Maximum Number of Cycles and Hamiltonian Cycles in Sparse Graphs

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In this talk we concentrate to the maximum number of cycles in the union of two trees. In order to prove non-trivial bounds we also need some upper bounds on the number of Hamiltonian cycles in 3- and 4-regular graphs.

We examine some important special cases of the following problem. Let $\mathcal{H}$ be a set of graphs. A graph is called $\mathcal{H}$-free, if it does not have any subgraph isomorphic to any member of $\mathcal{H}$. Let $f^k_{\mathcal{H}}(n)$ denote the maximum number of subgraphs isomorphic to a member of $\mathcal{H}$ in any graph that is a union of k $\mathcal{H}$-free graphs on the vertex set $V := \{1, 2, \ldots, n\}$.

We mainly focus the problem, where $\mathcal{H} = \mathcal{C}$, here $\mathcal{C}$ denotes the set of all cycles. Thus in this problem every G_i is a tree (forest) and we are going to calculate (bound) the maximum number of cycles in their union. Our main goal is to prove upper and lower bounds for $f(n) := f^2_{\mathcal{C}}(n)$. Interestingly, giving a non-trivial upper bound for $f(n)$ needs non-trivial upper bound for the number of Hamiltonian cycles in a 4-regular graph. And, for proving this bound, we need an upper bound on the number of Hamiltonian cycles in a 3-regular graph.

Theorem 1 (Folklore) *If $G = (V, E)$ is a connected multigraph on n vertices having m edges, then the number of Eulerian subgraphs is exactly 2^{m-n+1}. Consequently $2^{m-n+1} - 1$ is an upper bound on the number of cycles.*

Let $h(G)$ denote the number of Hamiltonian cycles and c(G) denote the number of cycles in graph G. Fix $\alpha = \sqrt[8]{8} \approx 1.2968$, $\beta = \sqrt[4]{2} \approx 1.1892$, $c_1 = 3/(2\beta) \approx 1.2613$.

Theorem 2 *If G is a 3-regular graph on n vertices then $h(G) \le c_1 \cdot \alpha^n$.*

Theorem 3 *If G is a graph with m edges then $h(G) \le c_1 \cdot (2 - \varepsilon_1)^{m-n}$, where $\varepsilon_1 = 2 - \left(2^{\frac{11}{12}} \cdot \gamma^{\frac{1}{12}}\right) \approx 0.0287$.*

Theorem 4 *If G is a connected graph with $m = 2n$ edges, then $c(G) \le c \cdot n^2 \cdot (2-\varepsilon)^n$ for some positive constants c and ε. Consequently $f(n) \le c \cdot n^2 \cdot (2-\varepsilon)^n$.*

We will also prove a theorem about union of k trees, and show some constructions giving the best known lower bounds. Detailed version can be found in *Egres Technical Report TR-2009-03*, www.cs.elte.hu/egres/