \(P\) be an \((s_1, t_1)\)-path and \(Q\) be an \((s_2, t_2)\)-path such that \(P \cup Q\) does not contain a directed cycle and \(X \subseteq V(P) \cup V(Q)\) be a vertex set where each vertex is from a distinct partite set. Then \(P\) and \(Q\) can be quasi-merged into a single \((s_i, t_j)\)-path \(R\) covering \(X\), where \(i, j \in \{1, 2\}\). Such a quasi-merging can be performed in \(O(|V(P)\cup V(Q)|)\) time.

**Proof.** First observe that it is sufficient to prove the lemma for vertex-disjoint pairs of paths, since intersecting paths that do not induce directed cycles can be quasi-merged piece-wise if every pair of disjoint sub paths \(P[s,t]\) and \(Q[s,t]\) between subsequent intersection points \(s\) and \(t\) can be quasi-merged.

The proof is by induction on \(|V(P)| + |V(Q)|\). During this proof we call a quasi-merging of \(P\) and \(Q\) containing \(X\) a solution.

Trivial paths \(P = s_1 = t_1\) and \(Q = s_2 = t_2\) can easily be quasi-merged: If \(s_1 \in H(s_2)\) then at most one of \(s_1, s_2\) can be in \(X\), say \(s_2 \notin X\) and \(R = s_1\) is a solution. If there is an arc between \(s_1\) and \(s_2\), say \(s_1s_2 \in A\), then \(R = s_1s_2\) is a solution.

So assume \(P = v_1v_2 \ldots v_p\) and \(Q = u_1u_2 \ldots u_q\), with \(v_1 = s_1\), \(v_p = t_1\), \(u_1 = s_2\) and \(u_q = t_2\), and the lemma holds for all pairs of paths with total number of vertices less than \(p + q\). We now split into two cases:

**Case 1:** If \(s_1 \notin H(s_2)\) then there is an arc between \(s_1\) and \(s_2\), say \(s_1s_2\). If \(P = s_1\) the path \(s_1Q\) is a solution. If \(P\) is non-trivial then by induction we can quasi-merge the paths \(P - s_1\) and \(Q\) into a single \((v_2, t_1)\)-path or \((s_2, t_1)\)-path \(R\) containing \(X - s_1\), but \(s_1\) dominates both \(v_2\) and \(s_2\) so \(s_1R\) is a solution.

**Case 2:** If \(s_1 \in H(s_2)\) then at most one of \(s_1, s_2\) can be in \(X\), say \(s_1 \notin X\). If \(P = s_1\) then \(Q\) is a solution. If \(P\) is non-trivial we can quasi-merge the paths \(P - s_1\) and \(Q\) into a single \((v_2, t_1)\)-path or \((s_2, t_1)\)-path \(R\) containing \(X\). If \(R\) starts in \(s_2\) it is already a solution. If \(R\) starts in \(v_2\) then \(s_1R\) is a solution.

If we preprocess by marking the vertices of \(X\) every iteration of the algorithm can be performed in constant time by evaluating the adjacency between \(s_1\) and \(s_2\) and checking if they are marked. Then we recursively solve a problem of size \(p + q - 1\), giving a total runtime of \(O(p + q)\). 

\[\square\]
Consider the shortest (z,x) and (z,y)-paths from the table. Since z \notin H(x) at least one of them is just an arc. Assume, by directional duality, xz \in A and let P be the (z,x)-path. There exists u \in C \cap P such that P[z,u] \cap C = \emptyset (possibly u = x). Find such an u.

By Lemma 5.1 we can quasi-merge P[z,u] and C[x,u] into a path R containing X \cap (V(P[z,u]) \cup V(C[x,u])). Now xRC[u,x] is a cycle containing X \cap V(C) and z. The algorithm constructs the path R by quasi-merging, sets C = xRC[u,x] and repeats.

The table computation takes time O(|A|). The time for each iteration is dominated by the time of a quasi-merging, which is O(n). Given that there are at most |X| iterations, we have that the total running time is O(n^2).

Corollary 5.3. Let D = (V,A) be a strong semicomplete multipartite digraph and X \subseteq V be a vertex set where each vertex is from a distinct partite set. Then there exists a quasi-hamiltonian cycle containing X.

Proof. Just extend X with one vertex from every partite set disjoint from X and apply Theorem 5.2.

For further structure we refer to earlier work on decomposing SMDs into strong components.

Lemma 5.4. Let D = (V,A) be a semicomplete multipartite digraph with partite sets V_1, \ldots, V_c. Then there exists a unique partition of V into R_1, \ldots, R_c, where D[R_i] is either a strong component of D or R_i \subseteq V_j for some 1 \leq j \leq c, such that R_i \Rightarrow R_j for all 1 \leq i < j \leq c and there are x_i \in R_i and y_i \in R_{i+1} such that x_iy_i \in A for 1 \leq i < c. We call this unique partition the linear decomposition of D.

Notice that there is an O(n^2) time algorithm to find the linear decomposition of a given SMD: find the strong components, find an acyclic ordering of the strong components and then group together all the vertices from the same partite set that form consecutive components in the ordering.

Applying Lemma 5.1 we get a short proof of the the following variation of Lemma 2.4 in [? ].

Lemma 5.5. Let R_1,\ldots,R_k be the linear decomposition of a connected non strong semicomplete multipartite digraph D, let x \in R_1, y \in R_k and X \subseteq V - H(\{x,y\}) be a vertex set where each vertex is from a distinct partite set. Then there exists an (x,y)-path covering X.

Proof. Since D is not strong we have 1 < k. X \cap R_1 is either an empty set or R_1 is a strong component and by Theorem 5.2 there exists a cycle C_1 in R_1 covering (x \cup X) \cap R_1. In the second case we can easily construct an (x,y)-path P_x that contains X \cap R_1 by following C_1 as far as needed and then exploiting that the arc xy exists for all vertices u \in X \cap R_1. Similarly construct an (x,y)-path P_y that contains X \cap R_k if X \cap R_k \neq \emptyset. Every vertex x_i \in X - (R_1 \cup R_k) lies on an (x,y)-path P_i of length 2, since x dominates all vertices of X - \{R_1\} and y is dominated by all vertices of X - R_k.

Now we simply quasi-merge the internally disjoint (x,y)-paths P_x, P_y (if they exist) and all the paths P_i into one (x,y)-path R containing all of X.
With the algorithm implied by the proof above such an \((x, y)\)-quasi-hamiltonian path (which always exists) can be identified in \(O(n^2)\) time.

With these tools established we are now ready to approach the problem of deciding if a given semi-complete multipartite digraph contains an \([x, y]\)-quasi-hamiltonian path in a very similar way to the approach used by C. Thomassen in [7] , which is used as inspiration for the partial characterization that lies at the core of our algorithm.

**Lemma 5.6.** Let \(D = (V, A)\) be a non-strong semicomplete multipartite digraph with \(x, y \in V\), let \(R_1, \ldots, R_k\) be its linear decomposition and let \(x \in R_i\), \(y \in R_j\). If \(R_i \cup \ldots \cup R_j\) intersects all partite sets of \(V\) and \(i \neq j\), then there exists an \((x, y)\)-quasi-hamiltonian path in \(D\) and we can find such a path in polynomial time. If \(R_i \cup \ldots \cup R_j\) does not intersect some partite set of \(V\), then there exists no \((x, y)\)-quasi-hamiltonian path in \(D\).

**Proof.** It follows directly from Lemma 5.5 that if \(i \neq j\) then \(D(R_i \cup \ldots \cup R_j)\) contains an \((x, y)\)-quasi-hamiltonian path \(P\), and if \(R_i \cup \ldots \cup R_j\) intersects all partite sets of \(V\) then \(P\) is also quasi-hamiltonian in \(D\). The implied algorithm runs in \(O(n^2)\) time.

On the other hand if \(R_i \cup \ldots \cup R_j\) does not intersect a partite set \(X\) then it is impossible for any \((x, y)\)-path to intersect \(X\), simply by the structure of the linear decomposition. \(\square\)

Notice that an open case remains where \(x\) and \(y\) are in the same strong component which intersects all partite sets of \(D\), in this case the answer to the problem depends solely on this strong component since no \((x, y)\)-path will ever leave it. So we can now focus exclusively on strong SMDs.

**Lemma 5.7.** Let \(D = (V, A)\) be a strong semicomplete multipartite digraph with \(x, y \in V\) such that \(D - x\) is not strong, let \(R_1, \ldots, R_k\) be the linear decomposition of \(D - x\) and let \(y \in R_i\). There exists an \([x, y]\)-quasi-hamiltonian path in \(D\) if and only if either \(R_1 \cup \ldots \cup R_i\) or \(R_i \cup \ldots \cup R_k\) intersect all partite sets of \(V - H(x)\). We can find such a path in polynomial time if it exists.

**Proof.** Since \(D\) is strong \(x\) must have at least one out-neighbour \(a \in R_1\) and one in-neighbour \(b \in R_k\). Now by Lemma 5.5 we can find an \((a, y)\)-quasi-hamiltonian path \(P\) in \(D(R_1 \cup \ldots \cup R_i)\) and a \((y, b)\)-quasi-hamiltonian path \(Q\) in \(D(R_i \cup \ldots \cup R_k)\). By the condition either \(xP\) or \(Qx\) is an \([x, y]\)-quasi-hamiltonian path in \(D\) giving sufficiency.

Necessity is obvious since if both \(R_1 \cup \ldots \cup R_{i-1}\) and \(R_{i+1} \cup \ldots \cup R_k\) contain partite sets not represented elsewhere in \(D\) then any quasi-hamiltonian path in \(D\) that starts or ends at \(y\) will have to contain \(x\) as an internal vertex. \(\square\)

**Lemma 5.8.** Let \(D = (V, A)\) be a strong semicomplete multipartite digraph with \(x, y \in V\) and let \(D - x\) and \(D - y\) be strong. If \(D - \{x, y\}\) is not strong then there exists an \([x, y]\)-quasi-hamiltonian path and we can identify such a path in polynomial time.

**Proof.** Let \(R_1, \ldots, R_k\) be the linear decomposition of \(D - \{x, y\}\). Since \(D - x\) and \(D - y\) are strong \(x\) must have at least one out-neighbour \(a \in R_1\) and \(y\) must have at least one in-neighbour \(b \in R_k\). By Lemma 5.5 there exists an \((a, b)\)-quasi-hamiltonian path \(P\) in \(D - \{x, y\}\) so \(xPy\) is our \((x, y)\)-quasi-hamiltonian path. \(\square\)

The above lemmas settle all cases except strong SMDs which remain strong after removing \(x, y\) or \(\{x, y\}\). In [7] this remaining case for tournaments can be characterized nicely, but for SMDs a full characterization is quite cumbersome and will thus be omitted here. Instead we prove some structural results on the remaining case and argue that we either find an \([x, y]\)-quasi-hamiltonian path directly in polynomial time or that there exists such a path if and only if a short one exists.

**Theorem 5.9.** Let \(D = (V, A)\) be a strong semicomplete multipartite digraph with vertices \(x, y \in V\). Let \(C\) be a cycle in \(D - \{x, y\}\) intersecting all the partite sets of \(V - H(\{x, y\})\) and consider a subset of cyclically enumerated vertices \(U = \{u_1, \ldots, u_k\} \subset V(C) - H(\{x, y\})\) that intersects all the partite sets of \(V - H(\{x, y\})\). In time \(O(n^2)\) we can either find an \([x, y]\)-quasi-hamiltonian path or establish that (a)-(e) below hold.
(a) \(|U|\) is even.

(b) We can cyclically relabel \(U\) such that
\[ E \Rightarrow x \Rightarrow O, \quad O \Rightarrow y \Rightarrow E, \tag{1} \]
where \(E, O\) denotes the set of vertices of \(U\) with even or odd index respectively.

(c) For every \(u \in V(C)\) there does not exist both an \((x, u)\)-path and a \((y, u)\)-path or both a \((u, x)\)-path and a \((u, y)\)-path internally disjoint from \(V(C) \cup \{x, y\}\).

(d) \(H(u) = \{u\}\) for \(u \notin H(\{x, y\})\) and \(V - H(\{x, y\}) = U\).

(e) \(C[u_i, u_{i+1}]\) has length at most two, for \(1 \leq i \leq |U|\).

Furthermore if (a)-(e) hold and \(D\) contains an \([x, y]\)-quasi-hamiltonian path then \(D\) contains an \([x, y]\)-quasi-hamiltonian path of length at most \(2|U| + 2\) provided the following holds.

(f) There exists a \((u_i, u_j)\)-path \(P\) of length 3 internally disjoint from \(V(C) \cup \{x, y\}\), with \(i \neq j\) of different parity.

Proof. The straightforward way of checking if conditions (a)-(e) hold takes \(O(n^2)\) time. So we are left with proving that any violation of the conditions will enable us to find an \([x, y]\)-quasi-hamiltonian path in \(O(n^2)\) time.

(a): Assume \(U = \{u_1\}\). If there is no \([x, y]\)-quasi-hamiltonian path of length 2, then \(\{x, y\} \Rightarrow u_1\) or \(u_1 \Rightarrow \{x, y\}\). Assume, by directional duality, that \(\{x, y\} \Rightarrow u_1\). Since \(D\) is strong there exists a \((u_1, \{x, y\})\)-path \(P\): if \(P\) ends in \(x\), then \(yP\) is a \((y, x)\)-quasi-hamiltonian path, if \(P\) ends in \(y\), then \(xP\) is a \((x, y)\)-quasi-hamiltonian path.

Assume \(|U| > 1\), but still odd. We can cyclically relabel \(u_1, \ldots, u_n\) such that \(u_1y \in A\) (reversing all arcs if necessary). If \(ux_2 \in A\), then \(xP[u_2, u_1y]\) is an \((x, y)\)-quasi-hamiltonian path, so assume \(u_2 \Rightarrow y\). Now symmetrically if \(yu_3 \in A\) we find an \((y, x)\)-quasi-hamiltonian path, so assume \(u_3 \Rightarrow y\). By repeating this argument we find an \([x, y]\)-quasi-hamiltonian path unless \(E \Rightarrow x, O \Rightarrow y\) and \(|U|\) is even.

(b): From the proof of (a) we already have \(O \Rightarrow y\) and \(E \Rightarrow x\). Assume that \(u_i x \in A\) for some odd \(i\) or \(u_j y \in A\) for some even \(j\). By the above reasoning we find an \([x, y]\)-quasi-hamiltonian path unless \(O \Rightarrow x\) and \(E \Rightarrow y\), but then \(U \Rightarrow \{x, y\}\). Since \(D\) is strong we can find an \((\{x, y\}, C)\)-path \(P\). Assume without loss of generality that \(P\) is an \((x, u)\)-path and let \(u_t\) be the last vertex, when following the direction of the cycle, of \(U\) before \(u\). Then \(P[C[u_t, u_i]y\} is an \((x, y)\)-quasi-hamiltonian path.

(c): Assume that we have an \((x, u)\)-path \(P\) and a \((y, u)\)-path \(Q\) (the other case is symmetric) internally disjoint from \(\{C, x, y\}\), for some \(u \in V(C)\). Let \(u_t\) be the last vertex, when following the direction of the cycle, of \(U\) before \(u\). Since we already assumed (b) holds there are only two possibilities: either \(u_t y \in A\) or \(u_t x \in A\): in the first case \(P[C[u_t, u_i]y\}\) is a \((x, y)\)-quasi-hamiltonian path, in the second case \(Q[C[u_t, u_i]x\}\) is a \((y, x)\)-quasi-hamiltonian path.

(d): Assume that there exists a partite set in \(V - H(\{x, y\})\) with at least two vertices \(v, w\). If \(v, w \in C\) then one of the sets \(U \cup w, U - w\) has an odd number of elements while still intersecting all the partite sets, so we get an \([x, y]\)-quasi-hamiltonian path from (a). Therefore we can assume that \(w \notin C\) and \(H(w) \cap C = v\).

Suppose \(\{x, y\} \Rightarrow w\) and let \(P\) be a shortest \((w, C)\)-path. If \(x, y \notin P\) then \(xP, yP\) violate (c). Otherwise we may assume without loss of generality that \(x \in P\) and \(y \notin P[w, x]\). Since \(|P| > 1\) and \(P\) was shortest we have \(C \Rightarrow w\). For any odd \(u_i \in C\) we have \(u_i y \in A\) by (b) so \(u_i w \notin A\) since otherwise \(u_i y\) and \(u_i P[w, x]\) would violate (c), but \(w u_i, u_i w \notin A\) implies \(u_i \in H(w) \cap C = v\). So every odd \(u_i\) is \(v\) implying \(|U| = 2\), but now \(yu_2 P[w, x]\) is a \((y, x)\)-quasi-hamiltonian path. Symmetrically we can find an \([x, y]\)-quasi-hamiltonian path if \(w \Rightarrow \{x, y\}\).
Therefore we may assume, by directional duality, that \( y \Rightarrow w \Rightarrow x \). Now observe that a vertex \( u \in V(C) - H(\{x, y, w\}) \) such that \( x \Rightarrow u \Rightarrow y \) cannot exist, for otherwise \( xu, ywu \) or \( uy, uwx \) would form two paths contradicting (e). Thus, by (b), there are only two vertices of \( U \) on \( C \): \( v \) and \( v' \), with \( y \Rightarrow v' \Rightarrow x \). Now either \( yv'wx \) or \( ywu'x \) is a \( (y, x) \)-quasi-hamiltonian-path.

(e): Assume \( C[\{\text{u}_i, \text{u}_{i+1}\}] \) has length at least 3. If \( i \) is odd then by (b) \( u_i \Rightarrow y \Rightarrow u_{i+1} \). Consider a vertex \( x \in V(C[\{\text{u}_i, \text{u}_{i+1}\}]) - H(x) \): if \( xw \in A \), then \( yC[\{u_{i+1}, \text{v}\}]x \) is a \( (y, x) \)-quasi-hamiltonian path; if \( xw \in A \), then \( xC[\text{v}, \text{u}_i]y \) is a \( (x, y) \)-quasi-hamiltonian path. If \( i \) is even, a symmetric argument applies where we consider a vertex \( v \in V(C[\{\text{u}_i, \text{u}_{i+1}\}]) - H(y) \) instead.

Now to prove the last claim we assume that \( D \) satisfies (a)-(e).

(f): Let \( D \) contain the \( (\text{u}_i, \text{u}_j) \)-path \( P \), with \( i \equiv j \mod 2 \), such that \( P \) has length at least 3 and is internally disjoint from \( C \cup \{x, y\} \). Suppose \( i \) is odd, so \( xu_i \in A \), and \( j \) is even, so \( u_jx \in A \). Since \( D \) satisfies (d) we know that \( V(P[u_i, u_j]) \subset H(\{x, y\}) \), so one of the two internal vertices of \( P \) must be in \( H(y) \), say \( a \in V(P[u_i, u_j]) \cap H(y) \). If \( xw \in A \) then \( D \) contains \( xP[a, u_j]C[u_j, u_{j-1}]y \) by (b), which is an \( (x, y) \)-quasi-hamiltonian path of length at most \( 2|U| + 2 \) by (e). On the other hand if \( ax \in A \) then \( D \) contains \( yC[\text{u}_{i+1}, \text{u}_i]P[u_i, a]x \), which is a \( (y, x) \)-quasi-hamiltonian path of length at most \( 2|U| + 2 \). The case for even \( i \) follows by symmetric arguments.

To limit the length of a shortest \( [x, y] \)-quasi-hamiltonian path to a constant we must now only get \( U \) down to constant size. To achieve this we employ a result from \([?]\) on strong multipartite tournaments, instead of reproving it for strong SMDs we will use that:

**Theorem 5.10.** \([?]\) Any strong semicomplete multipartite digraph with at least three partite sets contains a spanning strong multipartite tournament.

**Proof.** Corollary 2.4 in \([?]\) gives this result for any digraph \( D = (V, A) \) where \( D = \{uv, vu\} \) is connected for all \( u, v \in V \). Any SMD with three or more partite sets remains connected if all arcs between an arbitrary pair of vertices are removed.

Identifying a strong MT in the given strong SMD takes only \( O(n^3) \) time \([?]\). Now we are able to use:

**Lemma 5.11.** (Corollary 2.10 in \([?]\) for SMDs) For any three distinct vertices of a strong semicomplete multipartite digraph there is a quasi-hamiltonian path connecting two of them.

The algorithm implied by the proof of this lemma in \([?]\) can find such a path in time \( O(n^3) \).

Applying this we get:

**Corollary 5.12.** Let \( D, x, y, C, U \) be as in Theorem 5.9. If \( |U| \notin \{2, 4\} \) then \( D \) contains an \( [x, y] \)-quasi-hamiltonian path and we can find such a path in \( O(n^3) \) time.

**Proof.** We can assume that conditions (a) and (b) in Theorem 5.9 are met since we otherwise find an \( [x, y] \)-quasi-hamiltonian path in \( D \).

So \( |U| \) is even. If \( |U| > 4 \), then by Theorem 5.9 (b), we have that \( x \Rightarrow \{u_1, u_3, u_5\} \Rightarrow y \). By Lemma 5.11 we can find a quasi-hamiltonian path \( P \) between two of \( \{u_1, u_3, u_5\} \) in \( D - \{x, y\} \) in \( O(n^3) \) time and since \( P \) is a \( [u_i, u_j] \)-path for some distinct \( i, j \in \{1, 3, 5\} \) we get that \( xPy \) is an \( (x, y) \)-quasi-hamiltonian path in \( D \).

Excluding all of the cases we showed to be decidable in time \( O(n^3) \) we are left with a very specific type of graph:

**Lemma 5.13.** Let \( D, x, y, C, U \) be as in Theorem 5.9, suppose they satisfy conditions (a)-(e) and let \( |U| \in \{2, 4\} \). Then there exists an \( [x, y] \)-quasi-hamiltonian path if and only if there exists an \( [x, y] \)-quasi-hamiltonian path of length at most 27.
Figure 2: A semicomplete multipartite digraph in which all \([x,y]\)-quasi-hamiltonian paths are long: the vertices \(x, y, a, b, c, d\) are in different partite sets, \(P_1\) and \(P_2\) are paths of length \(l\) alternating between \(H(x)\) and \(H(y)\), all arcs on the subdigraphs induced by \(aP_1d\) and \(cP_2b\) distinct from the arcs of the path go backwards. \(\{x, y\} \Rightarrow P_1\), \(P_2 \Rightarrow \{x, y\}\) and \(P_3 \Rightarrow P_1\). In a \([x,y]\)-quasi-hamiltonian path of length smaller than \(l\) the successor of \(a\) must be \(y\), because the only other way out of \(a\) is the entire path \(P_1\). Similarly one can see that \(b\) must be the successor of \(y\), but this would imply that \(y\) is an internal vertex of the path, which is impossible.

Proof. Let \(P\) be a shortest \([x,y]\)-quasi-hamiltonian path and assume by directional duality that it starts in \(x\). By Theorem 5.9(d) and the fact that \(P\) is quasi-hamiltonian we get that \(U \subseteq P\). By (e) we get \(|C| \leq 2|U| \leq 8\).

Assume there exist three consecutive internal vertices \(w_1, w_2, w_3 \in (V(P) - V(C))\) on \(P\). We examine different placements of these vertices on \(P\).

- Suppose \(w_1, w_2, w_3 \in P[u_i, y]\), with \(U \cap P[x, u_i] = \{u_i\}\). By minimality of \(P\) and Theorem 5.9(h) \(i\) is even, so \(u_{i-l} y \in A\). Observe that \(A\) contains at least one of the arcs \(w_2x\) and \(w_3x\). Hence if \(u_{i-l} w_2 \in A\), then \(u_{i-l} w_2x\) or \(u_{i-l} w_2 w_3x\) is a \((u_{i-l}, x)\)-path, but then this path and \(u_{i-l} y\) violate Theorem 5.9(c), contradiction. So \(w_2 u_{i-l} \in A\), and now \(u_i w_1 w_3 u_{i-l} \) is a path of length 3 between vertices of \(U\) of different parity, hence, by Theorem 5.9 (f), there exists a \([x,y]\)-quasi-hamiltonian path of length at most \(2|U| + 2 \leq 10\).

- Suppose \(w_1, w_2, w_3 \in P[u_i, y]\), with \(U \cap P[u_i, y] = \{u_i\}\). By an argument symmetric to the one above, one can prove that this implies \(|P| \leq 10\).

- Suppose \(w_1, w_2, w_3 \in P[u_i, y]\), with \(U \cap P[u_i, y] = \{u_i, u_j\}\). We distinguish two cases: \(i\) and \(j\) have different parity, in which case let \(Q\) be the path \(u_i w_2 w_3 u_i\); \(i\) and \(j\) have the same parity, in which case if \(u_{i+1} w_2 \in A\), let \(Q\) be \(u_{i+1} w_2 w_3 u_i\), otherwise let \(Q\) be \(u_i w_1 w_2 u_{i+1}\). In any case \(Q\) is a path \((u_j w_1, u_j w_2, w_3 u_i \in A\) by minimality of \(P\)) of length 3 between vertices of \(U\) of different parity and, again by Theorem 5.9 (f), we have \(|P| \leq 10\).

So we can assume that there exist no three consecutive internal vertices \(w_1, w_2, w_3 \in (V(P) - V(C))\) on \(P\), but this implies that \(P\) has length at most \(2(|C| + 1) + |C| + 1 \leq 27\).

Notice that it is not true that if there is an \([x,y]\)-quasi-hamiltonian path, then there is an \([x,y]\)-quasi-hamiltonian path of length at most 27, even if the digraph is very close to the hypothesis of Lemma 5.13. Figure 2 shows how to construct a class of digraphs that satisfy conditions (a)-(d), but not (e) and contain only (arbitrarily) long \([x,y]\)-paths.

Since the remaining case could be solved by a brute force search for a short \([x,y]\)-quasi-hamiltonian path in polynomial time we are now ready to present a polytime algorithm that runs through all possible cases.
**Theorem 5.14.** Given a semicomplete multipartite digraph $D$ and two vertices $x, y \in V(D)$, there exists a polynomial algorithm that either outputs an $[x,y]$-quasi-hamiltonian path or correctly decides that such a path does not exist.

**Proof.** The following algorithm, whose correctness follows from the previous lemmas, has a polynomial running time.

1. Check if $D$ is strong. If so go to step 2. If not find the linear decomposition of $D$: if $x$ and $y$ are in the same component $R_i$ and $R_i$ intersects all the partite sets, then run recursively the algorithm on input $R_i, x, y$. Otherwise decide the problem according to Lemma 5.6.

2. Check if $D - x$ is strong. If so go to step 3. If not find the linear decomposition of $D - x$ and decide the problem according to Lemma 5.7.

3. Check if $D - y$ is strong. If so go to step 4. If not find the linear decomposition of $D - y$ and decide the problem according to Lemma 5.7.

4. Check if $D - \{x, y\}$ is strong. If so go to step 5. If not output the $[x,y]$-quasi-hamiltonian path given by Lemma 5.8.

5. Find a quasi-Hamiltonian-cycle $C$ of $D - \{x, y\}$ and a subset $U \subseteq V(C) - H(\{x, y\})$ that contains one vertex from each partite set of $V(C) - H(\{x, y\})$: if $|U| \not\in \{2, 4\}$ output the $[x,y]$-quasi-hamiltonian path given by Corollary 5.12. Check if condition (a)-(e) of Theorem 5.9 hold. If so go to step 6. If not output the $[x,y]$-quasi-hamiltonian path given by Theorem 5.9.

6. Look for an $[x,y]$-quasi-hamiltonian path among all the $(x,y)$-paths of length at most 27, if one is found output it, otherwise output that no $[x,y]$-quasi-hamiltonian path exists.

As it is not difficult to see all but the last step of the algorithm have an $O(n^3)$ runtime. Using a more careful approach which avoids Theorem 5.10 it is possible to lower it down to $O(n^2)$. The complexity of a brute-force search for an $[x,y]$-quasi-hamiltonian path of length at most 27 in the last step is $O(n^{26})$. This can also be reduced to $O(n^2)$ using a variation of standard color coding techniques as described by Alon, Yuster and Zwick in [? ]. You tackle the quasi-hamiltonian property by applying a second layer of coloring: First by fixing a color sequence the solution path has to follow and second by fixing a partite set sequence it has to intersect.

6. Conclusions and open problems

From our algorithm for the $[x,y]$-quasi-hamiltonian path problem, it is not difficult to derive an (polynomial) algorithm that finds an $[x,y]$-path maximizing the number of partite sets intersected. This generalizes the original problem since it is a longest, in terms of partite sets, $[x,y]$-path problem. If we ask for a longest path in terms of number of vertices on the paths we have a different generalization of the longest $[x,y]$-path problem in semicomplete digraphs.

**Conjecture 6.1.** There exists a polynomial algorithm for finding the longest $[x,y]$-path in a semicomplete multipartite digraph.

Alternatively one could ask which path is the longest among the quasi-hamiltonian paths.

**Problem 2.** Is there a polynomial algorithm that, given a semicomplete multipartite digraph $D$ and $x, y \in V$, finds a longest $[x,y]$-quasi-hamiltonian path?
Both of these problems generalize the \([x, y]\)-hamiltonian path problem in semicomplete multipartite digraphs whose complexity is not known.

On the other hand it is possible to show that there always exists an \([x, y]\)-path intersecting at most 3 different partite sets, this implies a polynomial algorithm to find an \([x, y]\)-path that minimizes the number of partite sets intersected. The shortest path version of Problem 2 seems harder, though.

**Problem 3.** Is there a polynomial algorithm that, given a semicomplete multipartite digraph \(D\) and \(x, y \in V\), finds a shortest \([x, y]\)-quasi-hamiltonian path?

Or one could consider a condition between quasi-hamiltonian and hamiltonian and demand a certain number of vertices from each partite set.

**Problem 4.** Is there a polynomial algorithm that given an integer \(k\) and a semicomplete multipartite digraph \(D\) with vertices \(x, y\) and partite sets \(V_1, \ldots, V_c\), decides whether \(D\) has an \([x, y]\)-path covering at least \(\min\{k, |V_i|\}\) vertices from each \(V_i\), \(i = 1, \ldots, c\)?

Problem 4 seems non-trivial even in a milder version:

**Problem 5.** Is there a polynomial algorithm that given an integer \(k\) and a semicomplete multipartite digraph \(D\) with partite sets \(V_1, \ldots, V_c\), decides whether \(D\) has a path covering at least \(\min\{k, |V_i|\}\) vertices from each \(V_i\), \(i = 1, \ldots, c\)?

Note that a collection of cycles and a path would be enough, since they can be merged into a single path spanning the same vertices, by a result in [2]. When \(k = 1, 2\) it is not difficult to find the desired path in polynomial time.


