Outline

1. Parallel Machine Models

Parallel Machine Models

Lecture 13

Outline

1. Parallel Machine Models

- $P_m | | C_{max}$ (without preemption)
  - LPT heuristic, approximation ratio: $\frac{4}{3} - \frac{1}{3m}$
  - $P_\infty | prec | C_{max}$ CPM
  - $P_m | prec | C_{max}$ strongly NP-hard, LNS heuristic (non optimal)
  - $P_m | p_j = 1, M_j | C_{max}$ LFJ-LFM (optimal if $M_j$ are nested)
Not NP-hard:

- Linear Programming (exercise)
- Construction based on $LWB = \max \left\{ p_1, \sum_{j=1}^{n} \frac{p_j}{m} \right\}$
- Dispatching rule: longest remaining processing time (LRPT) optimal in discrete time

Dantzig-Wolfe decomposition

If model has “block” structure
\[
\begin{align*}
\max & \quad c^1 x^1 + c^2 x^2 + \ldots + c^K x^K \\
\text{s.t.} & \quad A^1 x^1 + A^2 x^2 + \ldots + A^K x^K = b \\
& \quad D^1 x^1 + D^2 x^2 + \ldots + D^K x^K \leq d_1 \\
& \quad \vdots \\
& \quad x^1 \in \mathbb{Z}^n_+ \quad x^2 \in \mathbb{Z}^n_+ \quad \ldots \quad x^K \in \mathbb{Z}^n_+
\end{align*}
\]

Substituting each set $X^k, k = 1, \ldots, K$ in original model
getting Master Problem
\[
\begin{align*}
\max & \quad c^1 (\sum_{i \in T_1} \lambda_i x_i^1) + c^2 (\sum_{i \in T_2} \lambda_i x_i^2) + \ldots + c^K (\sum_{i \in T_K} \lambda_i x_i^K) \\
\text{s.t.} & \quad A^1 (\sum_{i \in T_1} \lambda_i x_i) + A^2 (\sum_{i \in T_2} \lambda_i x_i) + \ldots + A^K (\sum_{i \in T_K} \lambda_i x_i) = b \\
& \quad \sum_{i \in T_k} \lambda_i = 1 \quad k = 1, \ldots, K \\
& \quad \lambda_{ij} \in \{0, 1\}, \quad t \in T_k \quad k = 1, \ldots, K
\end{align*}
\]

Strength of linear master model

Solving LP-relaxation of master problem, is equivalent to (Wolsey Prop 11.1)
\[
\begin{align*}
\max & \quad c^1 x^1 + c^2 x^2 + \ldots + c^K x^K \\
\text{s.t.} & \quad A^1 x^1 + A^2 x^2 + \ldots + A^K x^K = b \\
& \quad x^1 \in \text{conv}(X^1) \quad x^2 \in \text{conv}(X^2) \quad \ldots \quad x^K \in \text{conv}(X^K)
\end{align*}
\]

Strength of Lagrangian relaxation

- $z_{\text{LP}}$ be LP-solution value of master problem
- $z_{\text{LD}}$ be solution value of lagrangian dual problem

(Theorem 11.2) $z_{\text{LP}} = z_{\text{LD}}$
Delayed column generation, linear master

*(minimization problem)*

Run Simplex algorithm as if complete master problem was known

- Start with a basis solution
- Solve $x_B = A_B^{-1} b$
  and find dual variables $y = c_B A_B^{-1}$

- When choosing entering variable solve pricing problem which minimizes reduced costs $c_{rj} = c_j - y A_j$

- If $c_{rj} < 0$ add corresponding column $A_j$ to model and repeat
- If $c_{rj} \geq 0$ stop

Cutting Stock Problem

*(minimization problem)*

$a_{ij}$ is number of pieces of type $i$ cut from pattern $j$

Master problem,

$$\min \sum_{j=1}^{n} u_j$$

s.t. $\sum_{i=1}^{m} a_{ij} u_j \geq b_i \quad i=1, \ldots, m, \quad j=1, \ldots, n$

Solving linear master through delayed column generation

- Start with patterns which only contain one type $i$
- Solve restricted master
- Dual variables $y_i$ say how “attractive” a type $i$ is
- Pricing problem

$$z^S = \min \sum_{i=1}^{m} y_i x_i$$

s.t. $\sum_{i=1}^{m} w_i x_i \leq L$

$x \geq 0$, integer

- stop if $z^S \geq 0$

Terminology

- Master Problem
- Restricted Master Problem
- Subproblem or Pricing Problem
- Branch and cut: Branch-and-bound algorithm using cuts to strengthen bounds.
- Branch and price: Branch-and-bound algorithm using column generation to derive bounds.
- One says that discarded columns are “priced out”.

Branch-and-price

- LP-solution of master problem may have fractional solutions
- Branch-and-bound for getting IP-solution
- In each node solve LP-approximation of master
- Subproblem may change when we add constraints to master problem
- Branching strategy should make subproblem easy to solve
Branch-and-price, example

The matrix $A$ contains all different cutting patterns

\[
A = \begin{pmatrix}
4 & 0 & 1 & 2 & 3 \\
0 & 7 & 5 & 4 & 2
\end{pmatrix}
\]

Problem

minimize $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5$

subject to

\[
\begin{align*}
4\lambda_1 + 0\lambda_2 + 1\lambda_3 + 2\lambda_4 + 3\lambda_5 & \geq 7 \\
0\lambda_1 + 7\lambda_2 + 5\lambda_3 + 4\lambda_4 + 2\lambda_5 & \geq 3
\end{align*}
\]

$\lambda_j \in \mathbb{Z}_+$

LP-solution $\lambda_1 = 1.375, \lambda_4 = 0.75$

Branch on $\lambda_1 = 0, \lambda_1 = 1, \lambda_2 = 2$

- Column generation may not generate pattern (4,0)
- Pricing problem is knapsack problem with pattern forbidden

Tailing off effect

Column generation may converge slowly in the end
- We do not need exact solution, just lower bound
- Solving master problem for subset of columns does not give valid lower bound (why?)
- Instead we may use Lagrangian relaxation of joint constraint
- "guess" lagrangian multipliers equal to dual variables from master problem

Heuristic solution

- Restricted master problem will only contain a subset of the columns
- We may solve restricted master problem to IP-optimality
- Restricted master is a "set-covering-like" problem which is not too difficult to solve