MATEMATIK 1 JANUAR 2008 FACITLISTE

<u>OPG. 1 a)</u> $P_1(x) = f(0) + f'(0)\cdot(x-0) = \underline{2 + 0.1x}$

$f(0.2) \approx P_1(0.2) = 2 + 0.1\cdot 0.2 = \underline{2.02}$

<u>OPG. 1 b)</u> $f(x) = (32 + 8x^5 + 8\cdot\ln(x+1))^{\frac{1}{5}}$

$f(0) = (32)^{1/5} = 2$

$f'(x) = \frac{1}{5}\cdot(32+8x^5+8\cdot\ln(x+1))^{-\frac{4}{5}}\cdot(40x^4+\frac{8}{x+1})$

$f'(0) = \frac{1}{5}\cdot 32^{-4/5}\cdot 8 = \frac{1}{5}\cdot\frac{1}{16}\cdot 8 = \frac{1}{10}$

$f''(x) = \frac{1}{5}\cdot(-\frac{4}{5})\cdot(32+8x^5+8\ln(x+1))^{-\frac{9}{5}}\cdot$
$(40x^4+\frac{8}{x+1})^2 + \frac{1}{5}(32+8x^5+8\ln(x+1))^{-\frac{4}{5}}\cdot$
$(160x^3 + 8\cdot(-1)(x+1)^{-2})$

$f''(0) = -\frac{4}{25}\cdot\frac{1}{2^9}\cdot 8^2 + \frac{1}{5}\cdot\frac{1}{2^4}\cdot(-8)$
$= -\frac{1}{50} + (-\frac{1}{10}) = -\frac{1}{50} - \frac{5}{50} = \underline{-\frac{3}{25}}$

$P_2(x) = f(0) + f'(0)\cdot(x-0) + \frac{1}{2}\cdot f''(0)\cdot(x-0)^2$
$= 2 + 0.1x + \frac{1}{2}(-\frac{3}{25})x^2$
$= \underline{2 + 0.1x - 0.06x^2}$

$f(0.2) \approx P_2(0.2) = 2 + 0.1\cdot 0.2 - 0.06\cdot(0.2)^2$
$= 2.02 - 0.0024$
$= \underline{2.0176}$

Lommeregner-udregning giver :
$f(0.2) = 2.017939433\ldots$

<u>OPG. 2 a)</u> $\frac{\partial y}{\partial x} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} = -\frac{2xy^3}{5y^4+3x^2y^2} = \underline{-\frac{2xy}{5y^2+3x^2}}$

$\frac{\partial y}{\partial x}$ i $(5,5) = -\frac{2\cdot5\cdot5}{5\cdot5^2+3\cdot5^2} = -\frac{2}{8} = \underline{-\frac{1}{4}}$

<u>OPG. 2 b)</u> $El_x(y) = \frac{x}{y}\cdot\frac{\partial y}{\partial x} = -\frac{2x^2y}{5y^3+3x^2y} = \underline{\frac{-2x^2}{5y^2+3x^2}}$

$El_x(y)$ i $(5,5) = \frac{5}{5}\cdot(-\frac{1}{4}) = \underline{-\frac{1}{4}}$

y vokser med $(-\frac{1}{4})\cdot 8\% = \underline{-2\%}$

$y \approx 5 - \frac{2}{100}\cdot 5 = 5 - 0.1 = \underline{4.9}$

eller $y(5.4) \approx y(5) + y'(5)\cdot 0.04 = 5 - \frac{1}{4}\cdot 0.04$
$= \underline{4.9}$

OPG. 3a

$$\int (x^4+3x^2)dx = \underline{\tfrac{1}{5}x^5 + x^3 + c}$$

$$\int_0^5 (x^4+3x^2)dx = \left[\tfrac{1}{5}x^5+x^3\right]_0^5 = \tfrac{1}{5}\cdot 5^5 + 5^3 - 0$$

$$= 625 + 125 = \underline{750}$$

$$PS = -\int_0^5 (x^4+3x^2)dx + 5\cdot 700$$

$$= -750 + 3500 = \underline{\underline{2750}}$$

OPG. 3b

$$\int_a^5 \left(500 + \frac{1000}{x}\right)dx = \left[500x + 1000\cdot \ln x\right]_a^5$$

$$= \underline{2500 + 1000\cdot\ln 5 - 500a - 1000\ln(a)}$$

$$\lim_{a\to 0}\int_a^5 \left(500 + \frac{1000}{x}\right)dx =$$

$$2500 + 1000\cdot\ln 5 - 500\cdot 0 - 1000\cdot(-\infty) =$$

$$2500 + 1000\cdot\ln 5 - 0 + 1000\cdot\infty = \infty$$

$$CS = \lim_{a\to 0}\int_a^5 \left(500 + \frac{1000}{x}\right)dx - 5\cdot 700$$

$$= \infty - 3500 = \underline{\infty}$$

OPG. 4a

$$X = \left(-b - \sqrt{b^2-4ac}\right)/2a$$

$$\frac{\partial X}{\partial b} = \left(-1 - \frac{2b}{2\cdot\sqrt{b^2-4ac}}\right)/2a = \underline{\left(-1 - \frac{b}{\sqrt{b^2-4ac}}\right)/2a}$$

$\frac{\partial X}{\partial b}$ for $a=2, b=5$ og $c=-25$:

$$\left(-1 - \frac{5}{\sqrt{25-4\cdot 2\cdot(-25)}}\right)/4 = \left(-1-\frac{5}{15}\right)/4$$

$$= -\frac{20}{15\cdot 4} = \underline{-\frac{1}{3}}$$

X er $(-5-15)/4 = \underline{-5}$

OPG. 4b

$$X \approx -5 - \tfrac{1}{3}(5.18-5) = -5 - 0.06 = -5.06$$

En lommeregner giver:

$$X = -5.18 - \sqrt{5.18^2 - 4\cdot 2\cdot(-25)} = -5.060239\ldots$$

OPG. 5a $\frac{\partial g}{\partial x} = \underline{y^2 z^3}$ $\frac{\partial g}{\partial y} = \underline{2xyz^3}$ $\frac{\partial g}{\partial z} = \underline{2z + 3xy^2z^2}$

I $(1,1,1)$: $\frac{\partial g}{\partial x} = \underline{1}$ $\frac{\partial g}{\partial y} = \underline{2}$ $\frac{\partial g}{\partial z} = 2+3 = \underline{5}$

$dg = \underline{y^2z^3 \cdot dx + 2xyz^3 \cdot dy + (2z+3xy^2z^2)dz}$

I $(1,1,1)$: $\underline{dg = dx + 2dy + 5dz}$

OPG. 5b $g(x,y,z) = 2$ giver lokalt $dg \approx 0$, dvs.

$y^2z^3dx + 2xyz^3dy + (2z+3xy^2z^2)dz \approx 0$, dvs.

$$dx \approx -\frac{2xyz^3}{y^2z^3}dy - \frac{2z+3xy^2z^2}{y^2z^3}dz$$

$$= -\frac{2x}{y}dy - \frac{2+3xy^2z}{y^2z^2}dz$$

Heraf aflæses $\frac{\partial x}{\partial y} = \underline{-\frac{2x}{y}}$ og $\frac{\partial x}{\partial z} = \underline{-\frac{2+3xy^2z}{y^2z^2}}$

Alternativt: $\frac{\partial x}{\partial y} = -\frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial x}}$ og $\frac{\partial x}{\partial z} = -\frac{\frac{\partial g}{\partial z}}{\frac{\partial g}{\partial x}}$

I $(1,1,1)$: $\frac{\partial x}{\partial y} = \underline{-2}$ og $\frac{\partial x}{\partial z} = \underline{-5}$

OPG 6a $\frac{\partial f}{\partial x} = 3(x-4)^2 + 2(x+y-4) - 8 = 3x^2 - 22x + 2y + 32$

$\frac{\partial f}{\partial y} = 2(x+y-4) - 5 = 2x + 2y - 13$

Stationære punkter: $3x^2 - 22x + 2y + 32 = 0$ to lign. med to ubek.
$2x + 2y - 13 = 0$

Fra Ligning 2 fås $2y = 13 - 2x$.
Ved indsættelse i Ligning 1 fås:

$3x^2 - 24x + 45 = 0$

$x^2 - 8x + 15 = 0$

$(x-3)\cdot(x-5) = 0$

$x = 3$ eller $x = 5$

$x = 3$ giver $y = (13-6)/2 = \frac{7}{2}$

$x = 5$ giver $y = (13-10)/2 = \frac{3}{2}$

Altså $\underline{(3,\frac{7}{2})}$ og $\underline{(5,\frac{3}{2})}$ er de stationære punkter!

OPG. 6b

$\frac{\partial^2 f}{\partial x^2} = \underline{6x - 22}$ $\frac{\partial^2 f}{\partial x \partial y} = \underline{2}$ $\frac{\partial^2 f}{\partial y^2} = \underline{2}$

$(3,\frac{7}{2})$: $AC - B^2 = (-4)\cdot 2 - 2^2 = -12$ <u>sadelpkt.</u>

$(5,\frac{3}{2})$: $AC - B^2 = 8 \cdot 2 - 2^2 = +12$ og $A = 8$ <u>lokalt minimum</u>

OPG. 7 c) Indre: $(5,\frac{3}{2})$ eneste kandidat

$$f(5,\tfrac{3}{2}) = 1^3+(5+\tfrac{3}{2}-4)^2-8(5-4)-5\cdot\tfrac{3}{2} = 1+\tfrac{25}{4}-8-\tfrac{15}{2} = -\tfrac{33}{4} = \underline{-8.25}$$

Rand: $x=4$ giver $f(4,y)=y^2-5y=g(y)$, hvor $y\geq 0$.

$g(y)$ har minimum for $2y-5=0$, dvs $y=\frac{5}{2}$

$$f(4,\tfrac{5}{2}) = (\tfrac{5}{2})^2-5\cdot\tfrac{5}{2} = -\tfrac{25}{4} = \underline{-6.25}$$

$y=0$ giver $f(x,0)=x^3-11x^2+32x-16=h(x)$, hvor $x\geq 4$.

$h(x)$ har minimum for $x=4$ eller $3x^2-22x+32=0$, dvs $x=4$ eller

$$x=\frac{+22\pm\sqrt{22^2-4\cdot3\cdot32}}{6}=\frac{22\pm10}{6}=\begin{cases}\frac{16}{3}=5.33\\ 2 \text{ udenfor intervallet}\end{cases}$$

$$f(4,0)=\underline{0} \qquad f(\tfrac{16}{3})=-\tfrac{176}{27}=\underline{-6.52}$$

min $f(x,y)$ under forudsætning af at $x\geq 4$ og $y\geq 0$ bliver dermed minimum af -8.25, -6.25, 0 og -6.52, dvs <u>minimum er $f(5,\frac{3}{2})=-8.25$</u>

Randundersøgelsen kan undgås ved henvisning til Theorem 13.2.1 i [S&H ny] = Theorem 13.1.2 i [S&H ge], idet området er konvekst og

$\frac{\partial^2 f}{\partial x^2}=6x-22\geq 0$, $\frac{\partial^2 f}{\partial x\partial y}=2\geq 0$ og

$$\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 = 2\cdot(6x-22)-4=12x-48\geq 0$$

i hele området.

OPG. 7 d) I det indre er $(5,\frac{3}{2})$ eneste kandidat (da $(3,\frac{7}{2})$ er et sadelpkt.).

Men $f(0,0)$ er -16, så $f(5,\frac{3}{2})=-8.25$ er <u>ikke</u> minimum.

Minimum forekommer altså på randen!

OPG. 7d fortsat Rand-undersøgelse:

$y=0$ giver $f(x,0) = x^3-11x^2+32x-16$. Minimum forekommer for $x=0$, $x=2$ eller $x=\frac{16}{3}$ (se 7c).
$f(0,0) = -16$, $f(2,0) = 12$, $f(\frac{16}{3}) = -6.52$, dvs. minimum på x-aksen $\wedge$ $(x \geq 0)$ er $f(0,0) = -16$

$x=0$ giver $f(0,y) = y^2-13y-16$, som har minimum for $2y-13=0$, dvs $y=\frac{13}{2}$.
$f(0,\frac{13}{2}) = (\frac{13}{2})^2 - 13\cdot\frac{13}{2} - 16 = -\frac{233}{4} = -58.25$
Minimum er dermed $f(0,\frac{13}{2}) = -58.25$

OPG. 8a) $L(x,y) = -x^2-4y^2-2xy+50x+100y-\lambda(5x+10y-K)$

$\frac{\partial L}{\partial x} = -2x-2y+50-5\lambda$ $\qquad$ $\frac{\partial L}{\partial y} = -8y-2x+100-10\lambda$

De tre ligninger er dermed:
$-2x-2y+50-5\lambda=0$, $-8y-2x+100-10\lambda=0$, $5x+10y=K$

Af ligning 1 og 2 fås:
$100-10\lambda = 4x+4y$ (lign 1) og $100-10\lambda = 8y+2x$ (lign. 2),
dvs. $4x+4y = 8y+2x$, dvs $x=2y$.
Af $5x+10y=K$ og $x=2y$ fås så $y=\frac{K}{20}$ og $x=2y=\frac{K}{10}$
Ved indsættelse i ligning 1 fås: $-\frac{K}{5}-\frac{K}{10}+50-5\lambda=0$,
dvs. $\lambda = 10-\frac{K}{25}-\frac{K}{50}$, dvs. $\lambda = 10-\frac{3K}{50}$
Løsningen er altså $(x,y,\lambda) = (\frac{K}{10}, \frac{K}{20}, 10-\frac{3K}{50})$

OPG. 8b Maximum fås som løsning til ligningssystemet, dvs. maximum $= f(\frac{K}{10},\frac{K}{20}) =$
$-\frac{K^2}{100} - 4\frac{K^2}{400} - 2\cdot\frac{K^2}{200} + 50\cdot\frac{K}{10} + 100\cdot\frac{K}{20} = -\frac{3K^2}{100}+10K$
For $K=100$ fås maximum $= -300+1000 = 700$
(Det ses at $(-\frac{3K^2}{100}+10K)' = -\frac{6K}{100}+10 = -\frac{3K}{50}+10$ netop er λ, hvilket teorien jo også siger!)
For $K=100$ er $\lambda = -\frac{300}{50}+10 = 4$. Forøges K fra 100 til 101 forøges maximum derfor med circa 4. (En udregning viser at det nye maximum bliver $-\frac{3(101)^2}{100}+10\cdot 101 = 703.94$).