Graph coloring, communication complexity, and the stubborn problem.

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A classical result of Graham and Pollak asserts that the edge set of the complete graph on n vertices cannot be partitioned into less than $n-1$ complete bipartite graphs. A natural question is then to ask for some properties of graphs G_ℓ which are edge-disjoint unions of ℓ complete bipartite graphs. An attempt in this direction was proposed by Alon, Saks and Seymour, asking if the chromatic number of G_ℓ is at most $\ell+1$. This wild generalization of Graham and Pollak's theorem was however disproved by Huang and Sudakov who provided graphs with chromatic number $\Omega(\ell^{6/5})$. The $O(\ell^{\log \ell})$ upper-bound being routine to prove, this leaves as open question the *polynomial Alon-Saks-Seymour conjecture* asking if an $O(\ell^c)$ coloring exists for some fixed c.

A well-known communication complexity problem introduced by Yannakakis, involves a graph G of size n and the usual suspects Alice and Bob. Alice plays on the stable sets of G and Bob plays on the cliques. Their goal is to exchange the minimum amount of information to decide if Alice's stable set S intersect Bob's clique K. In the nondeterministic version, one asks for the minimum size of a certificate one should give to Alice and Bob to decide whether S intersects K. If indeed S intersects K, the certificate consists in the vertex $x = S \cap K$, hence one just has to describe x, which cost is $\log n$. The problem becomes much harder if one want to certify that $S \cap K = \emptyset$ and this is the core of this problem. A natural question is to ask for a $O(\log n)$ upper bound. Yannakakis observed that this would be equivalent to the following *polynomial clique-stable separation conjecture*: There exists a c such that for any graph G on n vertices, there exists $O(n^c)$ vertex bipartitions of G such that for every disjoint stable set S and clique K, one of the bipartitions separates S from K.

A variant of Feder and Vardi celebrated dichotomy conjecture for Constrait Satisfaction Problems, the List Matrix Partition (LMP) problem asks whether all $(0, 1, *)$ CSP instances are NPcomplete or polytime solvable. The LMP was investigated for small matrices, and was completely solved in dimension 4, save for a unique case, known as the *stubborn problem*: Given a complete graph G which edges are labelled by 1,2, or 3, the question is to partition the vertices into three classes V_1, V_2, V_3 so that V_i does not span an edge labelled i. An easy branching majority algorithm computes $O(n^{\log n})$ 2-list-coloring of the vertices such that every solution of the stubborn problem is covered by at least one of these 2-list-coloring. The stubborn problem hence reduces to $O(n^{\log n})$ 2-SAT instances, yielding a pseudo polynomial algorithm. A polynomial algorithm was recently discovered by Cygan et al., but whether the original branching algorithm could be turned into a polynomial algorithm is still open. Precisely one can ask the *polynomial stubborn 2-list cover conjecture* asking if the set of solutions of any instance of the stubborn problem can be covered by $O(n^c)$ instances consisting of lists of size 2.

In this talk, I will show that the polynomial Alon-Saks-Seymour conjecture, the polynomial clique-stable separation conjecture and the polynomial stubborn 2-list cover conjecture are indeed equivalent. One of the implications linking the two first problems was already proved by Alon and Haviv.