Cutting Plane Algorithms

Jesper Larsen & Jens Clausen

jla,jc@imm.dtu.dk

Informatics and Mathematical Modelling
Technical University of Denmark

Introduction

- We have an optimization problem

$$\max\{cx : x \in X\}$$

  where $X = \{x : Ax \leq b, x \geq 0, x$ integer$\}$. From earlier we have that conv(X) is a polyhedron $\{x : \tilde{A}x \leq \tilde{b}, x \geq 0\}$.

- In general conv(X) is **impossible** to find if the problem at hand is NP-hard.

Valid inequality

- Find a **good approximation** – especially around the optimal solution.
- How?
- $\pi x \leq \pi_0$ is a **valid inequality** for X if and only if $\pi x \leq \pi_0$ for all $x \in X$.

Example

If $X = \{x : Ax \leq b, x \geq 0, x$ integer$\}$ and conv(X)$= \{x : \tilde{A}x \leq \tilde{b}\}$ then the constraints

- $a^i x \leq b$ are valid inequalities for X
- $\tilde{a}^i x \leq \tilde{b}$ are valid inequalities for X

Example – Knapsack

Consider the following 0-1-knapsack set:

$$X = \{x \in B^5 : 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leq -2\}$$

- $x_2 = x_4 = 0$ we get lhs $3x_1 + 2x_3 + x_5$ and rhs -2, which is impossible. So a valid inequality is $x_2 + x_4 \geq 1$.
- $x_1 = 1$ and $x_2 = 0$ we get lhs $3 + 2x_3 - 3x_4 + x_5 \geq 3 - 3 = 0$ and rhs -2, which is impossible. So a valid inequality is $x_1 \leq x_2$.

Example – CFL

Capacitated Facility Location has the feasible region:

$$\begin{array}{lcll} \sum_{i \in M} x_{ij} & \leq & b_j y_j & j \in N \\ \sum_{j \in N} x_{ij} & = & a_i & i \in M \\ x_{ij} \geq 0, \ y_j \in \{0,1\} & & & i \in M, j \in N \end{array}$$

Here we see that all feasible solutions satisfy $x_{ij} \leq b_j y_j$ and $x_{ij} \leq a_i$ with $y_j \in B^1$.
Therefore $x_{ij} \leq \min\{a_i, b_j\} y_j$ is a family of valid inequalities.

Example – Integer Rounding

- Consider
  $P = \{x \in Z_+^4 : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\}$
- Divide by 11 and we get a valid inequality (uninteresting):

$$\frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq 6\frac{6}{11}$$

Example – Integer Rounding (Cont.)

- As $x \geq 0$ we round up the lhs and get a weaker valid inequality:

$$2x_1 + 2x_2 + x_3 + x_4 \geq 6\frac{6}{11}$$

- As x is integer the lhs must be integer, therefore we can round up:

$$2x_1 + 2x_2 + x_3 + x_4 \geq 7$$

Valid inequalities for Linear Programs

- When is the inequality $\pi x \leq \pi_0$ valid for $P = \{x : Ax \leq b, x \geq 0\}$?
- **Proposition:** $\pi x \leq \pi_0$ is valid for $P = \{x : Ax \leq b, x \geq 0\} \neq \emptyset$ if and only if
  - there exists $u, v \geq 0$ such that $uA - v = \pi$ and $ub \leq \pi_0$, or
  - there exists $u \geq 0$ such that $uA \geq \pi$ and $ub \leq \pi_0$.

A priori addition of constraints

- **Idea:** Examine the initial formulation $P = \{x : Ax \leq b, x \geq 0\}$ with $X = P \cap Z^n$. Find a set of valid inequalities $Qx \leq q$ for X, add them to the formulation giving a new formulation P'.
- Solve the new formulation eg. by LP-relaxation and branch-and-bound.

- **Advantages:** Well chosen valid inequalities will reduce the formulation significantly, bounds will be improved and branch-and-bound *should* be more effective.
- **Disadvantages:** The family of inequalities we would like to add may be enourmous (exponential in size), so that the LP will become big (maybe to big to handle) and take long to solve (and we are not sure of getting the IP optimum!).

Just to give you an idea

- For a TSP of size 16
  - The number of subtour elimination constraints is 16,368.
  - The number of comb inequalities is 1,933,711,339,620.
  - These are not the only families of valid inequalities for TSP.
- For a TSP of size 120
  - The number of subtour elimination constraints is 0.6×10^{36}!
  - The number of comb inequalities is 2^{179}.

Decomposition

- Decomposition may be used when considering valid inequalities a priori.
- In many situations the feasible region X can be written as the intersection of two or more set with more structure, eg. $X = X_1 \cap X_2$. This is called a **decomposition**.
- How do we find such a decomposition? Be creative!!

The cutting plane algorithm

- Suppose $X = P \cap Z^n$ and suppose that we know a family F of valid inequalities $\pi x \leq \pi_0$, $(\pi, \pi_0) \in F$.
- Given a specific objective function one is not really interested in finding the complete convex hull, but only an approximation in the area around the optimal solution.

$t = 0, P^0 = P, f = 0$

repeat

solve $\bar{z}^t = \max\{cx : x \in P^t\}$ (x^t is an opt solution)

if $x^t \notin Z^n$ **then**

Solve the separation problem between x^t and F

(find inequality $(\pi^t, \pi_0^t) \in F$ so that $\pi^t x^t > \pi_0^t$)

if no inequality is found **then** $f = 1$

$P^{t+1} = P^t \cap \{x : \pi^t x \leq \pi_0\}$

until $x^t \in Z^n \vee f = 1$

if $f = 0$ **then** x^t optimal solution for IP

else use branch-and-bound to obtain IP