HAMILTONICITY IN SQUARES OF GRAPHS REVISITED

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Given $G = V \cup E$, the $k$-th power of $G$, $G^k$, is defined by

$$V(G^k) = V, \quad E(G^k) = \{xy : d_G(x, y) \leq k\}$$

**Theorem** (M. Sekanina, 1960). $G^3$ is hamiltonian for every connected graph $G$.

P. Rosenstiehl (1971) produced algorithmic proof yielding hamiltonian cycle in $O(|V|)$ steps.
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In proving Sekanina’s result, it suffices to consider trees. Unfortunately, $S(K_{1,3})^2$ is not hamiltonian, but we have

**Theorem (F. Neumann, 1964).** Let $T$ be a tree on at least 3 vertices. $T^2$ is hamiltonian iff $T$ is a caterpillar. (i.e., every vertex lies on a largest path or is adjacent to such vertex).
Plummer and Nash-Williams (plus possibly some other colleagues) conjectured: 
\[ G = V \cup E \text{ nonseparable, } |V(G)| \geq 3, \text{ then } G^2 \text{ is hamiltonian.} \]


Chartrand, Hobbs, Jung, Kapoor, Nash-Williams: \( G^2 \text{ is even hamiltonian connected} \) (1974).
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Which graphs have a hamiltonian square?

**Theorem** (P. Underground, 1978). The problem of deciding which graphs have a hamiltonian square is tantamount to the problem of deciding which graphs are hamiltonian. Thus the problem is NP-complete.

Original proof of F.H.’s Theorem relies on the existence of *EPS-graphs* in conn. bridgeless gr’s.
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Def. An EPS-graph of a conn. Graph $G$ is a conn. spanning subgraph $S = E \cup P$ such that $E$.....(not necessarily connected) eulerian graph; $P$.....linear forest (=every component a path); $E$ and $P$ are edge-disjoint.

EPS-graphs essential for constructing hamiltonian cycles in $DT$-graphs (every edge is incident to a vertex of degree two).

In particular,
Theorem (F.H. + A.M. Hobbs, 1975). Total graph $T(G)$, $G \neq K_1$, is hamiltonian iff $G$ has an EPS-graph. - Note $T(G) = S(G)^2$.

Theory of EPS-graphs yields a description of the most general block-cutpoint structure s.t. every graph satisfying this structure has a hamiltonian $T(G)$. If $G$ doesn’t satisfy this block-cutpoint structure, then there is a $G'$ with same block-cutpoint structure, s.t. $T(G')$ is not hamiltonian.
For this and F.H.’s Theorem (but also for pancyclicity and panconnectedness) special types of $EPS$-graphs needed: $[v,w]\text{-}EPS$-graph, $[v,w,w']\text{-}EPS$-graph, $[w_1,\ldots,w_5]\text{-}EPS$-graph, etc, but also $JEPS$-graphs, where $J$ is an open trail, $E$ and $P$ as above and $J \cap E = \emptyset$, $J \cap P \subseteq V(G)$. These various types of $EPS$-graphs, $JEPS$-graph, used to construct a hamiltonian cycle/path in square of $DT$-graphs.

----- Why $DT$-graphs? -----
2-connected DT-graphs are contained in a very special way in edge-critical blocks.

However, determining the block-cutpoint structure for hamiltonian connectedness (or just hamiltonicity) in graphs remained an open problem (possibly in view of P. Underground’s result). However, shorter proofs of F.H.’s theorem found by Riha (1991) and Georgakopoulos (2009). Muettel and Rautenbach (2013): Short proof of a more general result (see below).
Gek Ling Chia conjectured that given a block $G$ and $x, y, u, v \in V(G)$, there is a $HP(x,y)$ in $G^2$ which contains at least one edge of $G$ at $u$ and at least one edge of $G$ at $v$ (best possible if true). – Conjecture recently proved by Chia and H.F.: many cases to be considered w.r.t. DT-graphs and then prove general case (edge-critical blocks). There is also a best possible result re. hamiltonian cycles in blocks.
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**Theorem** (H.F., 1976). $G$ 2-conn., $v, w \in V(G)$ arbitrarily, there exists a ham. cycle $H$ in $G^2$ whose edges in $v$ belong to $G$ and at least one of its edges in $w$ belongs to $G$. -- These last two theorems serve as the basis for describing the most general block-cutpoint structure for a graph to have a hamiltonian/hamiltonian connected square.

From here one can devise
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a polytime algorithm for finding a hamiltonian path/cycle in the square of the corresponding graphs (H-T Lau developed a polytime algorithm for finding a hamiltonian cycle in the square of a block; 1980).

THANK YOU FOR YOUR ATTENTION!