USING RANDOM WALKS TO DETECT AMENABILITY IN F

Murray Elder, Andrew Rechnitzer, Buks van Rensburg, Cameron Rogers

Odense, August 2016

EXPERIMENTAL WORK PREVIOUSLY DONE ON F

[4] Burillo, Cleary and Wiest 2007
The authors randomly choose words and reduce them to a normal form to test if they represent the identity element. From this they estimate the proportion of words of length n equal to the identity, as a way to compute the asymptotic growth rate of the cogrowth function.

[1] Arzhantseva, Guba, Lustig, and Préaux 2008
The authors study the *density* or least upper bound for the average vertex degree of any finite subgraph of the Cayley graph; an m-generated group is amenable if and only if the density of the corresponding Cayley graph is $2m$. They use a computer program to find a finite subset in F with density 2.89577. (To be amenable one would need to find sets whose density $\longrightarrow 4$).

EXPERIMENTAL WORK PREVIOUSLY DONE ON F

[6] Elder, Rechnitzer and Wong 2012
Lower bounds on the cogrowth rates of various groups are obtained by computing the dominant eigenvalue of the adjacency matrix of truncated Cayley graphs. These bounds are extrapolated to estimate the cogrowth rate. As a byproduct the first 22 coefficients of the cogrowth series are computed exactly.

[10] Haagerup, Haagerup, and Ramirez-Solano 2015
Lower bounds on the norms of the Markov operator derived from Kesten's condition are obtained for F using C^*-algebraic methods. Coefficients of the cogrowth series are computed exactly to 48 terms.

[5] Elder, Rechnitzer and van Rensburg 2015
The *Metropolis Monte Carlo* method from statistical mechanics is adapted to estimate the asymptotic growth rate of the cogrowth function by running random walks on the set of all trivial words in a group. The results obtained for Thompson's group F suggest it to be non-amenable.

Justin Moore [12] (2013) has shown that if F were amenable then its Følner function would increase faster than a tower of $n-1$ twos,

$$2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}$$

This has been proposed as an obstruction to all computational methods for approximating amenability; a computationally infeasibly large portion of the Cayley graph must be considered before sets with small boundaries can be found.

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However, in all but one of the experimental algorithms listed above computing Følner sets was not the principle aim. Exactly how the growth of the Følner function controls the convergence properties of the respective limits in the Grigorchuk-Cohen, Kesten, Reiter, ... characterisations is not clear.

COGROWTH

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- conjugate by a generator, reduce
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- *left-insert* a relator, reduce

Moves are accepted/rejected with a carefully chosen probability (depending on the relative change in length of the current state, and parameters α, β).

Let w be the current state (a reduced trivial word).

- If w' was obtained from w via a conjugation it is accepted as the new state with probability

$$\min\left\{1, \left(\frac{|w'|+1}{|w|+1}\right)^{1+\alpha} \beta^{|w'|-|w|}\right\}.$$

- If w' was obtained from w via an insertion it is accepted as the new state with probability

$$\min\left\{1, \left(\frac{|w'|+1}{|w|+1}\right)^{\alpha} \beta^{|w'|-|w|}\right\}.$$

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Note:

- random walk on trivial words, not the Cayley graph
- just need presentation, no efficient normal form or even solvable word problem

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i.e. the probability that the algorithm reaches state w after N steps converges to $\pi(w)$.

Using this we can test for amenability by estimating the location of the asymptotic growth rate for c_n, as follows.

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Expected length of a state in the walk is

$$E(|w|) = \sum |w|\pi(w) = \sum |w| \frac{(|w|+1)^{1+\alpha}\beta^{|w|}}{Z}$$
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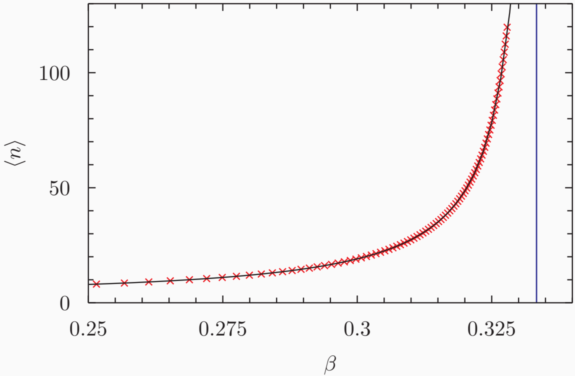
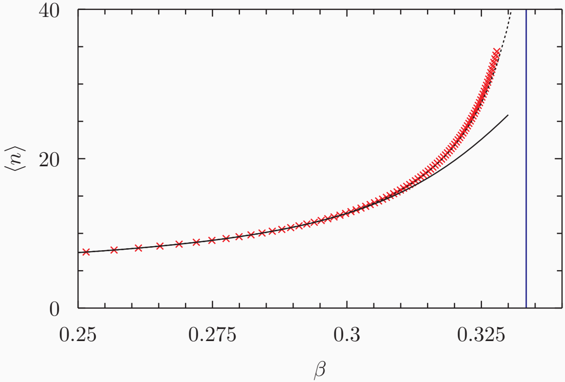
As $\beta \to$ reciprocal of the cogrowth rate, the mean length changes from finite to infinite.



Mean length in $\mathbb{Z}^2 = \langle a, b \mid bab^{-1}a^{-1} \rangle$ with $\alpha = 1$.

Reciprocal of $|S| - 1$ is $1/3$.



Mean length in $BS(1,2) = \langle a, b \mid bab^{-1}a^{-2} \rangle$ with $\alpha = 1$.

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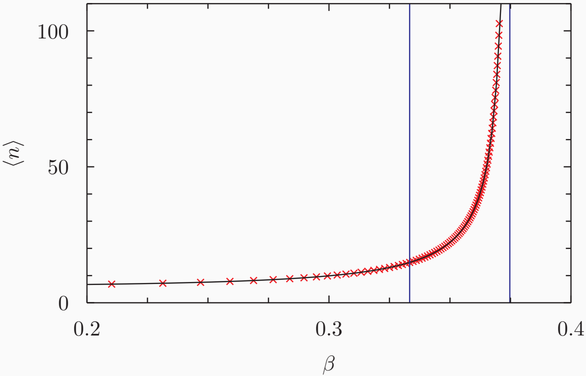

Mean length in $BS(2,2) = \langle a, b \mid ba^2b^{-1}a^{-2} \rangle$ with $\alpha = 1$.

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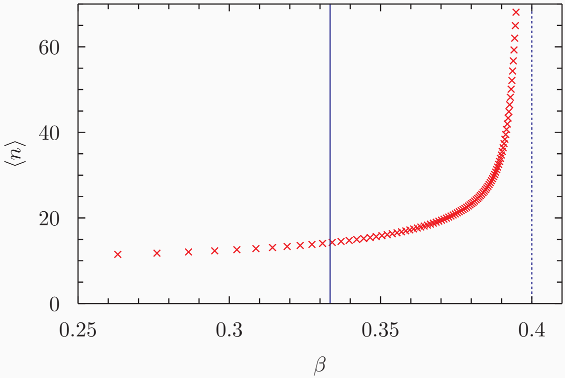

Mean length in $F = \langle a, b \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2] \rangle$ with $\alpha = 2$.

Reciprocal of $|S| - 1$ is $1/3$.

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Main objection to ERR:

Theorem (Moore [12])

If Thompson's group F is amenable it has a Følner function which grows like

$$2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}$$

Theorem (Følner)

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Vershik defined the following function as way of quantifying the rate of convergence of this limit.

$$\mathcal{F}(n) = min\left\{|F| \; : \; \frac{|\partial F|}{|F|} < \frac{1}{n}\right\}.$$

Cameron

- ran his code on more groups, and for larger α values
- collected extra data in addition to the mean length
- looked for a quantitative connection between the rate of convergence of ERR walks to their theoretical stationary distribution, and the Følner function.

Recall

- If w' was obtained from w via an insertion it is accepted as the new state with probability

$$\min\left\{1, \left(\frac{|w'|+1}{|w|+1}\right)^{\alpha} \beta^{|w'|-|w|}\right\}.$$

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$$\min\left\{1, \left(\frac{|w'|+1}{|w|+1}\right)^{\alpha} \beta^{|w'|-|w|}\right\}.$$

By varying α we can push the walk out to visit longer words.

PATHOLOGICAL EXAMPLE

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For large n the Metropolis rule will reject insertions of the long relator often. So you are essentially walking on the *wrong group*.

Recall: insert accepted with probability

$$\min\left\{1, \left(\frac{|w'|+1}{|w|+1}\right)^{\alpha} \beta^{|w'|-|w|}\right\}.$$

n	#steps	#short accepted	#long accepted
2	3.6×10^8	4420185	5579815
3	6.1×10^8	6323376	3676624
4	9.0×10^8	8016495	1983505
5	1.2×10^9	9088706	911294
6	1.4×10^9	9621402	378598
7	1.5×10^9	9850251	149749
8	1.7×10^9	9943619	56381
9	1.8×10^9	9977803	22197
10	1.9×10^9	9991680	8320
11	2.1×10^9	9997122	2878
12	2.2×10^9	9998720	1280
13	2.2×10^9	9999585	415
14	2.3×10^9	9999938	62
15	2.4×10^9	10000000	0
16	2.6×10^9	10000000	0
17	2.7×10^9	10000000	0

$$\langle a, b \mid aba = bab,\ a^n = b^{n+1} \rangle$$

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If $S = S^{-1}$ is a finite generating set, put $\mu(x) = \frac{1}{|S|}$ for each $x \in S$.

Then μ induces a random walk on G:

- start at $X_0 = e$;
- $X_{n+1} = X_n g$ where $g \in G$ is chosen with probability $\mu(g)$.

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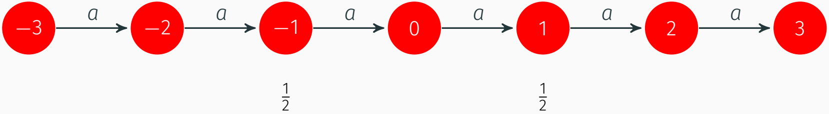

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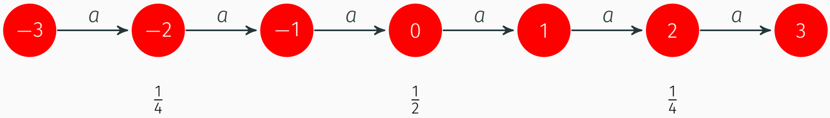

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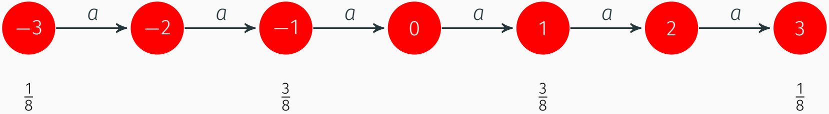

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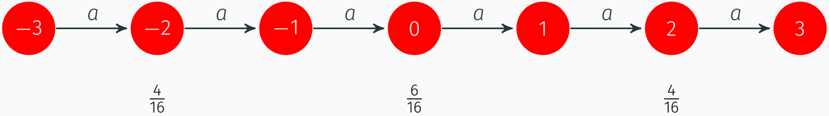

Recall $\sigma * \tau(g) = \sum_{h \in G} \sigma(h)\tau(h^{-1}g)$ is the *convolution* of two measures, so

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Note $\mu_n(e) = \frac{d_n}{|S|^n}$ when μ is the uniform measure on S.

Lemma

If μ is symmetric with $\text{supp}(\mu)$ generating G, then

- μ_{2n} *is maximised at* e
- $\mu_{2n}(e)$ *is non-increasing.*

Theorem (Avez [2])

G is amenable if and only if $\frac{\mu_{2n}(x)}{\mu_{2n}(e)} \longrightarrow 1$ *for all* $x \in G$.

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Definition (E, Rogers [8])

Let $p \in (0, 1)$. Define

$$A_{\mu,n,p} = \left\{ g \in G \mid \frac{\mu_{2n}(g)}{\mu_{2n}(e)} > p \right\}$$

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$A_{\mu,n,\frac{1}{2}}$ for $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$ with $\mu(a^{\pm 1}) = \mu(b^{\pm 1}) = \mu(e) = \frac{1}{5}$:

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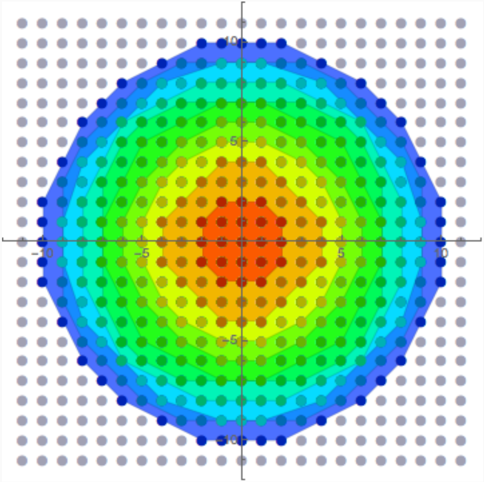

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Kamainovich and Vershik had shown a quantitative link the other direction:

Theorem (Kaimanovich, Vershik [11])

If F is a finite subset of G for which $|F \cup Fs \setminus F \cap Fs| < \epsilon$ for every $s \in S$ then for every $p < 1$

$$\mu_{2n}(e) \geq (1-p)^n \frac{1 - 2\epsilon/p^2}{|F|}.$$

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Conjecture

For any G, the set A_μ is the largest normal amenable subgroup of G

(the *amenable radical*).

Proofs use an alternative charaterisation of A_μ using:

Theorem (Reiter [13])

G is amenable iff for any finite subset K there is a sequence of unit vectors $f_n \in L^2(G)$ such that

$$\lim_{n \longrightarrow \infty} \|k \cdot f_n - f_n\|_2 = 0$$

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Where do we find with such a sequence of unit L^2 functions? How about $\mu_n(e)$?

Proposition

Let $\xi_n = \dfrac{\mu_n}{\|\mu_n\|_2}$, then $A_\mu = \left\{ g \in G \mid \lim_{n \longrightarrow \infty} \|g \cdot \xi_n - \xi_n\|_2 = 0 \right\}.$

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The following definition quantifies this sub-dominant behaviour in a way analogous to the Følner function.

Definition (E, Rogers [7])

Let $D = \limsup d_n^{1/n}$ (= $|S|$ when G is amenable). Define

$$\mathcal{R}(n) = \min \left\{ k \ : \ \frac{d_{2k+2}}{d_{2k}} > D^2 - \frac{1}{n} \right\}$$

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Example

For the trivial group with some finite symmetric generating set S we have $d_n = |S|^n$ so $\frac{d_{2k+2}}{d_{2k}} = |S|^2$, so $\mathcal{R}(n) = 0$.

Recall $d_n = \mu_n(e)|S|^n$ where μ is the uniform measure on S, so we can rephrase the set as

$$\mathcal{R}(n) = \min\left\{k \ : \ \frac{\mu_{2k+2}(e)}{\mu_{2k}(e)} > 1 - \frac{1}{|S|^2 n}\right\}.$$

Return probabilities have been computed for various groups, and from these we can derive asymptotic formulae for $\mathcal{R}(n)$:

G	$\mathcal{F}(n)$	$\mu_n(e)$	$\mathcal{R}(n)$
trivial, C_2	$\asymp$ constant	$\asymp$ constant	$\asymp$ constant
finite	$\asymp$ constant	$\asymp$ constant	$\asymp \ln n$
$\mathbb{Z}^k$	$\asymp n^k$	$\asymp n^{-k/2}$	$\asymp n$
$BS(1,N)$	$\asymp e^n$	$\asymp e^{-n^{1/3}}$	$\asymp n^{3/2}$
$\mathbb{Z} \wr \mathbb{Z}$	n^n	$\asymp \lvert S\rvert^{-n^{1/3}(\ln n)^{2/3}}$	$\asymp \ln(n)n^{3/2}$
$K \wr \mathbb{Z}$	$f(n)^n$	$\asymp e^{-n^{1/2}}$	$\asymp n^2$

K polycyclic with exponential growth, and Følner function $f(n)$.

BS$(1, n)$

The groups BS(1,N)= $\langle a, t \mid tat^{-1}a^{-N} \rangle$ are amenable groups.

$\mathrm{BS}(1, n)$

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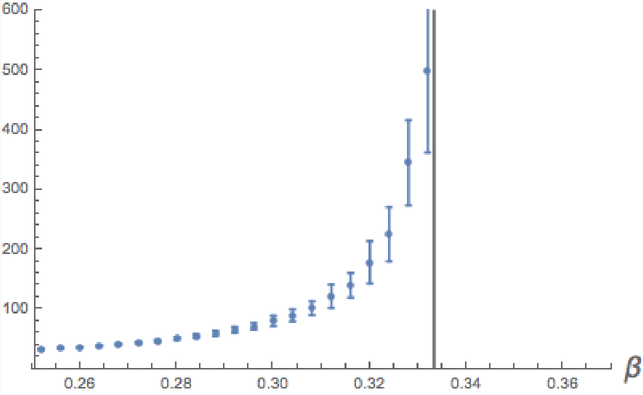
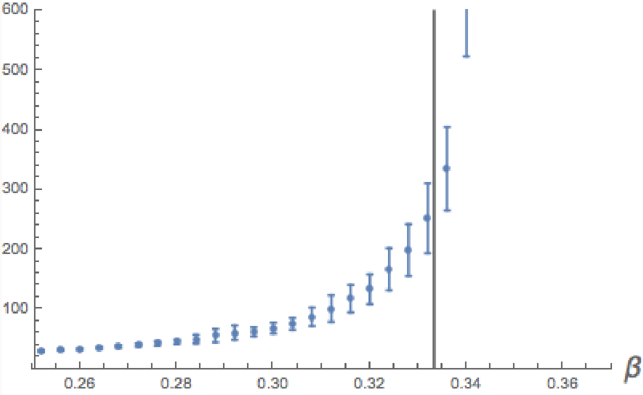
So how does the ERR algorithm perform for $BS(1, N)$ as N increases?

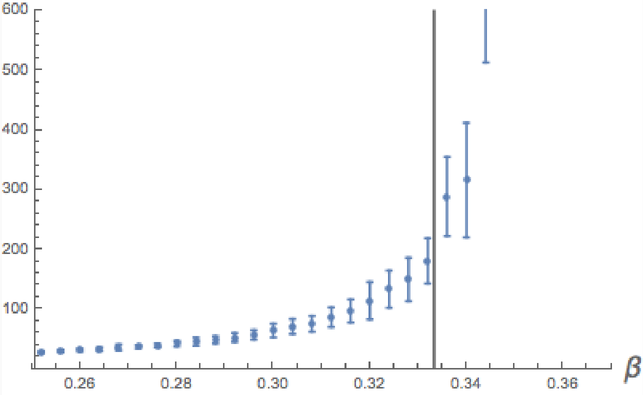
BS(1,2)



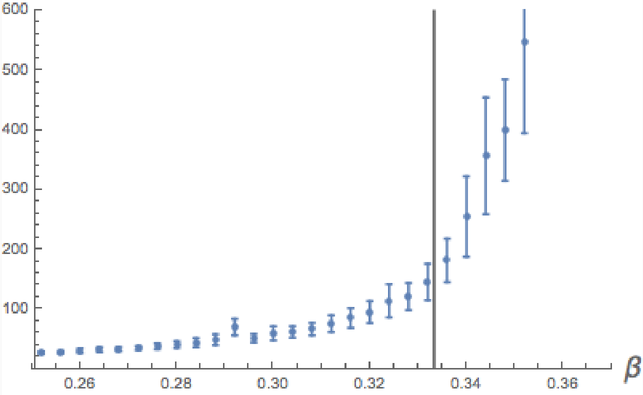
BS(1,3)



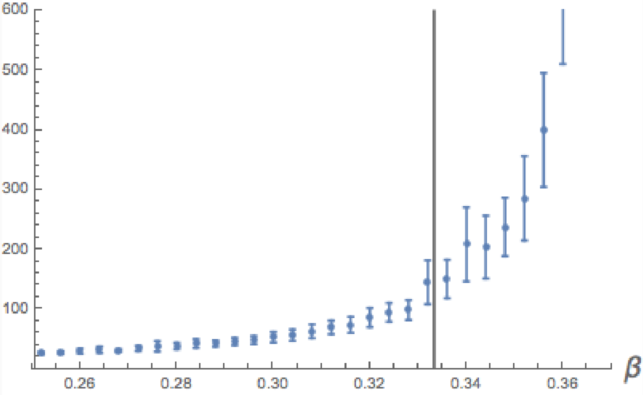
BS(1,4)



BS(1,5)



BS(1,6)



WHAT IS GOING WRONG?

Just one relator, so not the problem discussed before.

We postulate that the cause of error in ERR comes from groups with bad sub-dominant behaviour of cogrowth function, which is measured by $\mathcal{R}(n)$.

Cameron to give details tomorrow. Details in [7]

Thank you

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