# $(k,j)$-coloring of sparse graphs

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A graph $G$ is called *improperly* $(d_1, \dots, d_k)$*-colorable*, or just $(d_1, \dots, d_k)$*-colorable*, if the vertex set of $G$ can be partitioned into subsets $V_1, \dots, V_k$ such that the graph $G[V_i]$ induced by the vertices of $V_i$ has maximum degree at most $d_i$ for all $1 \leq i \leq k$. This notion generalizes those of proper $k$-coloring (when $d_1 = \dots = d_k = 0$) and $d$-improper $k$-coloring (when $d_1 = \dots = d_k = d \geq 1$). Proper and $d$-improper colorings have been widely studied. As shown by Appel and Haken [1, 2], every planar graph is 4-colorable, i.e. $(0,0,0,0)$-colorable. Eaton and Hull [4] and independently Škrekovski [5] proved that every planar graph is 2-improperly 3-colorable (in fact, 2-improper 3-choosable), i.e. $(2,2,2)$-colorable.

In this talk, we will focus on $(k,j)$-colorability of sparse graphs (in the meaning of small maximum average degree). We recall that the *maximum average degree* of a graph $G$, written $mad(G)$, is the largest average degree among the subgraphs of $G$:

$$mad(G) = \max\left\{\frac{2|E(H)|}{|V(H)|}, H \subseteq G\right\}$$

Let $k \geq 0$ be a integer. We will show that every graph with maximum average degree smaller than $\frac{3k+4}{k+2}$ is $(k,0)$-colorable. The key concepts in the proof are those of "soft components" and "feeding areas". These further develop those of soft cycles and feeding paths introduced by Borodin, Ivanova, Kostochka in [3]. A distinctive feature of the discharging in the proof is its "globality": a charge for certain vertices is collected from arbitrarily large "feeding areas", which is possible due to the existence of reducible configurations of unlimited size in the minimum counter-examples, called "soft components".

## References

[1] K. Appel and W. Haken. Every planar map is four colorable. Part I. Discharging. *Illinois J. Math.*, 21:429–490, 1977.

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[4] N. Eaton and T. Hull. Defective list colorings of planar graphs. *Bull. Inst. Combin. Appl.*, 25:79–87, 1999.

[5] R. Škrekovski. List improper coloring of planar graphs. *Comb. Prob. Comp.*, 8:293–299, 1999.