IPs with Total Unimodular matrices

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- **Definition**: The **Separation problem** for a COP $\max_{x \in X} c(x)$ is: Given an $x^* \in R^n$: Is x^* in conv(X)?
  - yes : ok
  - no : find an inequality $\pi x \leq \pi_0$ s.t. $\forall x \in X$ *but* $\pi x^* \leq \pi_0$

“Effective methods”

4 properties often seen together for $\max\{cx : x \in X \subset R^n\}$

1. Efficient Optimization Property
2. Strong Dual Property
3. Efficient Separation Property
4. Explicit Convex Hull Property

Total Unimodularity

Consider

$$\max\{cx : Ax \leq b, x \in Z^n_+\}$$

When is an optimal integer solution to the LP-relaxation (LP) guaranteed?

From LP theory we have $x_B, x_N = B^{-1}b, 0$, where B is an $m \times n$ non-singular submatrix of (A, I) and I is a square identity matrix of size m.

Sufficient condition: If the optimal basis B has det$(B) = \pm 1$, then the LP-relaxation solves the IP.

Definition: A matrix A is **totally unimodular** if every square submatrix of A has determinant $+1, -1, 0$.

Quick observation: If A is TU, $a_{ij} \in \{+1, -1, 0\}$ for all i, j.

Proposition:

A is TU

$\Updownarrow$

A^T is TU

$\Updownarrow$

(A, I) is TU.

Proposition: A is TU if,

1. $a_{ij} \in \{0, 1, -1\}$ for all i, j.
2. each column contains at most two non-zeros.
3. A partition of the rows exists, $M_1 \cup M_2 = M, M_1 \cap M_2 = \emptyset$ s.t.
    - if column j has two nonzeros of different sign these both belong to either M_1 or M_2.
    - if column j has two nonzeros of same sign these belong to one each of M_1 and M_2.

If the IP $\max\{cx : Ax \leq b, x \in Z_+^n\}$ has an A that is TU then we have

- Strong Dual Property
- Explicit Convex Hull Property
- Efficient Separation Property