Orientations of graphs with prescribed weighted out-degrees

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If we want to apply Galvin's kernel method to show that a graph G satisfies a certain coloring property, we have to find an appropriate orientation of G. This motivated us to investigate the complexity of the following orientation problem. The input is a graph G and two vertex functions $f, g : V(G) \to \mathbb{N}$. Then the question is whether there exists an orientation D of G such that each vertex $v \in V(G)$ satisfies $\sum_{u \in N_D^+(v)} g(u) \leq f(v)$.

On one hand, as proved by Hakimi in 1965, this problem can be solved in polynomial time if $g(v) = 1$ for every vertex $v \in V(G)$. On the other hand, the problem is NP-complete even if we restrict it to graphs which are bipartite, planar and of maximum degree at most 3 and to functions f, g where the permitted values are 1 and 2, only.

We also show that the analogue problem, where we replace g by an edge function $h : E(G) \to \mathbb{N}$ and where we ask for an orientation D such that each vertex $v \in V(G)$ satisfies $\sum_{e \in E_D^+(v)} g(e) \leq f(v)$, is NP-complete, too.

Furthermore, we discuss some new results related to the (f, g)-choosability problem, or in our terminology, to the list coloring problem of weighted graphs. In particular, we use Galvin's theorem to establish a generalization of Brooks's theorem for weighted graphs. We show that each graph G has a kernel perfect super-orientation D such that $d_D^+(v) \leq d_G(v) - 1$ for every vertex $v \in V(G)$, unless each block of G is a complete graph or an odd cycle.