A general white noise test based on kernel lag-window estimates of the spectral density operator

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Outline

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$\mathbb{H}$-valued time series

$\{X_t\}_{t\in\mathbb{Z}}$ is a stationary sequence of random elements with values in a real separable Hilbert space $\mathbb{H}$ such that $\mathrm{E}\,X_0 = 0$.

Definition

The autocovariance operators $\{\mathcal{C}(j)\}_{j\in\mathbb{Z}}$ of $\{X_t\}_{t\in\mathbb{Z}}$ are defined by

$$\mathcal{C}(j) = \mathrm{E}[X_j \otimes X_0] = \mathrm{E}[\langle \cdot, X_0\rangle X_j]$$

for $j \in \mathbb{Z}$.

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White noise and hypothesis

Definition

$\{X_t\}_{t\in\mathbb{Z}}$ is white noise if X_t's are uncorrelated, i.e. if $\mathcal{C}(j) = 0$ for each $j \neq 0$.

The hypotheses of interest are

$$H_0 : \mathcal{C}(j) = 0 \text{ for all } j \neq 0 \quad \text{versus} \quad H_1 : \mathcal{C}(j) \neq 0 \text{ for some } j \neq 0.$$

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Sample autocovariance operators

Definition

The sample autocovariance operators are defined by

$$\hat{\mathcal{C}}_n(j) = n^{-1} \sum_{t=j+1}^{n} X_t \otimes X_{t-j}$$

for $0 \leq j < n$ and by $\hat{\mathcal{C}}_n(j) = \hat{\mathcal{C}}_n^*(-j)$ for $-n < j < 0$.

Some remarks

- The autocovariance operator at lag $|j| < n$ is estimated using $n - |j|$ observations.
- A starting point could be the test statistic given by

$$\sum_{j=1}^{h} |||\hat{\mathcal{C}}_n(j)|||_2^2,$$

where $|||\cdot|||_2$ is the Hilbert-Schmidt norm and h should grow to infinity as $n \to \infty$.

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Idea

- We propose a test statistic that is based on the estimation of the spectral density function.
- Such an idea was proposed by Hong (1996) in the univariate setting.

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Spectral density function

Definition

The spectral density function is a discrete-time Fourier transform of $\{\mathcal{C}(j)\}_{j\in\mathbb{Z}}$ defined by

$$\mathcal{F}(\omega) = (2\pi)^{-1}\sum_{j\in\mathbb{Z}}\mathcal{C}(j)e^{-ij\omega}$$

for $\omega \in [-\pi, \pi]$ provided that $\sum_{j\in\mathbb{Z}} |||\mathcal{C}(j)|||_2 < \infty$, where $i = \sqrt{-1}$.

If $\{X_t\}_{t\in\mathbb{Z}}$ is white noise, then $\mathcal{F}(\omega) = (2\pi)^{-1}\mathcal{C}(0)$ for $\omega \in [-\pi, \pi]$.

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Distance between $\{X_t\}_{t\in\mathbb{Z}}$ and white noise

The distance is measured by

$$Q^2 = 2\pi \int_{-\pi}^{\pi} \|\|\mathcal{F}(\omega) - (2\pi)^{-1}\mathcal{C}(0)\|\|_2^2 d\omega$$
$$= \sum_{h\neq 0} \|\|\mathcal{C}(h)\|\|_2^2.$$

Hypothesis

The hypothesis that we want to test is as follows

$$H_0 \ : \ Q = 0 \quad \text{versus} \quad H_1 \ : \ Q > 0.$$

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Estimator of spectral density function

Definition

The kernel lag-window estimator of the spectral density function is defined by

$$\hat{\mathcal{F}}_n(\omega) = (2\pi)^{-1} \sum_{|j|<n} k(j/p_n)\hat{\mathcal{C}}_n(j)e^{-ij\omega}$$

for $\omega \in [-\pi, \pi]$, where $k : \mathbb{R} \to [-1, 1]$ is a kernel and $\{p_n\}_{n\geq 1}$ is a bandwidth.

Estimator of the distance to white noise

The estimator of Q is defined by

$$\begin{aligned}\hat{Q}_n^2 &= 2\pi \int_{-\pi}^{\pi} \||\hat{\mathcal{F}}_n(\omega) - (2\pi)^{-1}\hat{\mathcal{C}}_n(0)\||_2^2 d\omega \\ &= 2 \sum_{j=1}^{n-1} k^2(j/p_n)\||\hat{\mathcal{C}}_n(j)\||_2^2.\end{aligned}$$

Test statistic

We propose to use the test statistic T_n defined by

$$T_n = T_n(k, p_n) = \frac{\hat{\sigma}_n^4}{\|\hat{\mathcal{C}}_n(0)\|_2^2} \cdot \frac{2^{-1} n \hat{\sigma}_n^{-4} \hat{Q}_n^2 - C_n(k)}{\sqrt{2 D_n(k)}}$$

for $n \geq 1$, where $\hat{\sigma}^2 = n^{-1} \sum_{t=1}^{n} \|X_t\|^2$,

$$C_n(k) = \sum_{j=1}^{n-1} (1 - j/n) k^2(j/p_n),$$

$$D_n(k) = \sum_{j=1}^{n-2} (1 - j/n)(1 - (j+1)/n) k^4(j/p_n).$$

Asymptotic distribution of the statistic

Theorem

Suppose that

(i) $\{X_t\}_{t\in\mathbb{Z}}$ *are iid* $\mathbb{H}$*-valued random elements such that* $\operatorname{E} X_0 = 0$ *and* $\operatorname{E}\|X_0\|^4 < \infty$*;*

(ii) k *is an even function that is continuous at zero and at all but finite number of points, with* $k(0) = 1$ *and* $k(x) = O(x^{-\alpha})$ *for some* $\alpha > 1/2$ *as* $x \to \infty$*;*

(iii) $p_n \to \infty$ *and* $p_n/n \to 0$ *as* $n \to \infty$*.*

Then

$$T_n \xrightarrow{d} N(0,1)$$

as $n \to \infty$*.*

Consistency of the test

Theorem

Suppose that

(i) $\{X_t\}_{t\in\mathbb{Z}}$ *is a fourth order stationary sequence of zero mean* $\mathbb{H}$*-valued random elements such that* $\sum_{j=-\infty}^{\infty} |||\mathcal{C}(j)|||_1^2 < \infty$ *and* $\sup_{j\in\mathbb{Z}} \sum_{h=-\infty}^{\infty} |||\mathcal{K}_{h+j,h,j}|||_1 < \infty$*, where* $|||\cdot|||_1$ *is the nuclear norm and* $\{\mathcal{K}_{j_1,j_2,j_3}\}_{j_1,j_2,j_3\in\mathbb{Z}}$ *are the fourth order cumulant operators;*

(ii) $p_n \to \infty$ *and* $p_n/n \to 0$ *as* $n \to \infty$*.*

Then

$$(p_n^{1/2}/n)T_n \xrightarrow{p} \frac{2^{-1}Q^2}{|||\mathcal{C}(0)|||_2^2[2D(k)]^{1/2}}$$

as $n \to \infty$*.*

Outline

Simulation setup

We investigate the case when $\mathbb{H} = L^2([0,1], \mathbb{R})$.

The following data generating processes are considered

(i) IID-BM;

(ii) fGARCH$(1,1)$ (Aue, Horváth, and Pellatt (2016));

(iii) FAR$(1,S)$-BM with the kernel of the operator given by $\varphi_c(t,s) = c\exp\{(t^2+s^2)/2\}$ for $t,s \in [0,1]$ and the constant c is chosen so that $\|\varphi_c\| = S \in (0,1)$.

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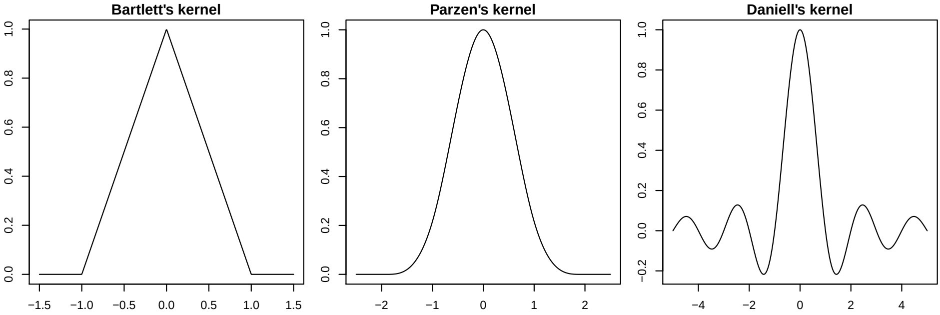
Kernels


Bartlett's kernel
Parzen's kernel
Daniell's kernel
0.0
0.2
0.4
0.6
0.8
1.0
−0.2
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
−2
−1
0
1
2
−4
−2
0
2
4

Bandwidth selection

Similarly as in Bühlmann (1996), we consider bandwidths of the form

$$p_n = n^{1/(2q+1)} \quad \text{and} \quad p_n = \hat{M} n^{1/(2q+1)},$$

where

- q is the largest positive integer such that

  $$\lim_{u \to 0} \{|u|^{-q}[1 - k(u)]\}$$

  exists, is finite and nonzero;
- $\hat{M}$ is a constant estimated from the data.

Monte Carlo simulation

DGP:	IID-BM				fGARCH(1,1)				FAR(1,0.3)-BM			
	$n = 100$		$n = 250$		$n = 100$		$n = 250$		$n = 100$		$n = 250$	
Stat/Nominal Size	5%	1%	5%	1%	5%	1%	5%	1%	5%	1%	5%	1%
$T_n(k_B, n^{1/3})$	50	23	67	34	113	71	142	78	824	749	995	993
$T_n(k_B, \hat{M}n^{1/3})$	58	24	70	43	110	72	118	73	860	786	998	997
$T_n(k_P, n^{1/5})$	57	23	63	38	110	72	130	74	868	788	997	996
$T_n(k_P, \hat{M}n^{1/5})$	58	22	68	41	111	73	121	71	851	776	997	997
$T_n(k_D, n^{1/5})$	54	24	63	36	112	69	134	76	860	782	997	995
$T_n(k_D, \hat{M}n^{1/5})$	58	21	71	29	112	70	127	71	833	761	998	994
$Z_n(10)$	48	9	49	11	50	12	41	5	708	386	992	913
BCD_n	9	0	25	5	24	4	37	4	124	43	376	174

Transformation of test statistic

Chen and Deo (2004) suggest to use the power transformation

$$T_n^{\beta} = \frac{(2^{-1} n \hat{\sigma}_n^{-4} \hat{Q}_n^2)^{\beta} - [C_n^{\beta}(k) + 2^{-1}\beta(\beta-1) C_n^{\beta-2}(k)\hat{\sigma}_n^{-8} \|\|\hat{\mathcal{C}}_n(0)\|\|_2^4 2D_n(k)]}{\beta C_n^{\beta-1}(k)\hat{\sigma}_n^{-4}\|\|\hat{\mathcal{C}}_n(0)\|\|_2^2 [2D_n(k)]^{1/2}}$$

for $n \geq 1$ and $\beta \neq 0$.

The delta method implies that $T_n^{\beta} \xrightarrow{d} N(0,1)$ as $n \to \infty$ for $\beta \neq 0$.

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Approximate skewness

The approximate skewness of T_n^{β} is equal to 0 provided that

$$\beta_1^* = 1 - \frac{2}{3}\frac{[\sum_{j=1}^{n-1} k^2(j/p_n)][\sum_{j=1}^{n-1} k^6(j/p_n)]}{[\sum_{j=1}^{n-1} k^4(j/p_n)]^2}.$$

Monte Carlo simulation (cont.)

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$T_n(k_D, \hat{M}n^{1/5})$	58	21	71	29	112	70	127	71	833	761	998	994
$T_n^{\beta_1^*}(k_B, n^{1/3})$	34	6	51	14	91	41	112	35	793	611	995	977
$T_n^{\beta_1^*}(k_B, \hat{M}n^{1/3})$	36	8	52	14	86	34	93	25	822	629	997	988
$T_n^{\beta_1^*}(k_P, n^{1/5})$	38	10	53	13	87	35	110	31	834	634	997	990
$T_n^{\beta_1^*}(k_P, \hat{M}n^{1/5})$	34	7	53	14	86	35	95	25	812	629	997	989
$T_n^{\beta_1^*}(k_D, n^{1/5})$	32	6	53	12	80	33	102	28	815	610	997	985
$T_n^{\beta_1^*}(k_D, \hat{M}n^{1/5})$	37	4	53	12	82	30	97	23	800	603	997	976
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Problem

Method and test statistic

Simulation study

Summary

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- A general test for white noise for $\mathbb{H}$-valued time series.
- The asymptotic distribution under independence and the consistency of the test.
- The transformed test statistic has good size.
- Better power against functional autoregressive alternatives compared to the existing tests.
- Not well sized for general weak white noise in function space such as for functional GARCH processes.

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