Some Representations of Thompson's groups and their coefficients

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Dedicated to the memory of Uffe Haagerup.

The constructions of this talk were inspired by thinking of the Thompson groups as local scale transformations of a quantum spin chain with spins located at rational (e.g dyadic rational) points in the interval [0,1] or the circle $\mathbb{T}^1$. But I will not explain this motivation.

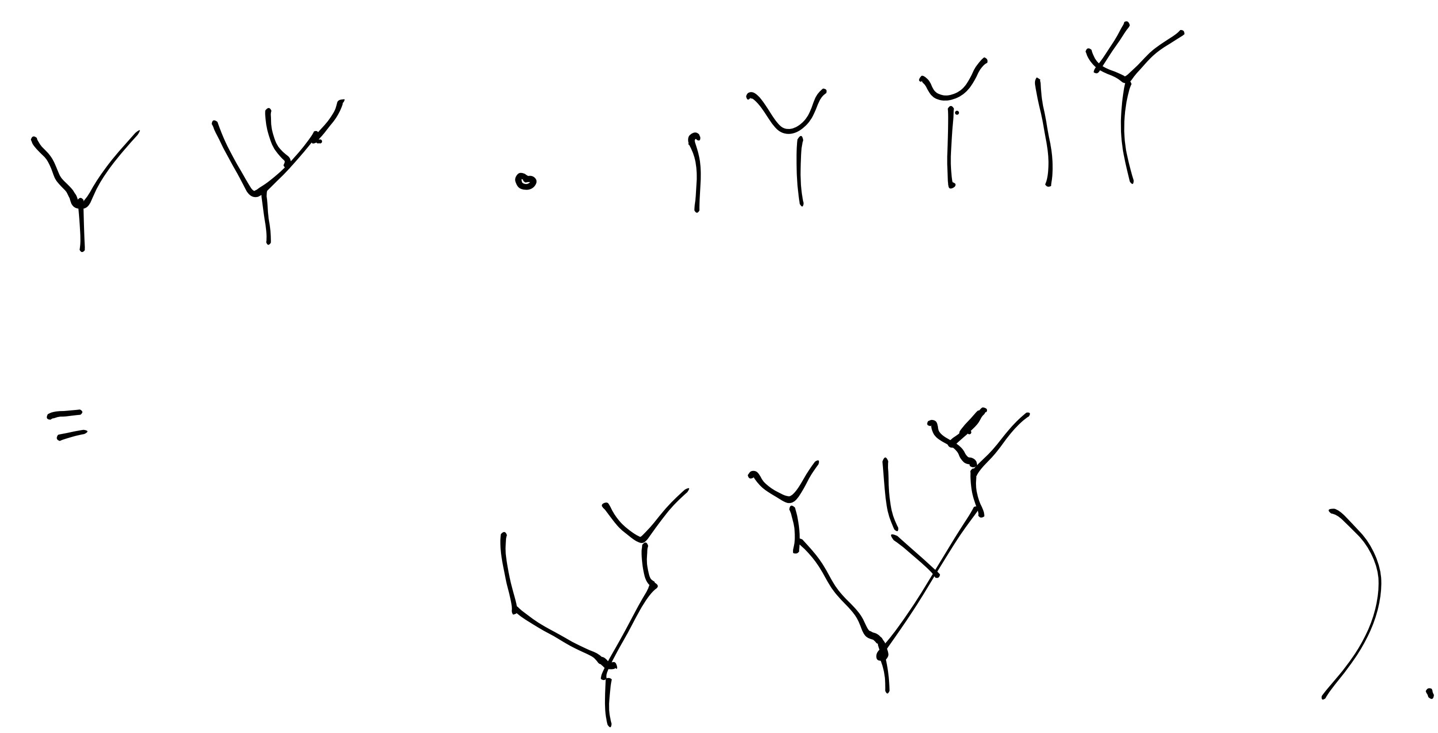
I will give the details of a general construction, first of the Thompson groups themselves. This construction will look just like everyone else's construction but if you continue to follow this particular path, a large family of actions and representations, unitary and otherwise, will appear almost tautologically. It is not clear that these representations are especially useful in solving analytical problems about the Thompson groups (amenability springs to mind) but the <u>coefficients</u> of these unitary representations sure are interesting. They are certainly good at producing subgroups (and Sapir/Golan and Yunxiang Ren are good at figuring out what subgroups these are....)

I apologise for the rather abstract nature of the construction which unfortunately completely obscures the intuition, but such an evolution has been known to occur before in mathematics....

Before going into the construction, let me state a question/conjecture for which I have very little evidence beyond the fact that it is true in the few cases I (or rather Sapir/Golan) have been able to decide it. In all these representations there will be a privileged "vacuum" vector Ω and the guess is that the representation of the Thompson group G is <u>irreducible</u> on $\overline{[G\Omega]}$.

Now down to business.

Let $\mathcal{C}$ be a small category with the following three properties (for examples you may think of $\mathbb{N}$ with one object, the semigroup of positive braids (one object) or the category whose morphisms are planar binary forests with stacking as composition of morphisms eg



properties:

1) There is an object $1 \in \mathcal{C}$ for which $Mor(1,a) \neq \phi$ for all $a \in Obects(\mathcal{C})$

Now let $\mathcal{D} = \coprod_{a \in Obj(\mathcal{C})} Mor(1,a)$

2) (Stabilisation) $\forall x,y \in \mathcal{D}$ $\exists p,q$ st. $px = qy$

3) (Cancellation) $px = qx$ for $x \in \mathcal{D}$ $\Rightarrow$ $p = q$

Conditions 2) and 3) are like Oré's semigroup conditions and one could easily trudge on to get a group of quotients. But I want to follow the party line by taking a <u>functor</u> $\Phi : \mathcal{C} \to \mathcal{K}$ where $\mathcal{K}$ is some category (<u>any</u> category).

To use Φ, first observe that property 2) makes $\mathcal{D}$ into a <u>directed set</u> with

$$x \le y \iff y = px \text{ for some } p \in \mathcal{C}.$$

<u>Proposition</u> The map $D_\Phi : \mathcal{D} \to$ sets given by $D_\Phi(x) = \mathrm{Mor}(\Phi(1), \Phi(x))$ is a direct system.

<u>Proof</u> We have to give $\iota_x^y : D_\Phi(x) \to D_\Phi(y)$ satisfying the obvious properties. Here we use property (3) (cancellation). For if $x \le y$ there is a <u>unique</u> p with $px = y$ and we put

$$\iota_x^y(f) = \Phi(p) \circ f$$

(book keeping $f : \Phi(1) \to \Phi(x)$, $\Phi(p) : \Phi(x) \to \Phi(y)$ ✓

Q.E.D.

Surprise surprise we are interested in the <u>direct limit</u> $\boxed{\varinjlim D_\Phi}$

We begin by examining it in the case Φ = identity.

<u>By definition</u> $\varinjlim D_\Phi$ is the quotient of

$$\coprod_{x \in D} \mathrm{Mor}(\Phi(1), \Phi(x)) = \coprod_{x \in D} \mathrm{Mor}(1, x)$$

$$= \{ (x, y) \mid \mathrm{target}(x) = \mathrm{target}(y) \}$$

modulo the equivalence relation

$$(x, y) \sim (px, py)$$

The notation $\frac{x}{y} \sim \frac{px}{py}$ is suggestive and it is routine to show that $\varinjlim D_{id}$ becomes a <u>group</u> under $(x, y)(y, z) = (x, z)$.

For $\mathbb{N}$ this gives $(\mathbb{Q}^+, \times)$, for B_n^+ this gives B_n and for binary forests this gives Thompson's group F. So far so good so ordinary.

The interest begins when we consider an arbitrary functor Φ. An element of $\varinjlim D_\Phi$ may be represented by

a pair (y, ξ) for $\xi \in \mathrm{Mor}(1, \Phi(y))$
and the same argument that showed that $\varinjlim D_{id}$ is a group shows that

$$(x, y)(y, \xi) \longmapsto (x, \xi)$$

defines an action of $\varinjlim D_{id}$ on $\varinjlim D_{\Phi}$.
The action is linear if the category K is, etc.

The direct limit can have surprises, especially if the ι_x^y are not injective.

example N - a (possibly non-invertible) matrix for each prime gives a representation of $(\mathbb{Q}^+, \times)$
(exercise)

B_n^+ - a family of matrices $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$
(not necessarily invertible!) gives a repn. of B_n

How? Probably look at Δ^2, the generator of the centre, and take $\bigcap_n \mathrm{Image}(\Delta^{2n})$. Δ^2 is invertible on it, so all the σ_i are also.

Main example category F of binary planar forests. Any functor gives an action of F.

This is quite versatile: all one needs is a morphism somewhere to represent Y.

<u>examples</u> a) $\mathcal{K}$ = Vect, v a unit vector in $\mathbb{C} \oplus \mathbb{C}$, represente Y as the map $z \to zv$ from $\mathbb{C}$ to $\mathbb{C} \oplus \mathbb{C}$ and extend it to linear maps $Y_i : \bigoplus_{j=1}^{n} \mathbb{C} \to \bigoplus_{j=1}^{n+1} \mathbb{C}$

by

$$Y_i\big((z_1, z_2, \dots, z_i, z_{i+1}, \dots, z_n)\big) = (z_1, z_2, \dots, Y(z_i), z_{i+1}, \dots, z_n) \in \mathbb{C}^{n+1}$$

One obtains a unitary rep of F on Hilbert space.

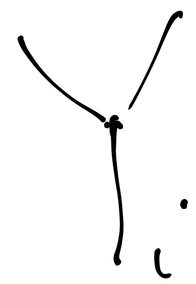
b) $\mathcal{K}$ = groups. Let $\mathbb{F}_n$ be the free group on n generators $x_1, x_2, \dots x_n$. Let w be a word on (x, y, x^{-1}, y^{-1}). Then define

$$Y_i : \mathbb{F}_n \to \mathbb{F}_{n+1} \text{ by}$$

$$Y_i(x_j) = \begin{cases} x_j & \text{if } j < i \\ w(x_i, x_{i+1}) & \text{if } j = i \\ x_{j+1} & \text{if } j > i \end{cases}$$

one obtains an acti[on] of F_2 on the inductive limit, whatever that is. . .

c) (The main examples). Suppose we are given a (finite dimensional) Hilbert space V and an isometry $R: V \to V \otimes V$.

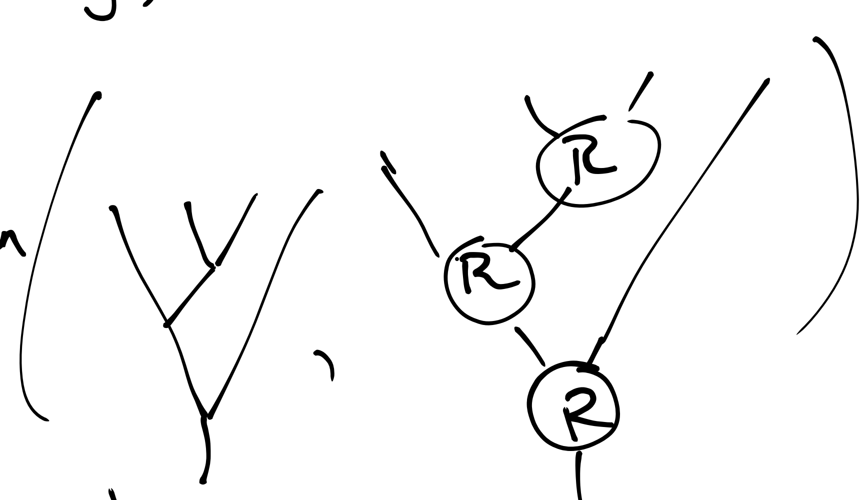
Define  $: \bigotimes^n V \to \bigotimes^{n+1} V$

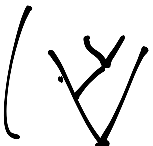
by $Y_i(v_1 \otimes v_2 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n)$

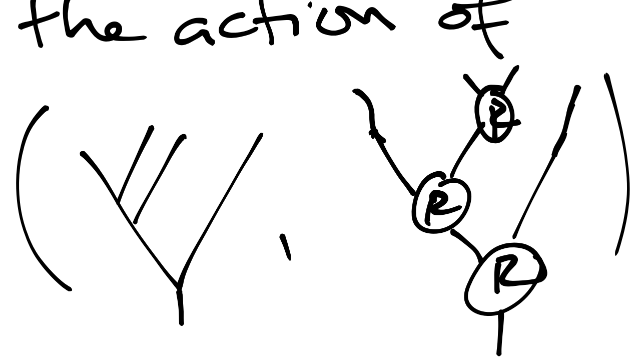
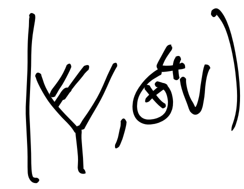
$= v_1 \otimes v_2 \otimes \cdots \otimes R(v_i) \otimes v_{i+1} \otimes \cdots \otimes v_n \in \bigotimes^{n+1} V.$

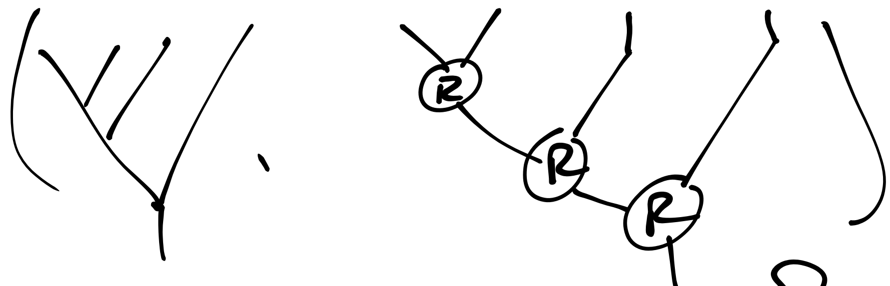
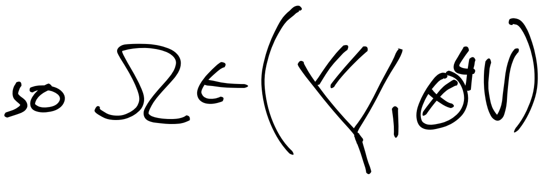
Again F acts by unitaries on $\varinjlim D_\Phi$

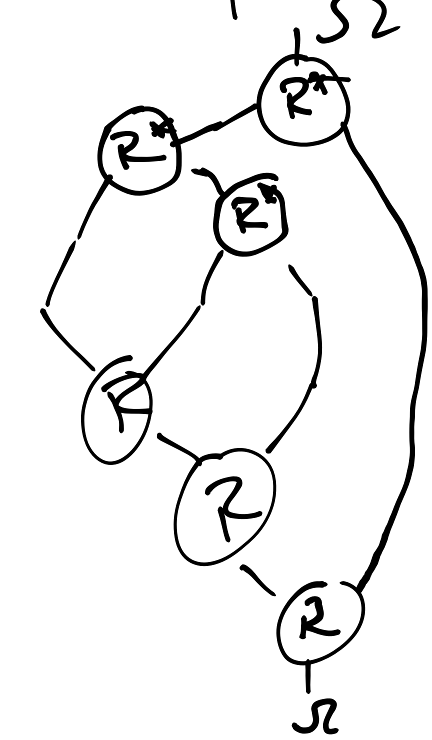
Let us dwell for a moment on the coefficients in this last example. Pick $\Omega \in V$, and an element g of F given by a pair of binary planar trees as usual, e.g.,

$g =$  , then  ,

an element of ( , $V \otimes V \otimes V \otimes V$). is a representative of $\Omega \in \varinjlim D_\Phi$ by definition.

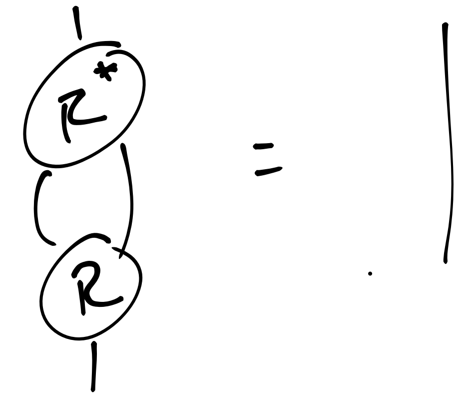
But by the definition of the action of g on $\varinjlim D_\Phi$, $g\Omega =$  $\in$ 

but  is also $\Omega \in$ 

$\langle g\Omega, \Omega \rangle =$ 

We have used a standard notation (from eg. spin network theory) for contracting systems of tensors. Ω is the first of a set of basis vectors and all the tensors are contracted according to the connection scheme ($R^*_{ijk} = \overline{R}_{kji}$).

This is a rather concrete formula for the coefficient. In order for the representation to be unitary it is necessary to have:



$\left(\sum R_{ijk} \overline{R_{ij\ell}} = \delta_{k,\ell} \right)$

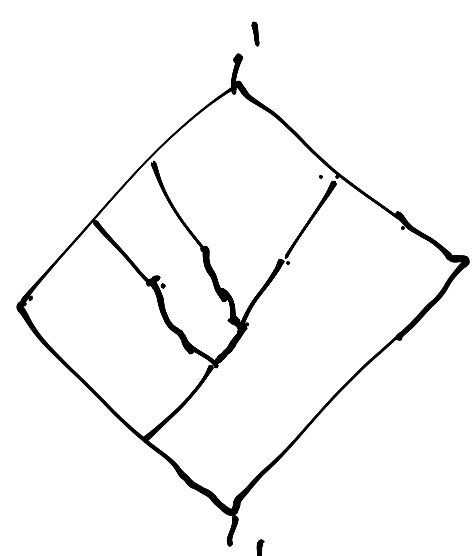
otherwise the choice of R is <u>completely</u> <u>arbitrary</u>! (unlike for instance the requirement of the Yang-Baxter equation to get braid group representations)

Here is a good example of such an R:

$$V = \mathbb{C}^3$$

$$R_{ijk} = \begin{cases} 0 & \text{if } i=j \text{ or } j=k \text{ or } i=k \\ \frac{1}{\sqrt{2}} & \text{if } |\{i,j,k\}| = 3 \end{cases}$$

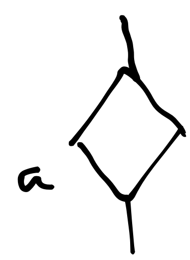
We see that (say $\Omega = (1,0,0)$) that $\langle g\Omega, \Omega \rangle$ is the number of ways of 3-colouring the graph



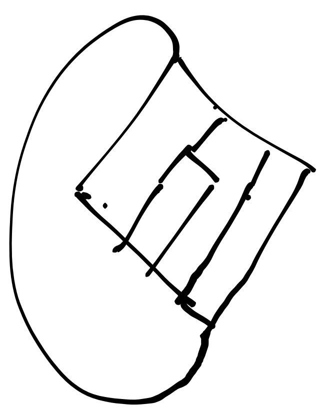
in such a way that all 3 colours at a given vertex are distinct. The positivity of this coefficient for all $g \in F$ is known to be equivalent to the 4-colour theorem.

The indices of the tensors in the above example lived on the edges of the graph ("vertex models" in statistical mechanics language). One may also use tensors where the spins live in the regions of the graph, and contraction happens by summing over spins in a given region.

If we use the same tensor R as before

 $b = 1$ for $b \neq c$, so normalisation different

we find that $\langle g\Omega, \Omega \rangle$ is the number of ways of three-colouring the map (up to a factor)

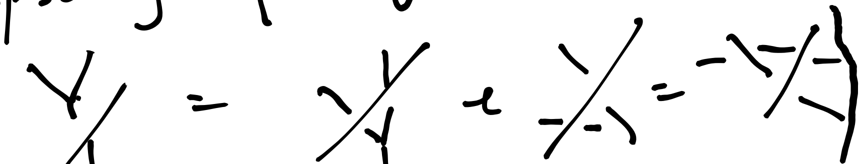


and since this is possible in 6 ways or not at all, the representation is the permutation one (quasi-regular) on $\ell^2(F/\Gamma)$, Γ being the stabilizer of Ω i.e. the subgroup of F for which the graph can be 3-coloured.

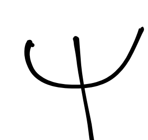
This group has been identified by Ren as Thompson_4.

A similar case was done earlier by Sapir-Golan and the group was Thompson_3.

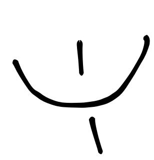
It is to be noted that the "groupoid/group of fractions" approach works (by enriching the objects in the categories) for Thompson's groups T, V, and the braided Thompson groups - and of course "2" can be replaced by any $n \in \mathbb{N}$.

The R tensor reps above extend immediately to T and V but the braided Thompson group requires extra data and properties (basically )

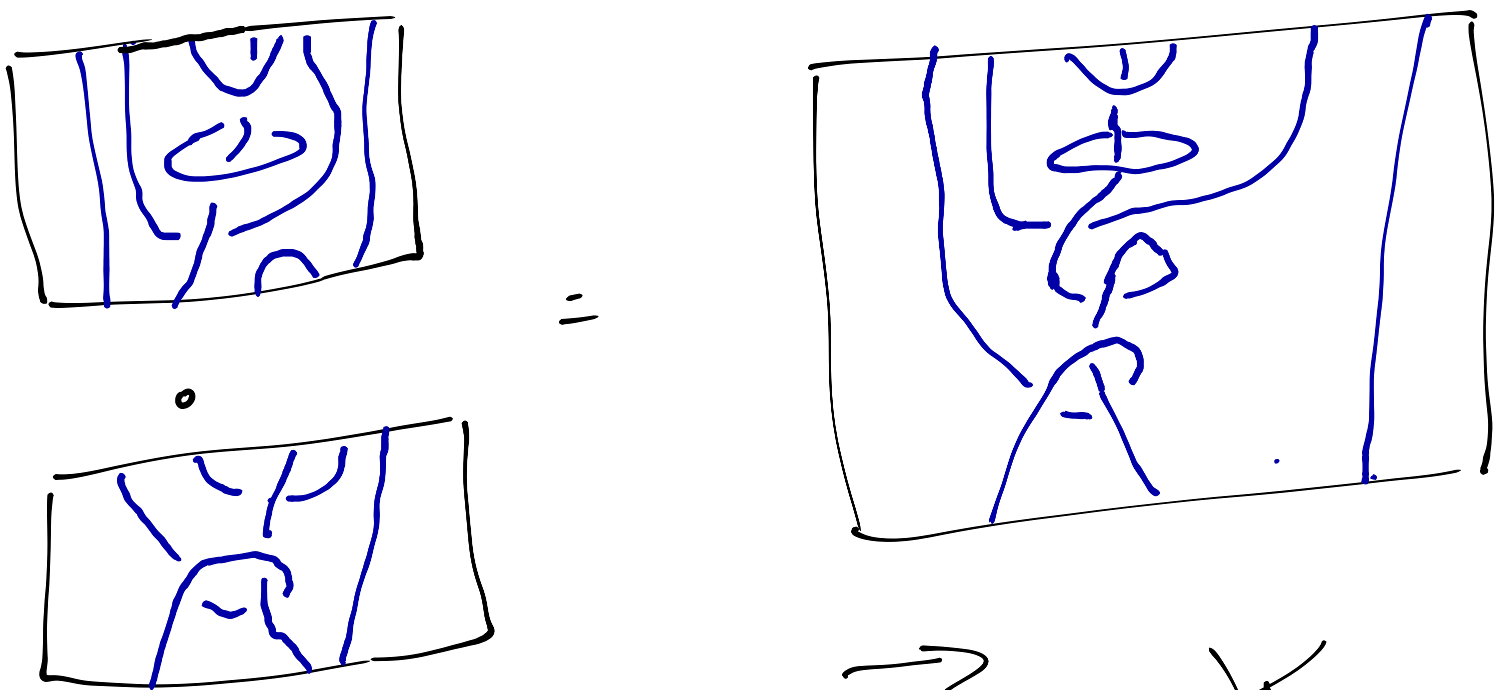
Such data are given by the fusion and braiding of conformal field theory.

Finally let us talk about knots. If we use $F_3 =$ Thompson$_3$, to find representations one needs elements somewhere looking like .

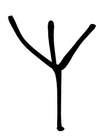
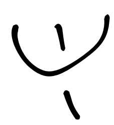
An obvious candidate is the crossing of knot theory:

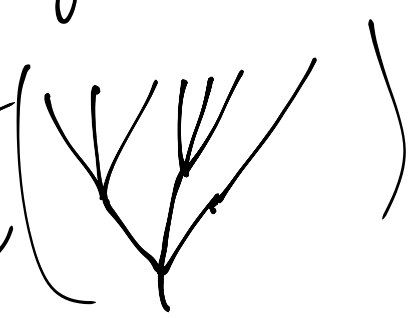
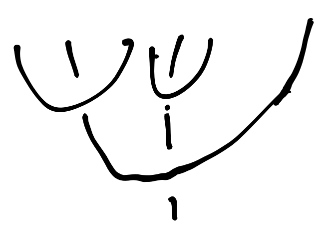


Indeed Conway's theory of tangles is a setup for this picture. If $\mathcal{K}$ is the category whose morphisms are "tangles" with an odd number of boundary points at the top and bottom of a rectangle, with stacking as composition, e.g.

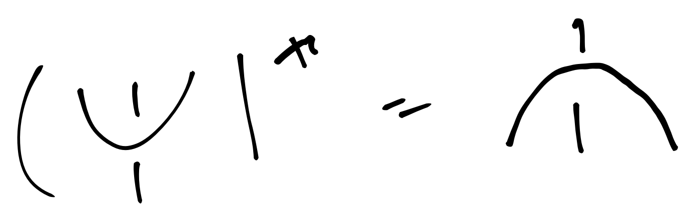


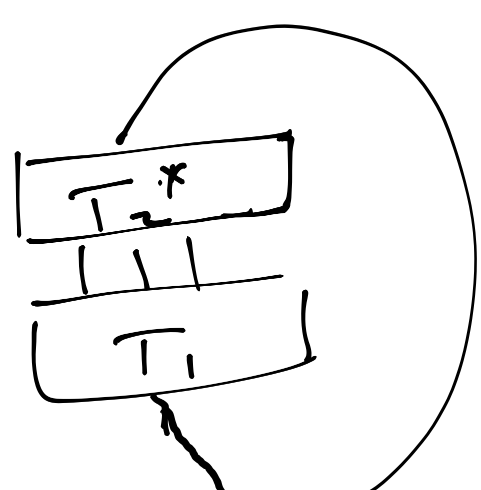
Then we may define $\Phi : \mathcal{F}_3 \to \mathcal{K}$

by sending  to  so for instance

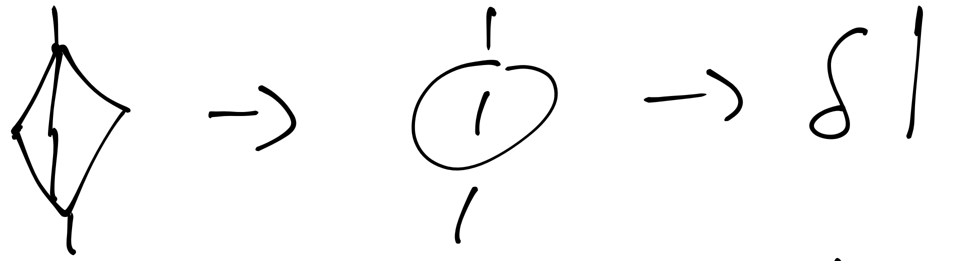
Φ  = 

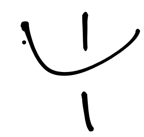
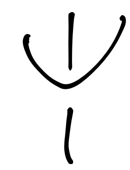
Tangles have a natural $*$ structure and link valued inner product :



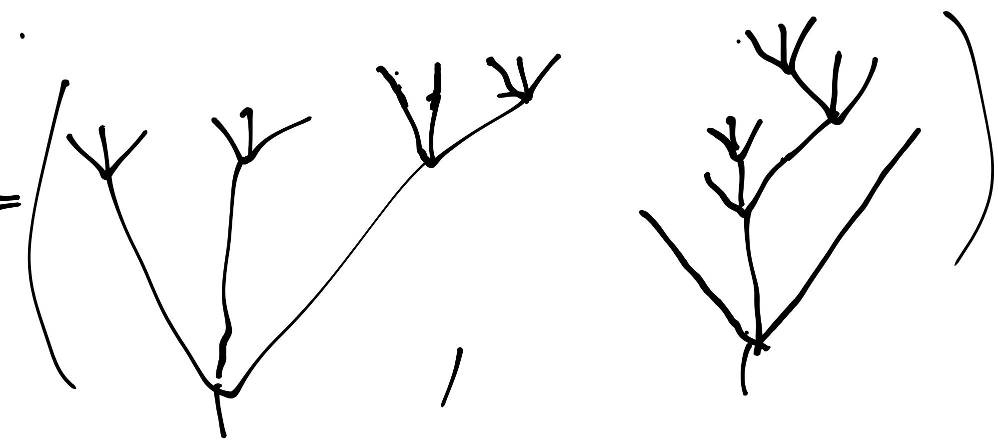
$\langle T_1, T_2 \rangle$ = 

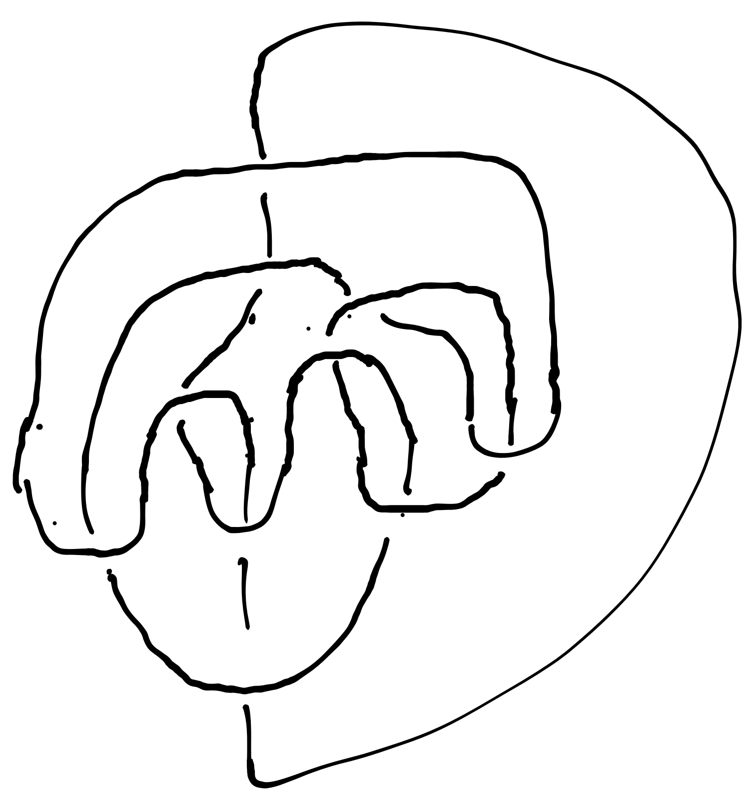
If we linearise by taking linear combinations of tangles we can also quotient by sending circles unlinked with anything to a number δ so

that 

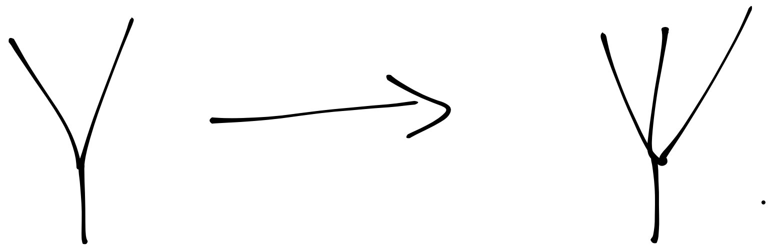
so we can normalise  to $R = \frac{1}{\sqrt{\delta}}$  and

the direct limit space has an inner product valued in linear combinations of links since all the ι_{x}^{y} are isometries.

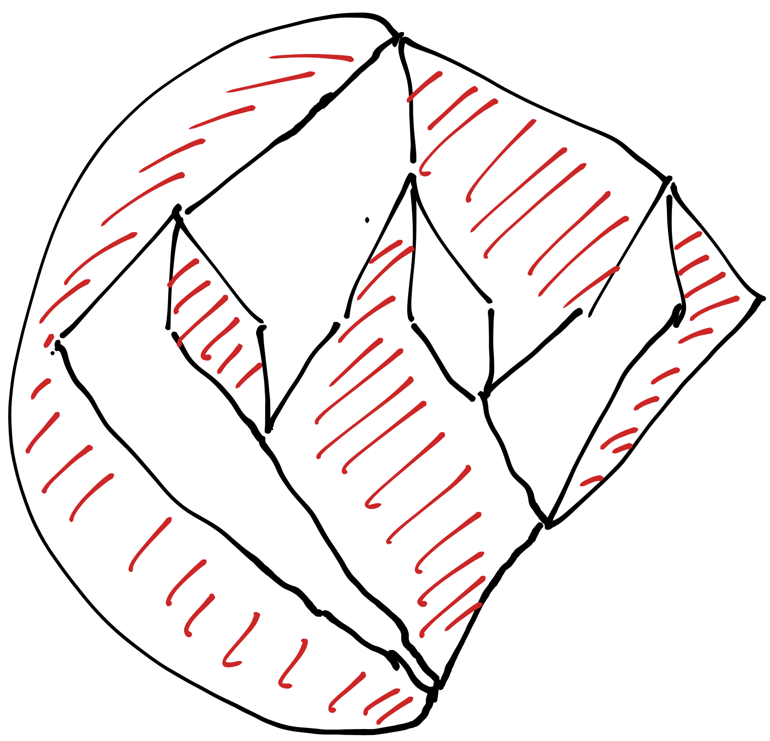
We see that if $g =$ 

then $\langle g\Omega, \Omega \rangle$ = 

All knots and links can be obtained in this way. Indeed one only needs the smaller subgroup of F_3 which is the image of F_2 under the homomorphism defined by



To handle oriented links one uses a subgroup $\vec{F}$ of F for which the surface obtained from the chequerboard shading is orientable:



← surface not orientable, not in subgroup.

For F_2, this is the subgroup identified by Sapir/Golan. We call it $\vec{F}$.

$\vec{F}$ can be obtained from a category $\vec{\mathcal{F}}$ where the objects and morphisms of $\mathcal{F}$ are enriched by data needed to orient the surfaces.