The maximum of the periodogram of a sequence of functional data

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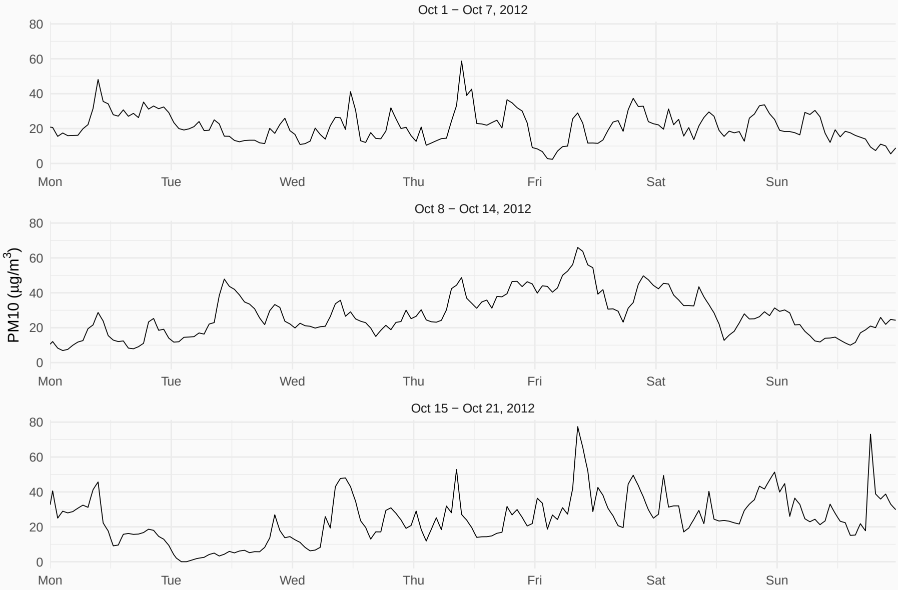
Outline

Motivation and problem

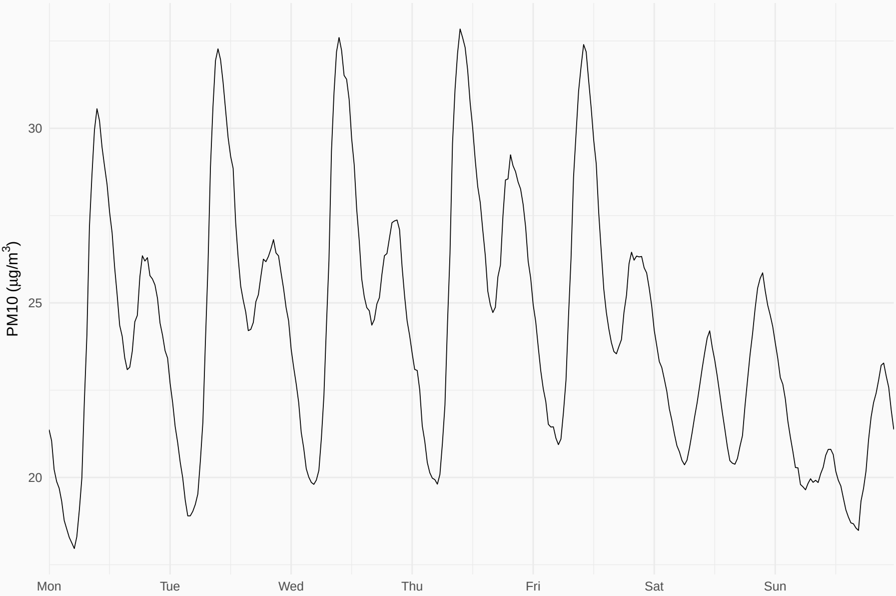
PM10 data

- Air quality data from Graz, Austria.
- The amount of particulate matter with a diameter of 10 µm or less (PM10) is measured.
- PM10 can settle in the bronchi and lungs and cause health problems.
- Starting on February 18, 2010, the amount of PM10 in µg/m^3 is recorded every 30 minutes resulting in 48 observations per day.

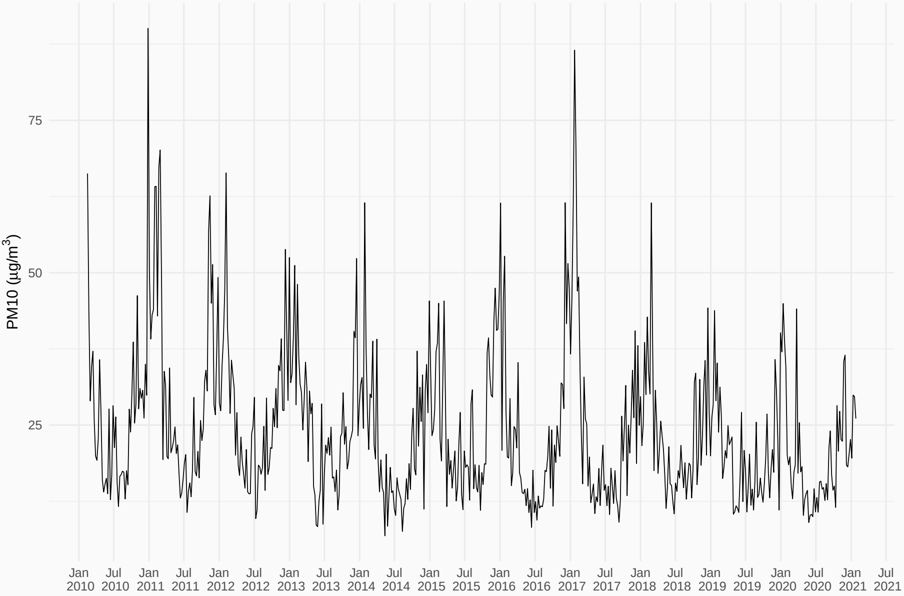
Raw data


Oct 1 − Oct 7, 2012
Oct 8 − Oct 14, 2012
Oct 15 − Oct 21, 2012
PM10 (µg/m3)
80
60
40
20
0
Mon
Tue
Wed
Thu
Fri
Sat
Sun

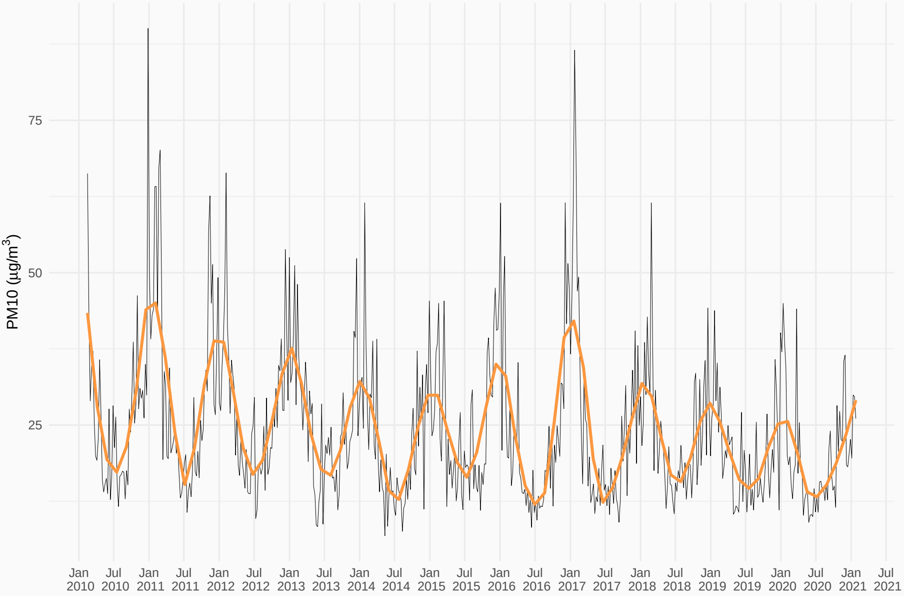
Weekly mean curve



Weekly averages



Weekly averages



The PM10 data as a sequence of curves

We investigate the PM10 data as a functional time series, i.e. as a sequence of daily curves.

Model

$\{X_t\}_{t\in\mathbb{Z}}$ is a time series with values in a real separable Hilbert space $\mathbb{H}$ defined by

$$X_t = \mu + s_t + Y_t$$

for each $t \in \mathbb{Z}$, where

- $\mu \in \mathbb{H}$;
- $\{s_t\}_{t\in\mathbb{Z}} \subset \mathbb{H}$ is a deterministic sequence such that

  $$s_t = s_{t+T} \quad \text{and} \quad \sum_{t=1}^{T} s_t = 0$$

  for all $t \in \mathbb{Z}$ with some $T \geq 2$;
- $\{Y_t\}_{t\in\mathbb{Z}}$ is a stationary sequence of zero mean random elements with values in $\mathbb{H}$.

Hypothesis testing

We develop a methodology to test

$$H_0 : X_t = \mu + Y_t \quad \text{versus} \quad H_1 : X_t = \mu + s_t + Y_t$$

with an unknown $T \geq 2$.

Main results

Frequency domain approach

Our methodology is based on the frequency domain approach to the analysis of functional time series.

Definition

The discrete Fourier transform (DFT) of $X_1, \ldots, X_n$ is defined by

$$\mathcal{X}_n(\omega_j) = n^{-1/2} \sum_{t=1}^{n} X_t e^{-it\omega_j}$$

for $n \geq 1$, where $\omega_j = 2\pi j/n$ with $j = -\lfloor (n-1)/2 \rfloor, \ldots, \lfloor n/2 \rfloor$ and $i = \sqrt{-1}$.

Maximum of periodogram

The test statistic is given by

$$\max_{1 \leq j \leq q} \|\mathcal{X}_n(\omega_j)\|^2$$

for $n > 1$, where

i) $\omega_j = 2\pi j/n$ with $1 \leq j \leq q = \lfloor n/2 \rfloor$;

ii) $\| \cdot \|$ is the norm of the complexification of $\mathbb{H}$.

Linear processes

Suppose that $\{Y_t\}_{t\in\mathbb{Z}}$ is a linear process with values in $\mathbb{H}$ given by

$$Y_t = \sum_{k=-\infty}^{\infty} a_k(\varepsilon_{t-k})$$

for each $t \in \mathbb{Z}$, where

- $\{a_k\}_{k\in\mathbb{Z}} \subset L(\mathbb{H})$;
- $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ are iid zero mean random elements with values in $\mathbb{H}$.

Assumptions

Assumption 1

i) $E\|\varepsilon_0\|^r < \infty$ where $r > 2$ if $\dim \mathbb{H} < \infty$ and $r \geq 4$ otherwise;

ii) the eigenvalues λ_k of $E[\varepsilon_0 \otimes \varepsilon_0]$ are distinct and the sequence $\{k\lambda_k\}_{k\geq 1}$ is ultimately non-increasing;

iii) some technical conditions on the decay rate of $\{\lambda_k\}_{k\geq 1}$.

Assumption 2

i) $\sum_{k\neq 0} \log(|k|)\|a_k\|_{op} < \infty$;

ii) $A^{-1}(\omega)$ exists for each $\omega \in [-\pi, \pi]$, where $A(\omega) = \sum_{k=-\infty}^{\infty} a_k e^{-ik\omega}$ with $\omega \in [-\pi, \pi]$;

iii) $\sup_{\omega\in[0,\pi]} \|A^{-1}(\omega)\|_{op} < \infty$.

Main result

Theorem

Under H_0 and Assumptions 1 and 2, we have that

$$\lambda_1^{-1}\Big(\max_{1\le j\le q}\|A^{-1}(\omega_j)\mathcal{X}_n(\omega_j)\|^2 - b_n\Big) \xrightarrow{d} G \quad \text{as} \quad n\to\infty,$$

where

- $A(\omega_j) = \sum_{k=-\infty}^{\infty} a_k e^{-ik\omega_j}$ with $j = 1, \ldots, q$;
- $b_n = \lambda_1 \log q - \lambda_1 \sum_{j=2}^{\infty} \log(1 - \lambda_j/\lambda_1)$;
- G is the standard Gumbel distribution with the CDF given by $F(x) = \exp\{-\exp\{-x\}\}$ for $x \in \mathbb{R}$.

High-dimensional Gaussian approximation

The core part of the proof is a high-dimensional Gaussian approximation for the DFT developed by Chernozhukov et al. (2017).

FAR(1)

$\{Y_t\}_{t\in\mathbb{Z}}$ is an FAR(1) model given by

$$Y_t = \rho(Y_{t-1}) + \varepsilon_t = \sum_{j=0}^{\infty} \rho^j(\varepsilon_{t-j})$$

for $t \in \mathbb{Z}$ with $\rho \in L(\mathbb{H})$.

Assumption 3

i) There is an $n_0 \geq 1$ such that $\|\rho^{n_0}\| < 1$;

ii) $\hat{\rho}$ is an estimator of ρ such that

$$\|\hat{\rho} - \rho\|_{op} = o_p(1/\tau_n')$$

as $n \to \infty$ with $\tau_n' \geq \log n$.

Residuals and their eigenvalues

- $\{\hat{\varepsilon}_k\}_{2\leq k\leq n}$ are the residuals given by

$$\hat{\varepsilon}_k = X_k - \hat{\rho}\left(X_{k-1}\right)$$

  for $k = 2, \ldots, n$.
- $\{\hat{\lambda}_j\}_{j\geq 1}$ are the eigenvalues of

$$\frac{1}{n-1}\sum_{k=2}^{n}\hat{\varepsilon}_k \otimes \hat{\varepsilon}_k.$$

Test statistic

Theorem

Under H_0 and Assumptions 1 and 3,

$$G_n := \hat{\lambda}_1^{-1} \max_{1 \le j \le q} \|(I - e^{-i\omega_j} \hat{\rho})(\mathcal{X}_n(\omega_j))\|^2 - \log q + \max\left\{ \sum_{j=2}^{\tau_n} \log(1 - \hat{\lambda}_j / \hat{\lambda}_1), c_n \right\} \xrightarrow{d} \mathcal{G}$$

as $n \to \infty$, where $\{\tau_n\}_{n \ge 1} \subset \mathbb{N}$ and $\{c_n\}_{n \ge 1} \subset \mathbb{R}$ are sequences that satisfy certain technical conditions.

Consistency

Theorem

Under H_1,

$$G_n/\ell_n \xrightarrow{p} \infty \quad as \quad n \to \infty$$

for any positive sequence $\ell_n = o(n)$ as $n \to \infty$ provided certain technical conditions are satisfied.

Empirical study

PM10 time series

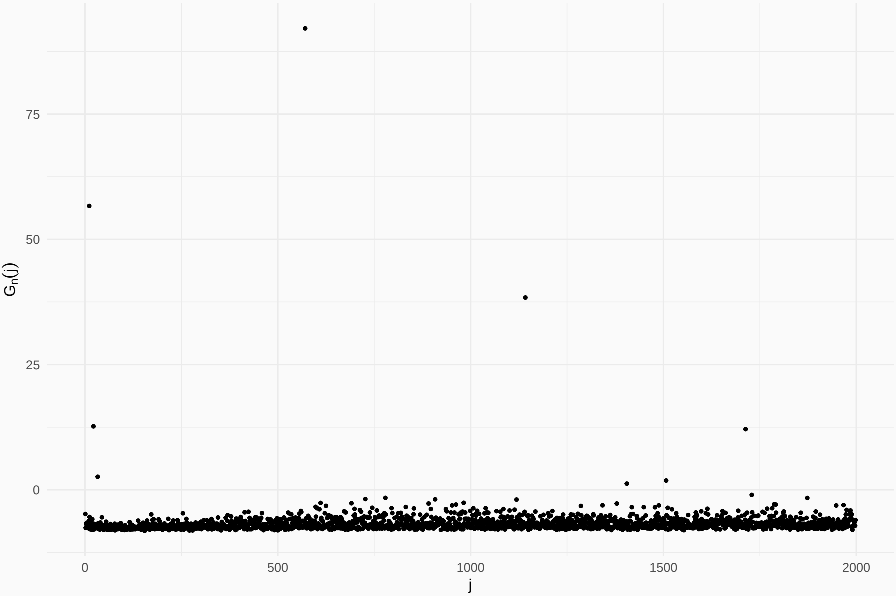
- We plot the points $(j, G_n(j))$ with $j = 1, \ldots, q = 1998$ and

$$G_n(j) := \lambda_1^{-1} \|(I - e^{-i\omega_j} \hat{\rho})(\mathcal{X}_n(\omega_j))\|^2 - \log q + \max\Big\{ \sum_{j=2}^{\tau_n} \log(1 - \hat{\lambda}_j / \hat{\lambda}_1), c_n \Big\},$$

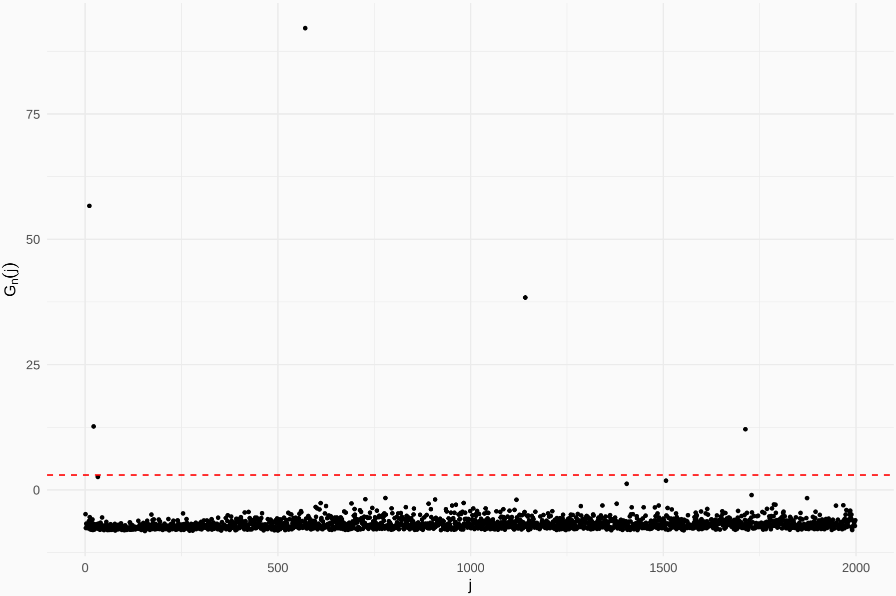
  where $n = 3997$.
- Observe that

$$G_n = \max_{1 \le j \le q} G_n(j).$$

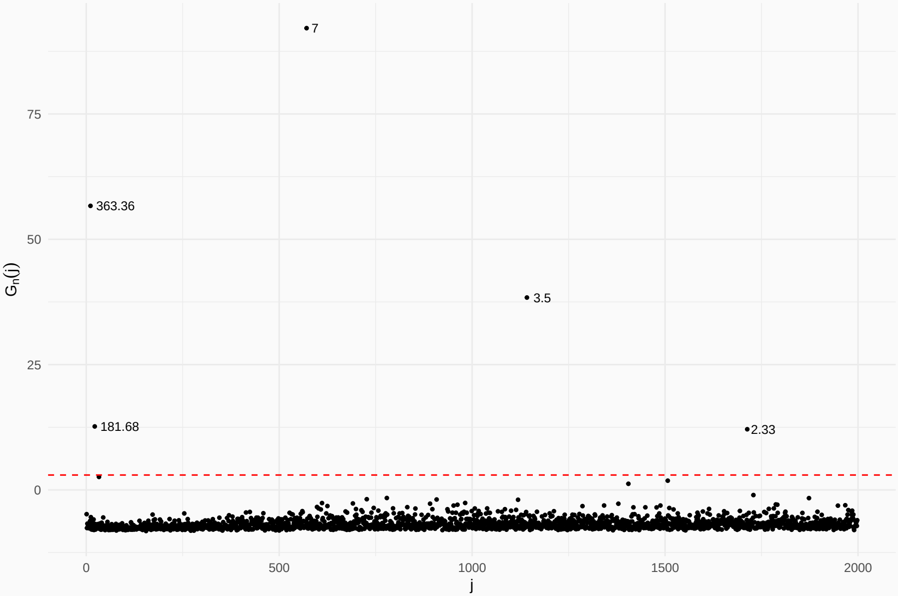
PM10 time series



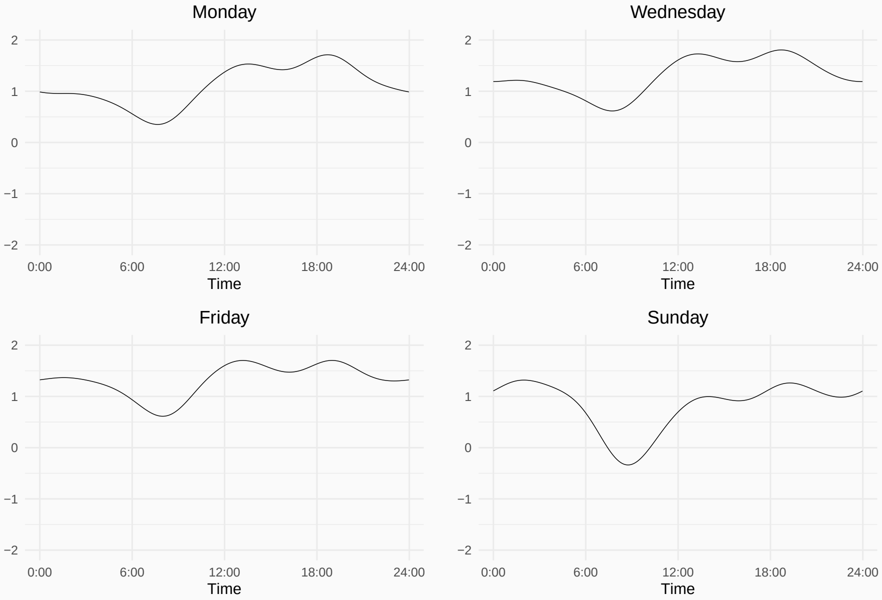
PM10 time series



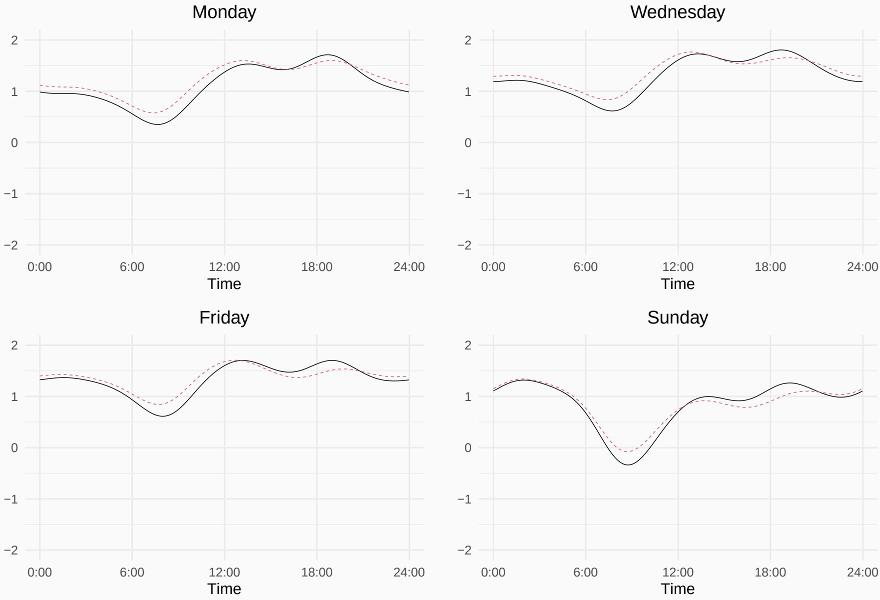
PM10 time series



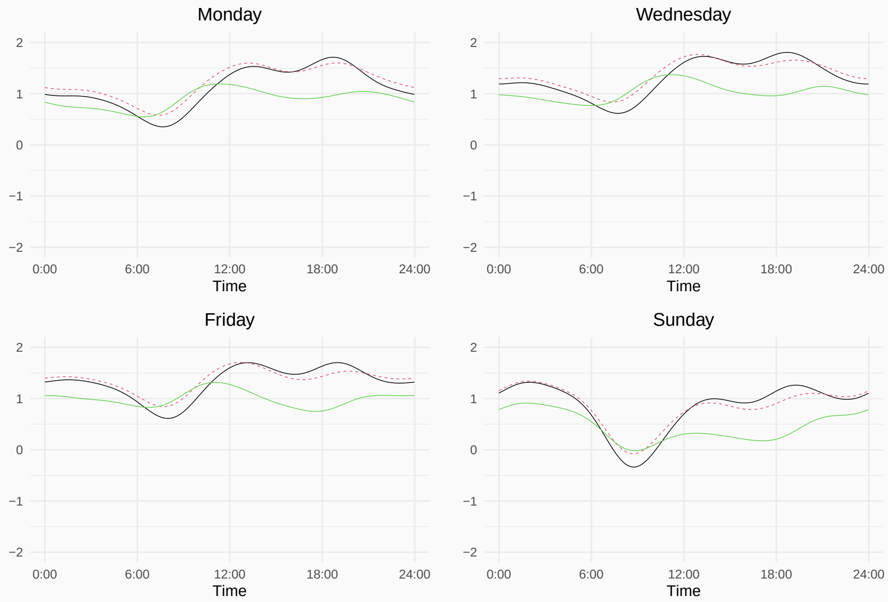
Yearly periodic component



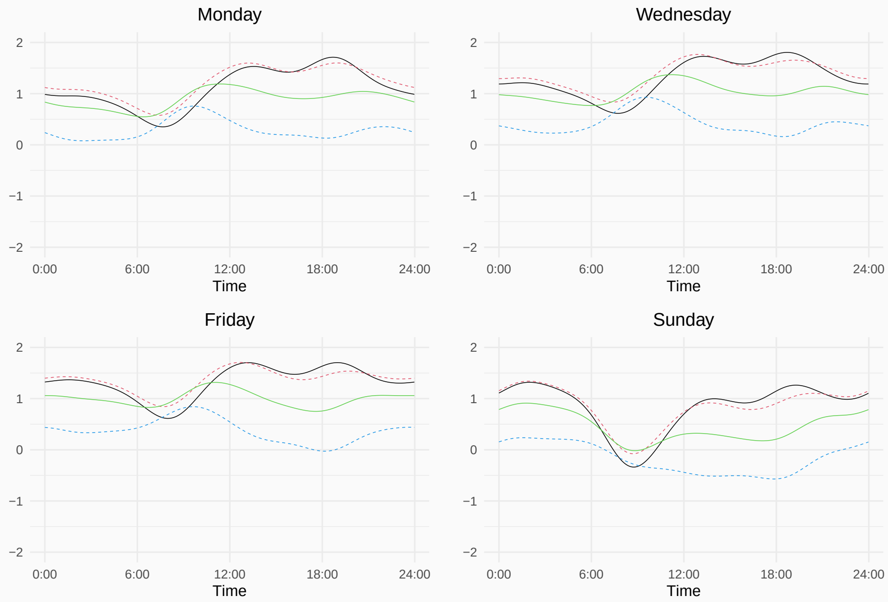
Periodic component



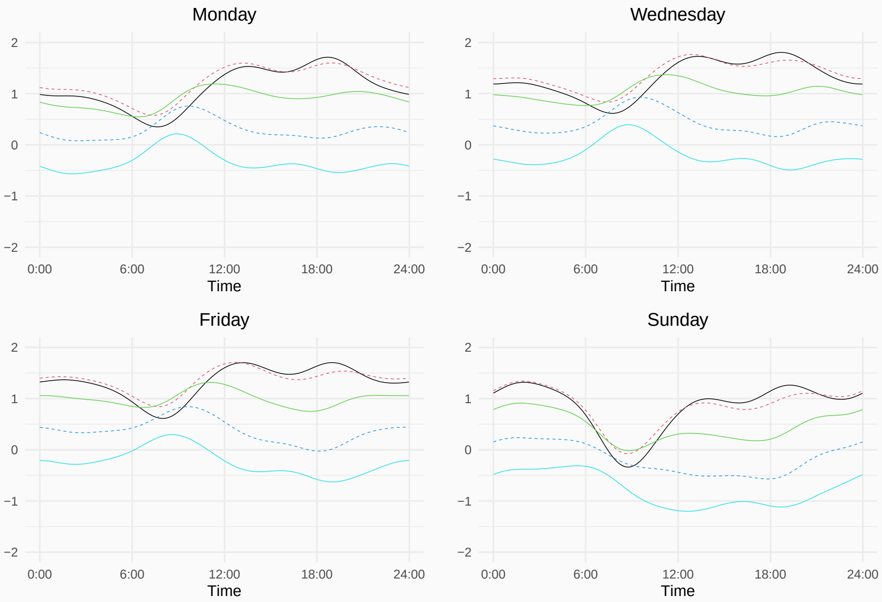
Periodic component



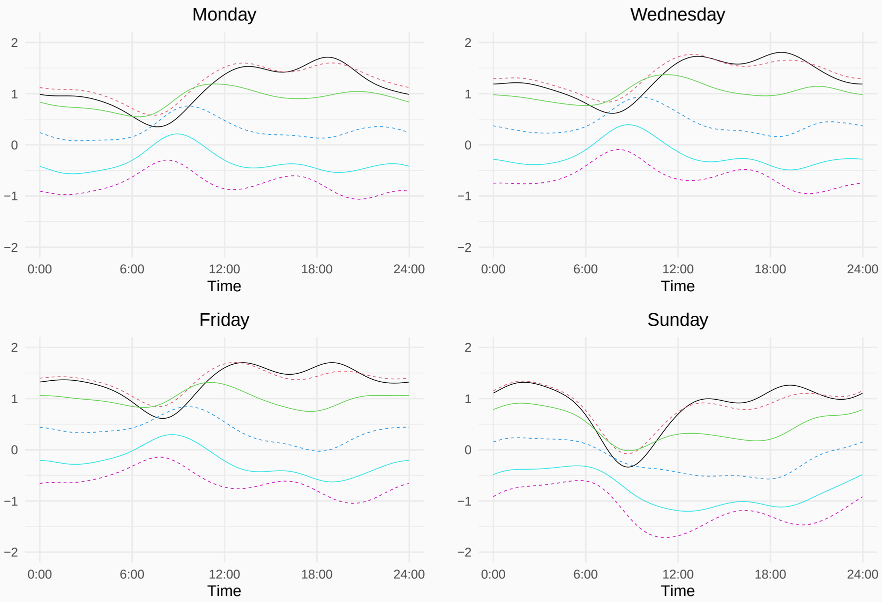
Periodic component



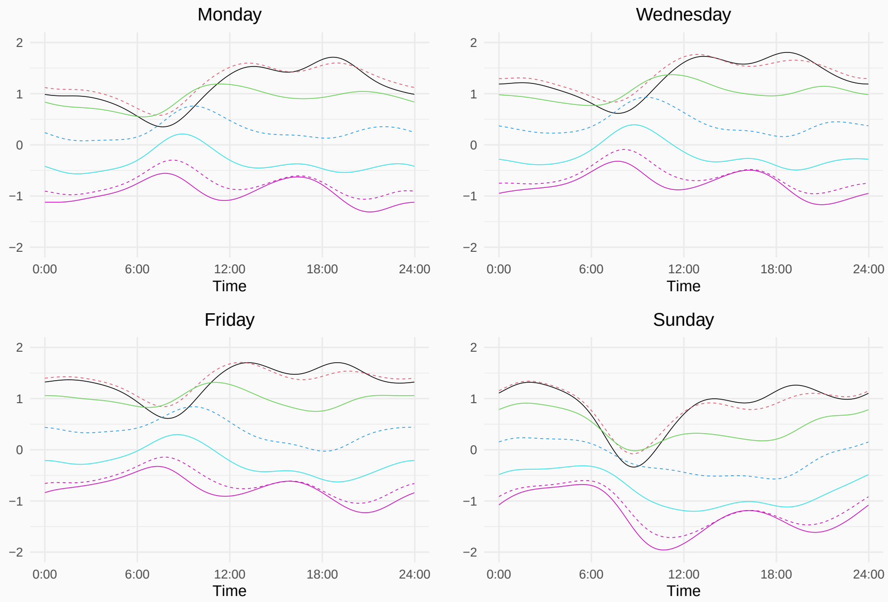
Periodic component



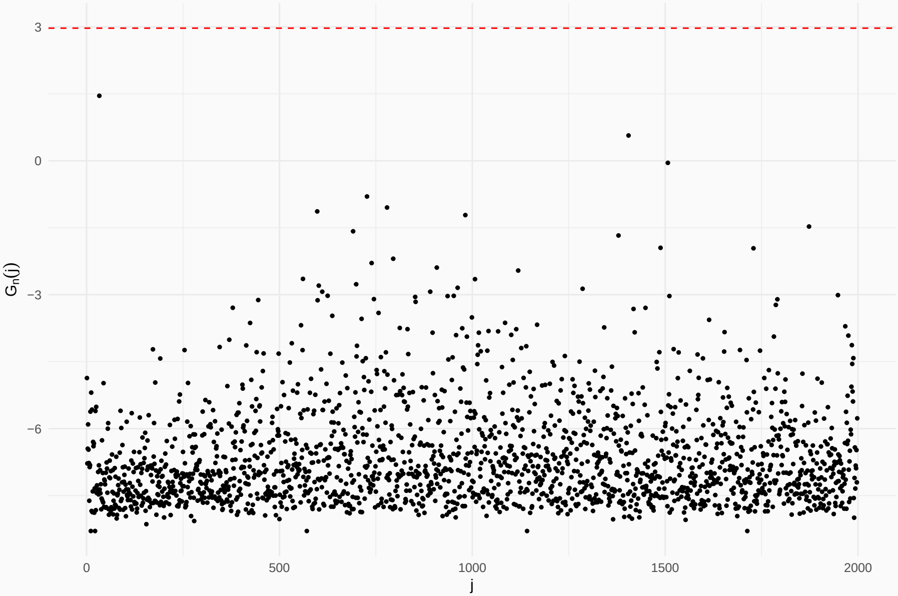
Periodic component



Periodic component



Deseasonalized data



Summary

Summary

- A general test for periodic signals in Hilbert space valued time series when the length of the period is unknown.
- The appropriately standardized maximum of the periodogram converges in distribution to the standard Gumbel distribution.
- A weekly as well as a yearly periodic components are detected in the PM10 data.
- The periodic signals in the PM10 data are not pure sinusoids but are actually driven by several sinusoids.

`https://imada.sdu.dk/~characiejus/`