Crossings and Colorings

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Mike Albertson conjectured that if a graph G has chromatic number r, then its crossing number is at least as much as the crossing number of K_r. Apparently, this conjecture is closely related to the Hajós conjecture. If a graph G satisfies the Hajós conjecture, then Albertson's conjecture also holds for G. Erdős and Fajtlowicz [2] proved that almost all graphs are counterexamples to the Hajós conjecture. Therefore, there should be enough ground against Albertson's conjecture as well. However, all results seem to talk for the validity of the conjecture so far.

For $k = 5$, it is equivalent to the 4CT.

Oporowski and Zhao [5] proved that graphs with at most two crossings are 5-colorable. Since $\textsc{cr}(K_6) = 3$, this implies the conjecture for $k = 6$.

Albertson, Cranston, and Fox [1] verified the conjecture for $7 \leq r \leq 12$. They also showed that any counterexample to Albertson's conjecture must have less than $4r$ vertices.

In this talk, we present the following improvements of these results:
we verify the conjecture for $13 \leq r \leq 16$, and
show that any counterexample has at most $3.57r$ vertices.

Our results are based on the theory of critical graphs settled by Gallai [3], Kostochka and Stiebitz [4] and on variations of the crossing lemma proved by Pach et al [6].

We recall an old conjecture by Ore on the structure (and the number of edges) of an r-critical graph on n vertices. Such a result appeared to be useful in attacking Albertson's conjecture. In this context, it is natural to raise the following restricted question:

Let G be an r-chromatic n-vertex graph without a topological K_r subgraph. What is the minimum number of edges G can have?

This is joint work with Géza Tóth (Rényi Institute, Budapest).

References

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