# Asymptotic Behaviour of Functional Linear Processes with Long Memory


Vaidotas Characiejus[1] Alfredas Račkauskas[2]

[1]Fakultät für Mathematik, Ruhr-Universität Bochum, Germany
<vaidotas.characiejus@gmail.com>

[2]Faculty of Mathematics and Informatics, Vilnius University, Lithuania
<alfredas.rackauskas@mif.vu.lt>



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# Outline

# Linear process

Suppose that

- $\mathbb{H}$ is a separable Hilbert space;
- $\{a_j\} = \{a_j : j \geq 0\} \subset L(\mathbb{H})$ are bounded linear operators;
- $\{\varepsilon_k\} = \{\varepsilon_k : k \in \mathbb{Z}\}$ are iid $\mathbb{H}$-valued random elements.

## Definition

A *linear process* is a sequence of $\mathbb{H}$-valued random elements $\{X_k\} = \{X_k : k \in \mathbb{Z}\}$ given by

$$X_k = \sum_{j=0}^{\infty} a_j(\varepsilon_{k-j})$$

for each $k \in \mathbb{Z}$.

# Partial sums and random polygonal functions

## Definition

$\{S_n\} = \{S_n : n \geq 1\}$ are the *partial sums* given by

$$S_n = \sum_{k=1}^{n} X_k$$

for each $n \geq 1$.

## Definition

$\{\zeta_n\} = \{\zeta_n : n \geq 1\}$ are the *random polygonal functions* given by

$$\zeta_n(t) = S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}$$

for each $n \geq 1$ and $t \in [0, 1]$, where $\lfloor \cdot \rfloor$ is the floor function.

# Asymptotic Behaviour

The convergence in some sense of the normalised partial sums and the normalised random polygonal functions as $n \to \infty$ is investigated.

The interesting question is whether the asymptotic behaviour of the linear process $\{X_k\}$ differs from the asymptotic behaviour of iid random elements.

# Absolute summability of $\{a_j\}$

The asymptotic behaviour of a linear process depends on the convergence of the series

$$\sum_{j=0}^{\infty} \|a_j\|_{op},$$

where $\|\cdot\|_{op}$ is the operator norm.

If $\sum_{j=0}^{\infty} \|a_j\|_{op} < \infty$, then the asymptotic behaviour of the linear process $\{X_k\}$ is essentially the same as that of iid random elements.

# CLT when $\sum_{j=0}^{\infty} \|a_j\|_{op} < \infty$

## Theorem

*Suppose that $\{X_k\}$ is an $\mathbb{H}$-valued linear process such that $\sum_{j=0}^{\infty} \|a_j\|_{op} < \infty$, $\mathrm{E}\,\varepsilon_0 = 0$ and $\mathrm{E}\,\|\varepsilon_0\|^2 < \infty$. Then*

$$\frac{S_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, AC_{\varepsilon_0}A^*) \quad \text{as} \quad n \to \infty$$

*in the space $\mathbb{H}$, where*

- *$\mathcal{N}$ is an $\mathbb{H}$-valued Gaussian random element;*
- *$C_{\varepsilon_0}$ is the covariance operator of $\varepsilon_0$;*
- *$A = \sum_{j=0}^{\infty} a_j$ and $A^*$ is the adjoint operator of $A$.*

Merlevède, Peligrad and Utev (1997); Račkauskas and Suquet (2010)

# FCLT when $\sum_{j=0}^{\infty} \|a_j\|_{op} < \infty$

## Theorem

*Suppose that $\{X_k\}$ is an $\mathbb{H}$-valued linear process such that $\sum_{j=0}^{\infty} \|a_j\|_{op} < \infty$, $\mathrm{E}\,\varepsilon_0 = 0$ and $\mathrm{E}\,\|\varepsilon_0\|^2 < \infty$. Then*

$$\frac{\zeta_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} W_{AC_{\varepsilon_0}A^*} \quad \text{as} \quad n \to \infty$$

*in the space $C([0,1];\mathbb{H})$, where*

- $W_{AC_{\varepsilon_0}A^*}$ *is the Wiener process with values in $\mathbb{H}$;*
- $C_{\varepsilon_0}$ *is the covariance operator of $\varepsilon_0$;*
- $A = \sum_{j=0}^{\infty} a_j$ *and $A^*$ is the adjoint operator of $A$.*

Račkauskas and Suquet (2010)

# Memory of a linear process

A linear process $\{X_k\}$ has *short memory* if

$$\sum_{j=0}^{\infty} \|a_j\|_{op} < \infty$$

in the sense that the asymptotic behaviour of $\{X_k\}$ is the essentially the same as that of iid random elements.

# Main problem

We investigate the asymptotic behaviour of the linear process $\{X_k\}$ with values in a infinite-dimensional separable Hilbert space $\mathbb{H}$ when the operator norms of $\{a_j\}$ are not summable, i.e.

$$\sum_{j=0}^{\infty} \|a_j\|_{op} = \infty.$$

The central limit theorem and the functional central limit theorem is investigated for a particular functional linear process.

# Hilbert space $L_2$

$L_2 = L_2[0,1]$ is the Hilbert space of square integrable functions $f : [0,1] \to \mathbb{R}$ with the inner product given by

$$\langle f, g \rangle = \int_0^1 f(r)g(r)\mathrm{d}r,$$

where $f, g \in L_2$.

# Linear process with values in $L_2$

Suppose that $\{X_k\}$ is a linear process with values in $L_2$ and

$$a_j = (j+1)^{-D}$$

for each $j \geq 0$.

The operators $\{(j+1)^{-D} : j \geq 0\}$ are multiplication operators such that

$$(j+1)^{-D} f = \{(j+1)^{-d(t)} f(t) : t \in [0,1]\}$$

for each $f \in L_2$, where $d : [0,1] \to (1/2, \infty)$ is a measurable function.

# Convergence of the series

## Proposition

The series

$$X_k = \sum_{j=0}^{\infty} (j+1)^{-D} \varepsilon_{k-j}$$

converges almost surely if

- $d(t) > 1/2$ for each $t \in [0,1]$;
- the integral
$$\int_0^1 \frac{\sigma^2(s)}{2d(s)-1} \mathrm{d}s$$
is finite, where $\sigma^2(t) = \mathrm{E}\,\varepsilon_0^2(t)$ for $t \in [0,1]$;
- $\mathrm{E}\,\varepsilon_0 = 0$ and $\mathrm{E}\,\|\varepsilon_0\|^2 < \infty$.

# Series $\sum_{j=0}^{\infty} \|(j+1)^{-D}\|_{op}$

## Proposition

If $1/2 < d(t) < 1$ for each $t \in [0,1]$, then

$$\sum_{j=0}^{\infty} \|(j+1)^{-D}\|_{op} = \infty.$$

# Simulated sample paths

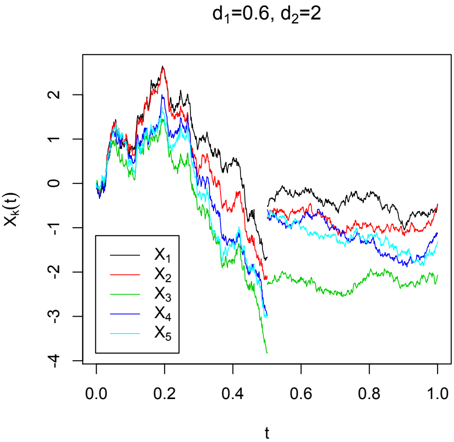
Let us asssume the following

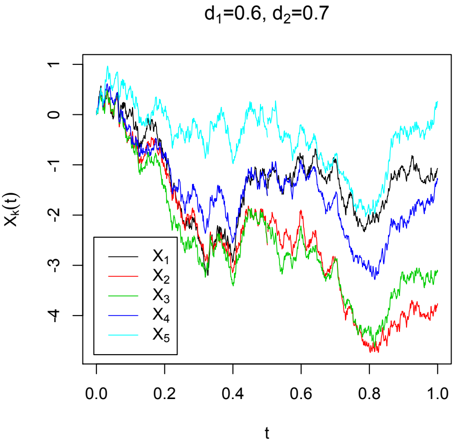
- $\{\varepsilon_k(t) : t \in [0,1]\}_{k\in\mathbb{Z}}$ are iid standard Wiener processes on the interval $[0,1]$;
- $d : [0,1] \to \mathbb{R}$ is a step function defined by

$$d(t) = d_1 \mathbf{1}_{[0,1/2)}(t) + d_2 \mathbf{1}_{[1/2,1]}(t),$$

where $d_1, d_2 \in (1/2, +\infty)$ and $\mathbf{1}_A$ is the indicator function of a set $A$.

# Simulated sample paths



# Norming sequence $\{n^{-H}\}$

$\{n^{-H}\} = \{n^{-H} : n \geq 1\}$ are multiplication operators such that

$$n^{-H} f = \{n^{-[3/2-d(t)]} f(t) : t \in [0,1]\}$$

for each $n \geq 1$ and for each $f \in L_2$.

# CLT for a linear process with values in $L_2$

**Theorem**

*If $1/2 < d(t) < 1$, $\operatorname{E} \varepsilon_0(t) = 0$, $\sigma^2(t) = \operatorname{E} \varepsilon_0^2(t) < \infty$ for each $t \in [0,1]$ and both of the integrals*

$$\int_0^1 \frac{\sigma^2(r)}{[1-d(r)]^2} \mathrm{d}r \quad \text{and} \quad \int_0^1 \frac{\sigma^2(r)}{[1-d(r)][2d(r)-1]} \mathrm{d}r$$

*are finite, then*

$$n^{-H} S_n \xrightarrow{\mathcal{D}} G$$

*in the space $L_2$, where $G = \{G(t) : t \in [0,1]\}$ is a zero mean Gaussian random element with values in $L_2$.*

Ch. and Račkauskas (2013)

# FCLT for a linear process with values in $L_2$

**Theorem**

*If* $1/2 < d(t) < 1$, $\mathsf{E}\,\varepsilon_0(t) = 0$, $\sigma^2(t) = \mathsf{E}\,\varepsilon_0^2(t) < \infty$ *for each* $t \in [0,1]$, *both of the integrals*

$$\mathsf{E}\Big[\int_0^1 \frac{\varepsilon^2(r)}{[1-d(r)]^2}\mathrm{d}r\Big]^{p/2} \quad \textit{and} \quad \int_0^1 \frac{\sigma^2(r)}{[1-d(r)][2d(r)-1]}\mathrm{d}r$$

*are finite and either* $p = 2$ *and* $\operatorname{ess\,sup} d < 1$ *or* $p > 2$, *then*

$$n^{-H}\zeta_n \xrightarrow{\mathcal{D}} \mathcal{G}$$

*in the space* $C([0,1];L_2)$, *where* $\mathcal{G} = \{\mathcal{G}(s,t) : (s,t) \in [0,1]^2\}$ *is a zero mean Gaussian random process with values in* $C([0,1];L_2)$.

Ch. and Račkauskas (2014)

Thank you!