Exact solution of the Erdős-Sós conjecture

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János Komlós,
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Alfréd Rényi Math Inst Budapest

Nyborg, 2015

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Starting right in the middle

N1: G_n, P_k, T_k.

Theorem (Ajtai Komlós Simonovits Szemerédi)

There exists a k_0 such that for $k > k_0$, for any tree T_k, if

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then $T_k \hookrightarrow G_n$.

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The general question:

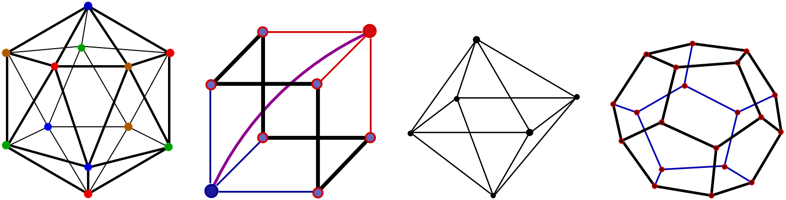
Given a sample graph L, how many edges can G_n have, without containing L.

N2: $\mathbf{ex}(n, L)$, = maximum number of edge ...
$\mathbf{EX}(\mathbf{n}, \mathbf{L})$. The family of **Extremal Graphs** = G_n attaining the maximum

Turán's questions

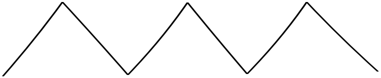
- Turán was motivated (basically) by **Ramsey**'s theorem
- Turán asked the extremal number for various excluded

subgraphs: cube, icosahedron, octahedron, dodecahedron,



For us the important case is:

path P_k.



and trees T_k

Erdős-Sós conjecture and its motivation

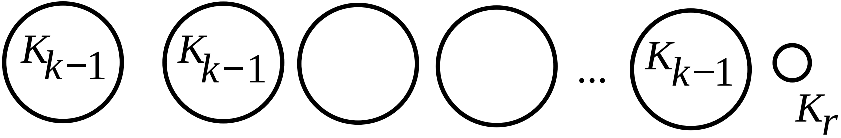


Figure : $Z_{n,k}$

Theorem (Erdős-Gallai)

$$\mathbf{ex}(n, P_k) \leq \frac{1}{2}(k-2)n.$$

The extremal graph is $Z_{n,k}$. (!)

If S_k is the star, then, trivially,

$$\mathbf{ex}(n, S_k) \leq \frac{1}{2}(k-2)n.$$

Erdős-Sós conjecture

For any T_k,

$$\mathbf{ex}(n,\ T_k) \leq \frac{1}{2}(k-2)n.$$

In other words,

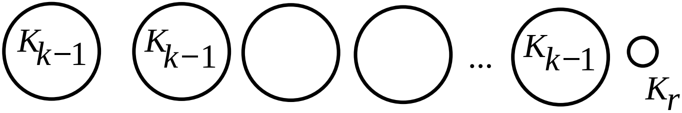
If

$$e(G_n) > \frac{1}{2}(k-2)n,$$

then G_n contains each k-vertex tree.

Easy:

$$\mathbf{ex}(n,\ T_k) \leq (k-2)n.$$



The conjecture/ other ≈-extremal structure

For any fixed tree T_k,

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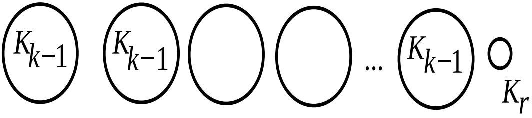
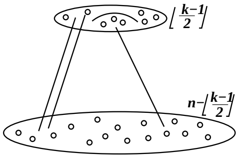


$Z_{n,k}$



$W_{n,\kappa}$

Motivation

Claim (Folklore)

If $d_{\min}(G_n) \geq k-1$, then $T_k \hookrightarrow G_n$, for every tree T_k.

Greedy embedding

- True for stars S_k: Trivial
- True for paths P_k: Erdős-Gallai.

- It would be trivial, if G_n were regular!

What is the difficulty?

That the vertices of G_n may have (very) different degrees, and embedding T_k step by step, we may arrive at a vertex $\mathbf{g} \in T_k$ having large degree, and when we try to put it down into $\mathbf{x} \in G_n$, all its neighbours are already used up.

Some known cases

Sidorenko, if there is an $x \in V(G_n)$ with $n/2$ leaves

Dobson, it the girth is "large"

Brandt-Dobson

Wozniak

Theorem (McLennan)

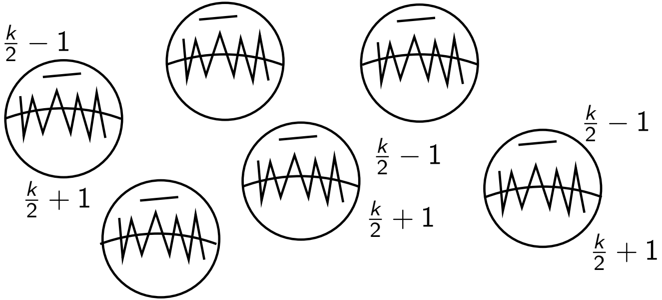
If **diameter**$(T_k) \leq 4$, *then ES Conjecture holds.*

⋮

The Loebl-Komlós-Sós conjecture

Conjecture (Loebl–Komlós–Sós Conjecture 1995)

Suppose that G is an n-vertex graph with at least $n/2$ vertices of degree more than $k-2$. Then G contains each tree of order k.



Motivation?

Erdős-Füredi-Loebl-Sós: Uniform distribution for graphs
Ramsey for moncohromatic trees
They needed the simplest form of this conjecture:
The Loebl Conjecture (i.e. $n = k$).
Komlós and Sós generalized the Loebl conjecture.
For paths there were already several similar results:
Woodall
Erdős-Faudree-Schelp-Simonovits results on the **Ramsey** numbers of a fixed graph versus a large tree.
Hao Li ...

What happend?

- Ajtai-Komlós-Szemerédi:
Proof of the Approximative weakening of the Loebl Conjecture.
- Yi Zhao: Exact solution for large k.
- Piguet-Stein / Oliver Cooley: a big step forward.
- Piguet-Hladký

Details?

Conjecture (Weaker, approximate version)

If at least $\frac{1}{2}(1+\eta)n$ vertices of G_n have degree at least $(1+\eta)k$, then $T_k \hookrightarrow G_n$.

- Ajtai-Komlós-Szemerédi
- Yi Zhao
- Piguet-Stein / Cooley

Theorem (Hladky-Komlós-Piguet-Simonovits-Stein-Szemerédi)

The Komlós-Sós Conjecture holds for $k > k_0$.

Arxiv (>160pp) + Short description

+ three out of four papers accepted for publications

Why is this problem difficult? II

Uniqueness of extremal graphs

Those problems are easy, where there is a main property of the (conjectured) extremal graphs "governing" the proof.
Here there are two (almost) extremal graphs, of completely different structures.

- Many graphs G_n
- many different trees T_k

Plan?

Is it easy for generalized random graphs?

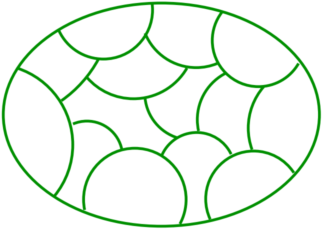
If YES, then Regularity Lemma may help.

- What is a **Generalized Random Graph**?
- What is the **Regularity Lemma**
- Why and when does the Regularity Lemma help?
- Does it help NOW?

What is a Generalized Random graph?

A matrix $A = (p_{ij})_{r \times r}$ of probabilities is given.

We divide n vertices into r classes U_i and join each $x \in U_i$ to $y \in U_j$ independently, with probability p_{ij}



$$d(X, Y) := \frac{e(X, Y)}{|X||Y|}$$

Definition (ε-regular pair (A, B) in G_n)

... if whenever $X \subseteq A$ and $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$, then

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

Important "test": Generalized Random Graphs

If we can solve an extremal graph problem "easily" for Generalized Random Graphs, then we probably can also solve it for any dense graphs sequence.

What is the Regularity Lemma?

Informally: Each graph can be approximated

by generalized random graphs / generalized quasi-random graphs

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For every $\varepsilon > 0$ and ν_0 there exists a $\nu_1(\varepsilon, \nu_0)$ such that for every G_n, $V(G_n)$ can be partitioned into ν sets $U_1, \dots, U_\nu$, for some $\nu_0 < \nu < \nu_1(\varepsilon, \nu_0)$, so that $||U_i| - |U_j|| \le 1$ for every $i, j > 0$, and $U_i U_j$ is ε–regular for all but at most $\varepsilon\binom{\nu}{2}$ pairs (i, j).

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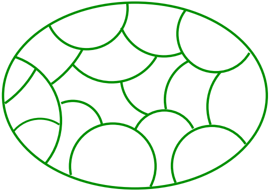

Cluster graph

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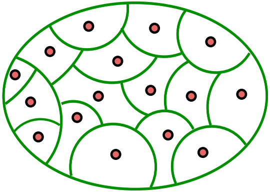

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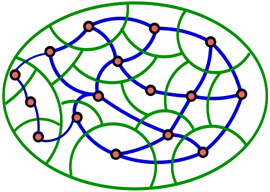

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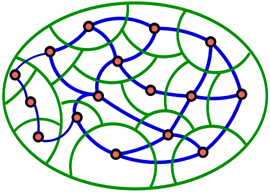
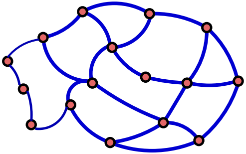


Cluster graph



Why and when does the Regularity Lemma help?

Basically, if

(a) (G_n) is a dense sequence: $e(G_n) > cn^2$.

(b) for the dense generalized random graph we can easily solve the problem.

However, the Tree problem is degenerate: the extremal graphs are not dense...

Does it help NOW?

YES and NO.

Our very simplified plan is:

- First we make the problem dense and solve only the approximate version:

Assuming that $n \leq \Omega k$ makes the considered graphs dense.

Adding ηkn edges create the approximate version.

Theorem (Approximate version)

There exists a k_0 such that for $k > k_0$, for any tree T_k, if

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Approximative weakening

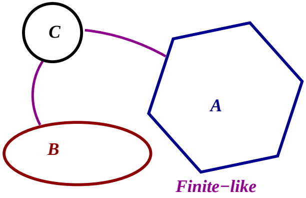
then $T_k \hookrightarrow G_n$.

So what is the plan?

- First we prove the approximate version:
  – First we get rid of the individual structure of T_k by a **slicing method**.
  – Next we get rid of the individual structure of G_n by using the **Szemerédi Regularity Lemma**

- We analyze the proof and gain or get structural information: **Using the stability method** we get the sharp theorem in the dense case.

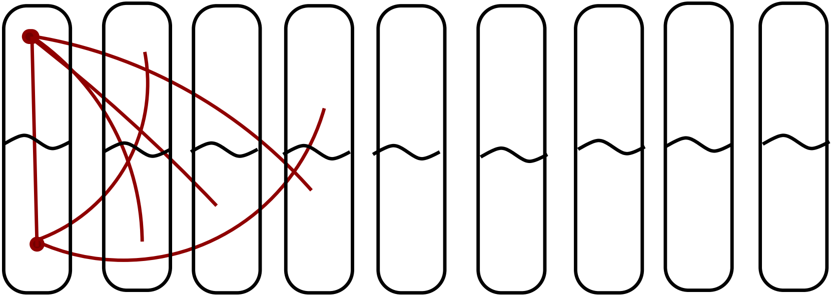
- To take care of the Sparse Case we partition $V(G_n)$ into three parts: $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ and show that only the case $V(G_n) = \mathbb{A}$ matters. There we can apply the methods used for the sparse case.



When would this proof be easy, using Regularity Lemma?

- If we had a 1-factor, or an almost-1-factor in the Reduced graph H_ν.

- Then the LKS Conjecture also would be easy, at least for the dense case.

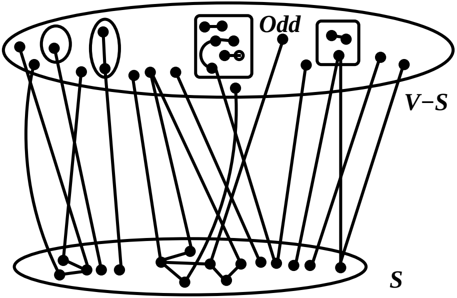


Cheating?

There are two extremal structures,

and the 1-factor case covers only one of them, the other is described by the

- The other is covered by a deeper analysis: **Gallai-Edmonds thm**
- Several specific embedding algorithms



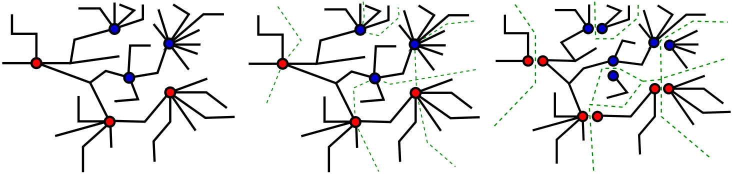
Where to read about it?

- The main part is under writing up, 3 very long papers
- On the Loebl-Komlós-Sós conjecture:

Arxiv: Hladký-Komlós-Piguet-Simonovits-Stein-Szemerédi

Slicing the tree T_k

We fix a very small ϑ, and cut T_k into subtrees of size smaller than ϑk.



The embedding of T_k becomes a special 2-coloured bin-packing problem: this way we can get rid of the special structure of T_k.

How to apply the Regularity Lemma?

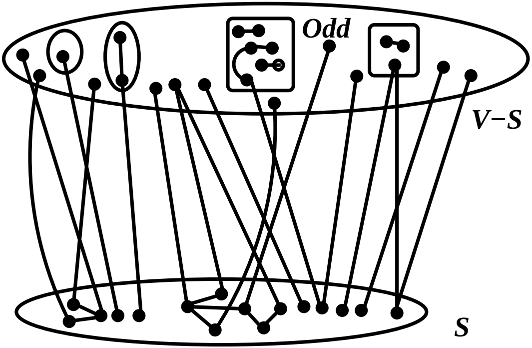
- First we assume that n is not too large: $n \leq \Omega k$.
- Build an auxilary graph, called **Reduced graph** H_ν.
- If H_ν has a 1-factor, we embed T_k into G_n.
- If H_ν has a generalized 1-factor, again $T_k \hookrightarrow G_n$.
- If H_ν does not have a 1-factor, we apply the Gallai-Edmonds Decomposition to H_ν and with the help of this $T_k \hookrightarrow G_n$.

- Stability argument

- But if G_n is sparse (i.e.n is very large)?

Establish a generalization of the regularity lemma

Gallai-Edmonds structural theorem



We can delete an $\mathbb{S}$ so that the connected odd components of $\mathbb{V}-\mathbb{S}$ are factor-critical: either they are small or have an almost-1-factor and $\mathbb{S}$ is joined to them by a 1-factor. The even components have a 1-factor.

Why do we need the Stability Method?

- Partly, because we loose some edges, whenever we use the Regularity Lemma: To get exact results with the regularity lemma we always (?) need the stability method.

- Even when we do not loose edges, the stability method makes the proofs more transparent:

  – Dodecahedron theorem
  – Icosahedron theorem
  – Babai-Sim-Spencer

  – Fano hypergraph result (Füredi-Sim / Keevash-Sudakov)

How do we apply Stability

Via 6-7 Lemmas (?)

Two elementary cases:
• Very High Degree

No Stability as yet!

Lemma

If

$$d_{\max}(G_n) \geq k-1 \qquad \text{and} \qquad d_{\min}(G_n) \geq \frac{k-1}{2}$$

and

$$d_{\max}(T_k) \geq \frac{3}{4}k,$$

then $T_k \subseteq G_n$.

Very High Degree, Stability

Not that Stability!

Lemma

There exists a $\omega_s > 0$ *for which, if*

$$d_{\max}(G_n) \geq k-1 \qquad \textit{and} \qquad d_{\min}(G_n) \geq \frac{k-1}{2}$$

and

$$d_{\max}(T_k) \geq \left(\frac{3}{4} - \omega_s\right) k,$$

then $T_k \subseteq G_n$.

Two-center trees

Lemma

There exist a $\beta > 0$ and a k_0 such that if $k > k_0$ and

$$e(G_n) > \frac{1}{2}(k-2)n,$$

and the tree T_k has two vertices, $\mathbf{g}_1$ and $\mathbf{g}_2$, of high degrees:

$$d_T(\mathbf{g}_1) + d_T(\mathbf{g}_2) > k - h \qquad \text{for some} \qquad h \leq \beta k,$$

then $T_k \subseteq G_n$. Moreover, if $d_G(\mathbf{x}) \geq k-1$, then there is an embedding that maps $\mathbf{g}_1$ to $\mathbf{x}$.

Even distance Odd distance

Sketch

AKSSz: THE STRUCTURE OF THE PROOF, [ApproxD3] May 27, 2008 16

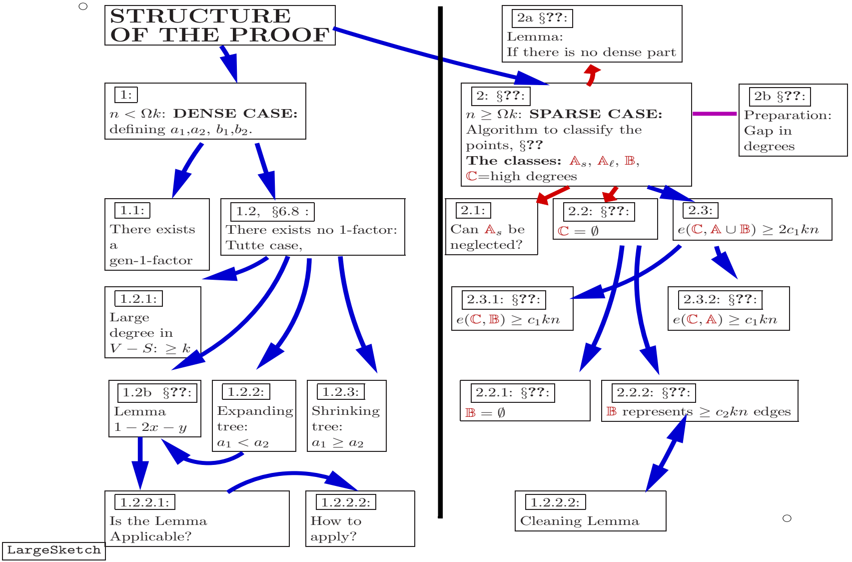


Figure 4: The structure of the proof. The actual proof follows a slightly different line.

How to prove the Erdős-Sós conjecture if we have its approximative version?

- Cut off some elementary cases,
- Analyze some general embedding situations.

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Clean preliminary results

High degree cases
1 high degree
2 high degrees

Small dense graphs: Blocks?

Sparse graphs
Pseudo−sparse graphs

Main result

Sharp form *If*

$$e(G_n) > \frac{1}{2}(k-2)n,$$

then for $k > k_0$, every k-vertex tree $T_k \subseteq G_n$.

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Approximate form *If*

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$\Longleftarrow$

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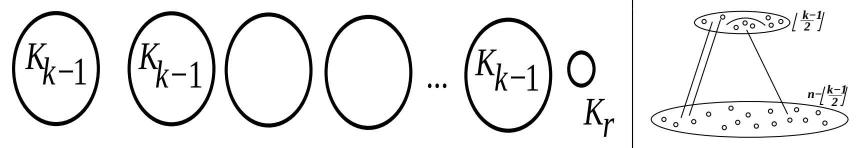
Conjectured graph sequences showing the (asymptotic) sharpness

Assuming that the conjecture holds, $Z_{n,k}$ is extremal if n is a multiple of $k-1$.



$Z_{n,k}$
(a) The extremal graphs

$W_{n,\kappa}$
(b) "bottleneck" graph

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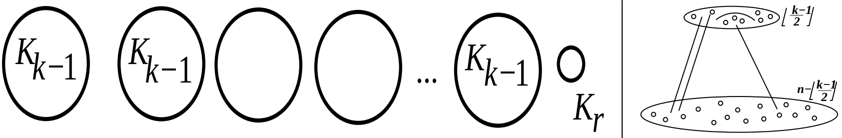


$Z_{n,k}$
(a) The extremal graphs

$W_{n,\kappa}$
(b) "bottleneck" graph

Difficulties come from

- Having many trees T_k,
- Having 2 extremal sequences.

Stability method

Theorem (Main Theorem, Approximative)

If $n, k > n_0(\eta)$ and for an arbitrarily fixed tree T_k, a graph G_n on n vertices contains no T_k, then

$$e(G_n) \leq \frac{1}{2}(k-2)n + \eta n.$$

- Analyze the special structure **when we really use $+\eta kn$.**
- Show that then we have a **very special structure.**
- Prove – using the special structure – the **Sharp Theorem.**

General cases, Stability

Apply the regularity lemma ($n < \Omega k$.) H_ν = cluster graph.

- There is a 1-factor in H_ν.
- There is a **Generalized 1_k-Factor** in H_ν.
- Shrinking Tutte
- Expanding Tutte

General cases, Stability

Tree-parameters: a_1, a_2, b_1, b_2

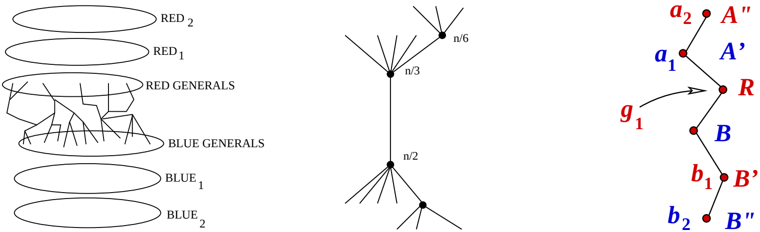
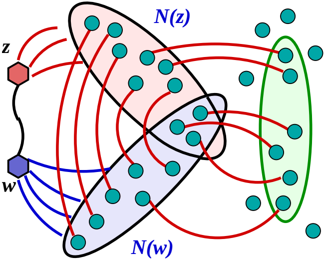


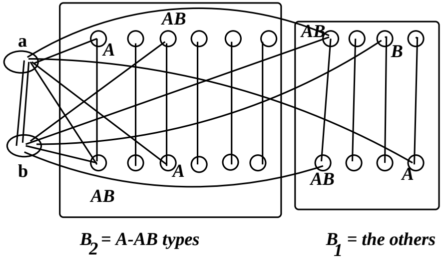
Figure : (a) The 4 parameters (b) High degree GENERALS (c) P_6

- Symmetry breaking: $a_1 + b_2 \leq \frac{1}{2}k$
- Shrinking: $a_2 < a_1$

Example for General cases, Stability

There is a **Generalized 1_k-Factor** in H_ν.





- We have a lot of cluster-edges in H_ν
- They are joined in 4 ways to the distinguished pair $(\mathbf{z}, \mathbf{w})$
- We define the Good and Bad parts.
- Fill in the **w**-neighbours
- Fill in the **z**-neighbours

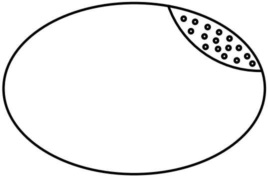
Dense Blocks: Almost complete graphs

Theorem

Fix $c^* = 10^{-10}$. *Let* T_k *be a* k*-vertex tree. If*

$$\ell \in [k-2, k+c^*k] \text{ and } e(G_\ell) > \frac{1}{2}(k-2)\ell$$

then $T_k \subset G_\ell$.



The graph itself is almost complete, with $\approx k$ vertices.

Graphs with Dense Blocks

Theorem (Almost complete blocks)

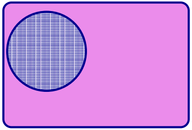
Fix $c^* = 10^{-10}$. Let T_k be a k-vertex tree. If $G_\ell \subseteq G_n$ for some $\ell \in [k-2, k+c^*k]$, G_n is connected, and

$$e(G_n) > \frac{1}{2}(k-2)n \qquad \text{and} \qquad e(G_\ell) > \frac{1}{2}(k-2)\ell - c^*k\ell,$$

then $T_k \subset G_n$, or there is a $G_m \subseteq G_n$ with

$$e(G_m) > \frac{1}{2}(k-2)m.$$

This means that the conjecture holds if G_n contains an almost complete block, with $\approx k$ vertices.



Dense Blocks, Broom-trees

second one?

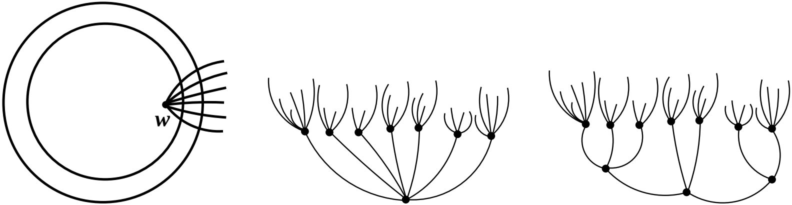
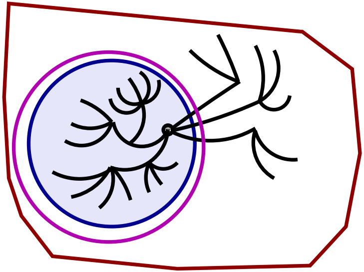


Figure : (a) Kernel (b) 2-level Broom-tree (c) Many-level broom-tree

Dense Blocks, Brooms, Basic idea



Outside we use a greedy algorithm; inside we use a pseudo-greedy embedding.

Kernel: Delete the low degrees from G_ℓ: H^*.

Extended Kernel: Add those ones sending $0.4k$ edges to H^*.

- The graph is basically cut into two parts: Extended Kernel and outside.
- The mindegree in the outside part is large.
- If many edges go out from the block, we build up a large part outside

Dense Blocks, Path-like-trees

When T_k has few ($< ck$) endvertices.

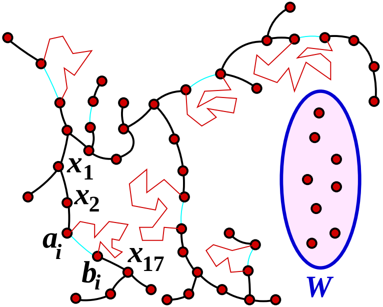


Figure : Shrinking and expanding the tree.

Using König-Hall

γ-sparse graphs

Theorem

Let $\gamma \leq 10^{-4}$. Assume that G_n does not contain γ-dense parts. There exists a constant $k_0(\gamma)$ for which, if $k > k_0(\gamma)$ and

$$d_{\max}(G_n) > d_{\max}(T_k) + 2\gamma k$$

and

$$d_{\min}(G_n) > d^*_{\max}(T_k) + 2\gamma k$$

then $T_k \subseteq G_n$. Moreover, if $d_{\min}(G_n) \geq \frac{k-1}{2}$, then the max-degree vertex $\mathbf{g}_1$ of T_k can be mapped onto any vertex of G_n of degree $\geq d_{\max}(T_k) + 2\gamma k$.

Pseudo-Sparse Graph Theorem

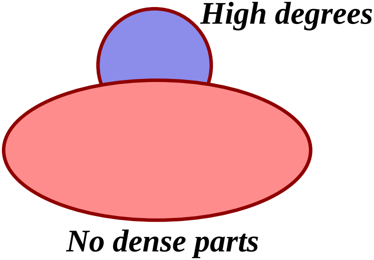
For any $\gamma < 10^{-4}$ *there exists a* $k_0(\gamma)$ *with the following property: Let* T_k *be any tree on* $k > k_0(\gamma)$ *vertices. Let* G_n *be a graph on* n *vertices with*

$$e(G_n) > \frac{1}{2}(k-2)n. \tag{1}$$

Assume that $V(G_n)$ *is partitioned into two classes* $\mathbb{C}$ *and* $\mathbb{B}$, *where all the vertices of* $\mathbb{C}$ *have degree* $> 100k$ *and all the vertices of* $\mathbb{B}$ *have degrees* $\geq \frac{1}{2}(k-2)$. *If* $G[\mathbb{B}]$, *i.e. the subgraph spanned by the vertices of* $\mathbb{B}$, *does not contain* γ*-dense parts, then* $T_k \subseteq G_n$.

Moreover, if $d_{\min}(G_n) \geq \frac{k-1}{2}$, *then the max-degree vertex* $\mathbf{g}_1$ *of* T_k *can be mapped onto any vertex* $\mathbf{x} \in V(G_n)$ *of degree* $d_G(\mathbf{x}) \geq d_{\max}(T_k)$ *and then one can extend this into an embedding* $T_k \hookrightarrow G_n$.

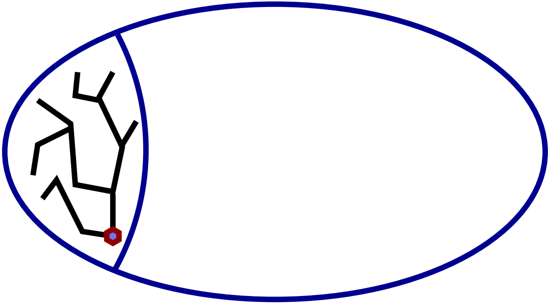
Pseudo-Sparse Graph Theorem (P)



Proof idea of the Sparse Graph Theorem

Since there are no dense pairs, when we have built up a $T_m \subseteq G_n$, most of the vertices send back to T_m only few edges.

We cut T_k into small subtrees. Embed them one by one.



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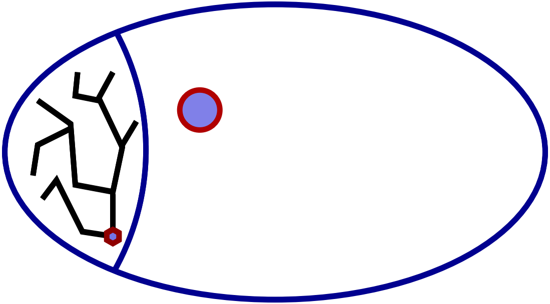

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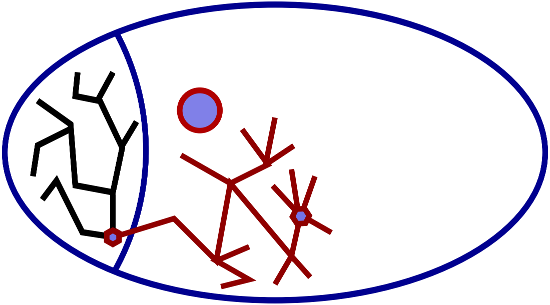

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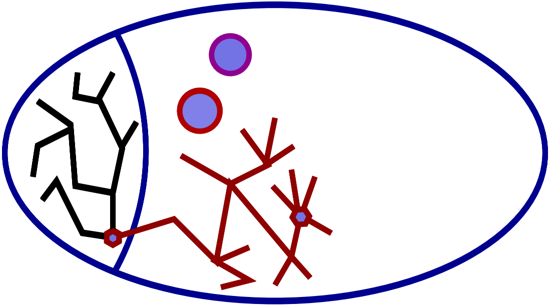

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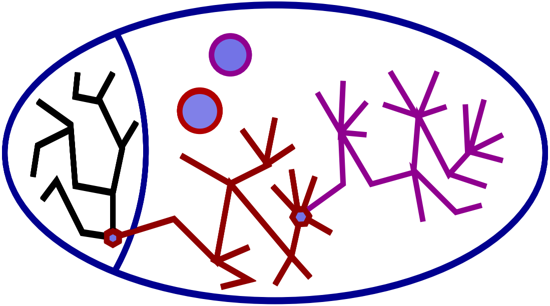


Generalizes results of Dobson and others, at least, for large k.

Further references

Szemerédi, Endre; Stein, Maya; Simonovits, Miklós; Piguet, Diana; Hladký, Jan; The approximate Loebl-Komlós-Sós conjecture and embedding trees in sparse graphs. Electron. Res. Announc. Math. Sci. 22 (2015), 1-11.

O. Cooley. Proof of the Loebl-Komlós-Sós conjecture for large dense graphs, preprint. cf. MR2551974

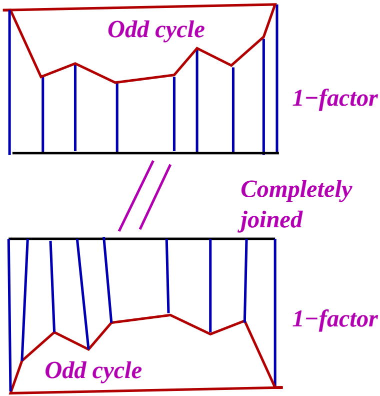
Piguet, Diana; Stein, Maya Jakobine; An approximate version of the Loebl-Komlós-Sós conjecture. J. Combin. Theory Ser. B 102 (2012), no. 1, 102-125.

Piguet, Diana; Stein, Maya Jakobine; The Loebl-Komlós-Sós conjecture for trees of diameter 5 and for certain caterpillars. Electron. J. Combin. 15 (2008), no. 1, Research Paper 106, 11 pp.

Happy Birthday, Bjarne

Bjarne and Bondy

The Toft graph



Erdős-Dirac: find a 4-colour-critical graph with many edges.

This led to interesting hypergraph extremal problems, solved by Toft/Simonovits and finally by Lovász.

Begining of the algebraic methods in extremal graph theory.

Happy Birthday, Bjarne