Partitioning a graph into a cycle and an anticycle, a proof of Lehel's conjecture

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Abstract

We prove that every graph G has a vertex partition into a cycle and an anticyle (a cycle in the complement of G). Emptyset, singletons and edges are considered as cycles. This problem was posed by Lehel and shown to be true for very large graphs by Łuczak, Rödl and Szemerédi [7], and more recently for large graphs by Allen [1].

Many questions deal with the existence of monochromatic paths and cycles in edge-colored complete graphs. Erdős, Gyárfás and Pyber asked for instance in [3] if every coloring with k colors of the edges of a complete graph admits a vertex partition into k monochromatic cycles. In a recent paper, Gyárfás, Ruszinkó, Sárközy and Szemerédi [5] proved that $O(k \log k)$ cycles suffice to partition the vertices. This question was also studied for other structures like complete bipartite graphs by Haxell [6]. One case which received a particular attention was the case $k = 2$, where one would like to cover a complete graph which edges are colored blue and red by two monochromatic cycles. A conjecture of Lehel, first cited in [2], asserts that a blue and a red cycle partition the vertices. This was proved for sufficiently large n by Łuczak, Rödl and Szemerédi [7], and more recently by Allen [1] with a better bound. Our goal is to completely answer Lehel's conjecture.

References

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