Oriented incidence colouring of digraphs

André Raspaud

(Joint work with Chris Duffy, Gary MacGillivray, Pascal Ochem)

LaBRI
Université de Bordeaux
France

GT2015
August 23-28, 2015
Nyborg, Denmark
Incidence coloring

An incidence of an undirected graph $G$ is a pair $(v, e)$ where $v$ is a vertex of $G$ and $e$ an edge of $G$ incident with $v$. Two incidences $(v, e)$ and $(w, f)$ are adjacent if one of the following holds:

- $v = w$,
- $e = f$,
- $vw = e$ or $f$. 

Incidence coloring

The set of all incidences in $G$ is denoted by $I(G)$.

A $k$-incidence coloring of a graph $G$ is a mapping $\phi$ from $I(G)$ into a set of colors $C = \{1, 2, \ldots, k\}$, such that adjacent incidence are assigned with distinct colors.
The set of all incidences in $G$ is denoted by $I(G)$.

A $k$-incidence coloring of a graph $G$ is a mapping $\phi$ from $I(G)$ into a set of colors $C = \{1, 2, \ldots, k\}$, such that adjacent incidence are assigned with distinct colors.

The minimum cardinality $k$ for which $G$ has a $k$-incidence coloring is the incidence chromatic number $\chi_i(G)$ of $G$. 

Incidence coloring
Incidence coloring

The notion of incidence colouring was introduced by Brualdi and Massey in 1993.

Theorem (Brualdi and Massey, 1993)

\[ \chi_i(K_n) = n, \quad n \geq 2 \]

For every graph \( G \),

\[ \Delta(G) + 1 \leq \chi_i(G) \leq 2\Delta(G) \]

Theorem (Guiduli, 1997)

For every graph \( G \),

\[ \chi_i(G) \leq \Delta(G) + 20 \log \Delta(G) + 84 \]
The notion of incidence colouring was introduced by Brualdi and Massey in 1993.

**Theorem (Brualdi and Massey, 1993)**

- $\chi_i(K_n) = n$, $n \geq 2$
- For every graph $G$, $\Delta(G) + 1 \leq \chi_i(G) \leq 2\Delta(G)$.

**Theorem (Guiduli, 1997)**

For every graph $G$,

$$\chi_i(G) \leq \Delta(G) + 20 \log \Delta(G) + 84.$$
For every arc $uv$ in a digraph $G$, we define two incidences:

- the *tail incidence* of $uv$ is the ordered pair $(uv, u)$
- the *head incidence* of $uv$ is the ordered pair $(uv, v)$
Oriented Incidence of digraphs

Two distinct incidences in a digraph $G$ are *adjacent* if and only if they correspond to one the following four cases:

- For every arc $uv$,
  - (1) the incidences $(uv, u)$ and $(uv, v)$ are adjacent.
- For every two related arcs $uv$ and $vw$,
  - (2) the incidences $(uv, v)$ and $(vw, v)$ are adjacent,
  - (3) the incidences $(uv, u)$ and $(vw, v)$ are adjacent,
  - (4) the incidences $(uv, v)$ and $(vw, w)$ are adjacent.
Oriented Incidence coloring of digraphs
Oriented Incidence colouring of digraphs

Let $I_G$ be the simple graph such that every vertex corresponds to an incidence of $G$ and every edge corresponds to two adjacent incidences.

An oriented incidence colouring of $G$ assigns a colour to every incidence of $G$ such that adjacent incidences receive different colours.

An oriented incidence colouring of $G$ is thus a proper vertex colouring of $I_G$.

For a digraph $G$, we define the oriented incidence chromatic number $\overrightarrow{\chi}_i(G)$ as the least $k$ such that $G$ has an oriented incidence $k$-colouring.
Observation

If $G$ has an orientation $\vec{G}$ then

$$\vec{\chi}_i(\vec{G}) \leq \chi_i(G)$$

Theorem (Brualdi and Massey)

For all $m \geq n \geq 2$ $\chi_i(K_{m,n}) = m + 2$
Observation

If $G$ has an orientation $\vec{G}$ then

$$\chi^i(\vec{G}) \leq \chi_i(G)$$

Theorem (Brualdi and Massey)

For all $m \geq n \geq 2$  $\chi_i(K_{m,n}) = m + 2$

Bipartite Tournament

$$\chi^i(T_{n,m}) = 4$$
Homomorphism

Let $G$ and $H$ be two digraphs a homomorphism is a mapping $f : V(G) \to V(H)$ such that $uv \in A(G)$ implies $f(u)f(v) \in A(H)$.

$$f : G \to H$$

Theorem

If $G$ and $H$ are digraphs such that $G \to H$, then

$$\overrightarrow{\chi_i}(G) \leq \overrightarrow{\chi_i}(H)$$
Oriented Incidence colouring and homomorphism

Oriented chromatic number

If $G$ is an oriented graph we denote $\chi_o(G)$ the oriented chromatic number of $G$. It is the minimum size of a tournament $T$ such that $G \rightarrow T$

Proposition

If $G$ is an oriented graph, then $\chi\hat{i}(G) \leq \chi_o(G)$.
Oriented Incidence colouring and homomorphism

Oriented chromatic number

If $G$ is an oriented graph we denote $\chi_o(G)$ the oriented chromatic number of $G$. It is the minimum size of a tournament $T$ such that $G \rightarrow T$.

Proposition

If $G$ is an oriented graph, then $\overrightarrow{\chi_i}(G) \leq \chi_o(G)$.

If $\chi_o(G) = k$ then

$G \rightarrow T_k$

$\overrightarrow{\chi_i}(T_k) \leq k$
Oriented Incidence colouring and homomorphism

Observation

If $G$ is an oriented bipartite graph: $\overrightarrow{\chi_i}(G) \leq 4$

$G \rightarrow \overrightarrow{K}_2$

Figure: \( \overrightarrow{K}_2 \)
Oriented Incidence colouring and homomorphism

Observation

If $G$ is an oriented bipartite graph: $\chi_i(G) \leq 4$

Figure: $\overrightarrow{K}_2$

Observation

For any integer $n$, it exists a bipartite graph $G$ such that $\chi_o(G) \geq n$. 
Oriented Incidence colouring and homomorphism

**Proposition**

If $G$ is an oriented forest, then $\vec{\chi}_i(G) \leq 3$.

The complete digraph $\vec{K}_k$ is obtained by replacing every edge $xy$ of the complete graph $K_k$ by the arcs $xy$ and $yx$.

**Proposition**

Let $\vec{G}$ be a digraph and $G$ be the underlying simple graph of $\vec{G}$. Then $\vec{\chi}_i(\vec{G}) \leq \vec{\chi}_i(\vec{K}_{\chi(G)})$. 
Symmetric complete digraphs

The complete digraph $\overrightarrow{K}_k$ is obtained by replacing every edge $xy$ of the complete graph $K_k$ by the arcs $xy$ and $yx$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^i (\overrightarrow{K}_n)$</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Table: Oriented incidence chromatic number of some symmetric complete digraphs
Symmetric complete digraphs

**Theorem**

If $k$ and $n$ are integers such that $n > \binom{k}{\lfloor k/2 \rfloor}$, then $\overrightarrow{\chi_i}(\overrightarrow{K_n}) > k$.

The Johnson graph $J(r, s)$ is the simple graph whose vertices are the $s$-element subsets of an $r$-element set and such that two vertices are adjacent if and only if their intersection has $s - 1$ elements.

**Theorem**

If $k$ and $n$ are integers such that $n \leq A(k, 4, \lfloor k/2 \rfloor)$, then $\overrightarrow{\chi_i}(\overrightarrow{K_n}) \leq k$.

$A(r, 4, s)$ is the independence number of the Johnson graph $J(r, s)$. 
Corollary

If \( n \geq 8 \), then
\[
\log_2(n) + \frac{1}{2} \log_2(\log_2(n)) \leq \overrightarrow{\chi_i}(\overrightarrow{K_n}) \leq \log_2(n) + \frac{3}{2} \log_2(\log_2(n)) + 2.
\]

Corollary

If \( G \) is a digraph then \( \overrightarrow{\chi_i}(G) \leq (1 + o(1)) \log_2(\chi(G)) \).
Observation

Let $G$ be a digraph with at least one arc, then $\overrightarrow{\chi_i}(G) = 2$ if and only if $G$ admits a homomorphism to $\overrightarrow{P_2}$. 
Theorem

Let $G$ be a digraph, then $\overrightarrow{\chi_i}(G) \leq 3$ if and only if $G$ admits a homomorphism to $H_5$.

Figure: The tournament $H_5$.
Two questions

• We have an oriented graph G so that an oriented graph admits a homomorphism to G if and only if it has an oriented chromatic number at most 3.
Is it possible to find a graph $G_k$ for any $k$ when $k \geq 4$, such that an oriented graph admits a homomorphism to $G_k$ if and only if it has an oriented incidence chromatic number at most $k$?
Two questions

- We have an oriented graph $G$ so that an oriented graph admits a homomorphism to $G$ if and only if it has an oriented chromatic number at most 3. Is it possible to find a graph $G_k$ for any $k$ when $k \geq 4$, such that an oriented graph admits a homomorphism to $G_k$ if and only if it has an oriented incidence chromatic number at most $k$?

- By the 4CT, the incidence oriented chromatic number of planar digraphs is at most 5. What is the incidence oriented chromatic number of planar oriented graphs? 4 or 5?