The Turán number of bipartite graphs plus an odd cycle

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Let $\mathcal{F}$ be a family of graphs. A graph is *$\mathcal{F}$-free* if it contains no copy of a graph in $\mathcal{F}$ as a subgraph. The *Turán number* $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices. The theory of Turán numbers of non-bipartite graphs is quite well-understood, but for bipartite graphs the field is wide open. Many of the main open problems here were raised in a series of conjectures by Erdős and Simonovits in 1982. One of these is as follows. Let C_k denote a cycle of length k, and let $\mathcal{C}_k$ denote the set of cycles C_ℓ, where $3 \le \ell \le k$ and ℓ and k have the same parity. Erdős and Simonovits conjectured that for any family $\mathcal{F}$ consisting of bipartite graphs there exists an odd integer k such that $\mathrm{ex}(n, \mathcal{F} \cup \mathcal{C}_k) \sim \mathrm{z}(n, \mathcal{F})$ as $n \to \infty$, where $\mathrm{z}(n, \mathcal{F})$ is the *Zarankiewicz number*: the maximum number of edges in an $\mathcal{F}$-free *bipartite* graph on n vertices. In joint work with Sudakov and Verstraëte we proved a stronger form of this conjecture, with stability and exactness, in the case when $\mathcal{F} = \mathcal{C}_{2\ell}$ with $\ell \in \{2, 3, 5\}$. Our proofs make use of pseudorandomness properties of nearly extremal graphs that are of independent interest. Also, in joint work with Allen, Sudakov and Verstraëte, we gave a general approach to the conjecture using Scott's sparse regularity lemma. This proves the conjecture for complete bipartite graphs $K_{2,t}$ and $K_{3,3}$, and moreover is effective for any $\mathcal{F}$ based on some reasonable assumptions on the maximum number of edges in an m by n bipartite $\mathcal{F}$-free graph, which are similar to the conclusions of another conjecture of Erdős and Simonovits.