Testing for white noise in functional time series

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Functional time series

$\{X_t\}_{t\in\mathbb{Z}}$ are stationary $L^2[0,1]$-valued random elements.

Definition

The autocovariance kernels $\{r_h\}_{h\in\mathbb{Z}}$ of $\{X_t\}_{t\in\mathbb{Z}}$ are defined by

$$r_h(\tau,\sigma) = \mathrm{Cov}[X_h(\tau), X_0(\sigma)]$$

for each $\tau,\sigma \in [0,1]$ and $h \in \mathbb{Z}$.

White noise and hypothesis

Definition

$\{X_t\}_{t\in\mathbb{Z}}$ is white noise if X_t's are uncorrelated, i.e. if $r_h = 0$ for each $h \neq 0$.

We are interested in testing the hypothesis that $\{X_t\}_{t\in\mathbb{Z}}$ is white noise.

Available tests

Time-domain tests for independence

- Gabrys and Kokoszka [2007], Gabrys, Horváth, and Kokoszka [2010];
- Horváth, Hušková, and Rice [2013].

Frequency-domain test for white noise

- Zhang [2016].

Some parameters need to be selected and/or bootstrap is needed to perform these tests.

Our test

We propose a frequency-domain based test.

The idea is to estimate the minimum distance between $\{X_t\}_{t\in\mathbb{Z}}$ and white noise.

Spectral density kernel

Definition

The spectral density kernel is a discrete-time Fourier transform of $\{r_h\}_{h\in\mathbb{Z}}$ defined by

$$f_\omega = \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \exp(-i\omega h) r_h$$

for $\omega \in [-\pi, \pi]$ provided that $\sum_{h\in\mathbb{Z}} \|r_h\|_2 < \infty$.

If $\{X_t\}_{t\in\mathbb{Z}}$ is white noise, then $f_\omega = (2\pi)^{-1} r_0$.

Distance to white noise

We measure the distance between f_ω, $\omega \in [-\pi, \pi]$, and a spectral density kernel $f \in L^2([0,1] \times [0,1])$ corresponding to white noise using the distance function

$$M^2(f) = \int_{-\pi}^{\pi} \|f_\omega - f\|_2^2 d\omega.$$

The minimum of distance function is given by

$$m^2 = \min_f M^2(f) = \int_{-\pi}^{\pi} \|f_\omega - \tilde{f}\|_2^2 d\omega,$$

where $\tilde{f}(\tau, \sigma) = (2\pi)^{-1} \int_{-\pi}^{\pi} f_\omega(\tau, \sigma) d\omega$ for each $\tau, \sigma \in [0, 1]$.

Hypothesis

The hypothesis that we test is as follows

$$H_0 \ : \ m^2 = 0 \quad \text{versus} \quad H_1 \ : \ m^2 > 0.$$

To perform this test, we need an estimator of the minimum distance m^2.

fDFT and periodogram kernel

Definition

The functional discrete Fourier transform (fDFT) is defined as

$$\widetilde{X}_\omega^{(T)} = \frac{1}{\sqrt{2\pi T}} \sum_{t=0}^{T-1} \exp(-i\omega t) X_t$$

for $\omega \in [-\pi, \pi]$ and $T \geq 1$.

Definition

The periodogram kernel is defined as

$$p_\omega^{(T)}(\tau, \sigma) = [\widetilde{X}_\omega^{(T)}(\tau)][\overline{\widetilde{X}_\omega^{(T)}(\sigma)}]$$

for each $\tau, \sigma \in [0, 1]$, where $\bar{x}$ is the complex conjugate of $x \in \mathbb{C}$.

Estimator of minimum distance

To estimate the minimum distance, we avoid direct estimation of the spectral density kernel and propose to use sums of periodograms.

The estimator is defined as

$$\hat{m}_T = 2\pi\left[\frac{2}{T}\sum_{k=2}^{\lfloor T/2\rfloor}\langle p_{\omega_k}^{(T)}, p_{\omega_{k-1}}^{(T)}\rangle - \left\|\frac{1}{T}\sum_{k=1}^{\lfloor T/2\rfloor}[p_{\omega_k}^{(T)} + \bar{p}_{\omega_k}^{(T)}]\right\|_2^2\right],$$

where ω_k are the Fourier frequencies defined by $\omega_k = 2\pi k/T$ for $1 \leq k \leq \lfloor T/2\rfloor$ and $T \geq 1$.

Asymptotic distribution of the estimator

Theorem

Suppose that

(i) $\{X_t\}_{t\in\mathbb{Z}}$ *is strictly stationary sequence of* $L^2[0,1]$*-valued random elements such that* $\mathrm{E}\,\|X_0\|_2^k < \infty$ *for each* $k \geq 1$*;*

(ii) $\int_0^1 \int_0^1 \sum_{t_1,t_2,t_3\in\mathbb{Z}} |\,\mathrm{E}[X_{t_1}(\tau)X_{t_2}(\sigma)X_{t_3}(\tau)X_0(\sigma)]|d\tau d\sigma < \infty$*;*

(iii) $\sum_{t_1,\dots,t_{k-1}\in\mathbb{Z}}(1+|t_j|)\|\,\mathrm{cum}(X_{t_1},\dots,X_{t_{k-1}},X_0)\|_2 < \infty$ *for* $j = 1,2,\dots,k-1$ *and all* $k \geq 2$*.*

Then

$$\sqrt{T}(\hat{m}_T - m^2) \xrightarrow{d} N(0,v^2) \quad \text{as} \quad T\to\infty,$$

where v^2 *is the asymptotic variance. Under the null hypothesis,* v^2 *is given by* $v_{H_0}^2 = 8\pi^2\|f_0\|_2^4$*.*

Rejection rule

A consistent estimator of the asymptotic standard deviation under the null hypothesis is given by

$$\widehat{v_{H_0}} = \frac{4\pi}{T} \sum_{k=2}^{\lfloor T/2 \rfloor} \langle p_{\omega_k}^{(T)}, p_{\omega_{k-1}}^{(T)} \rangle$$

for $T \geq 1$.

The null hypothesis is rejected if

$$\hat{m}_T > \frac{\widehat{v_{H_0}}}{\sqrt{T}} z_{1-\alpha},$$

where $z_{1-\alpha}$ is the $(1-\alpha)$-quantile of the standard normal distribution.

Functional time series under the null hypothesis

We simulate

- iid standard Brownian motions;
- iid Brownian bridges;
- the values of FARCH(1) process.

Empirical rejection probabilities under the null hypothesis

	Brownian motions			Brownian bridges			FARCH(1)		
T	10%	5%	1%	10%	5%	1%	10%	5%	1%
128	9.5 (11.0)	4.8 (4.2)	1.1 (0.8)	10.8 (11.0)	5.3 (5.4)	0.8 (1.1)	11.1 (10.7)	5.7 (5.9)	0.8 (0.9)
256	9.6 (10.0)	5.1 (4.2)	1.3 (0.9)	10.3 (9.5)	5.4 (4.8)	0.9 (0.7)	10.9 (11.1)	5.5 (5.2)	0.7 (0.9)
512	10.1 (9.9)	5.1 (4.7)	0.8 (0.6)	9.7 (10.3)	5.1 (5.9)	1.0 (1.3)	10.9 (11.1)	5.3 (4.9)	0.8 (0.7)

The numbers in brackets give the corresponding results of the test of Zhang (2016)

Functional time series under the alternative hypothesis

We simulate observations from the FAR(1) model

$$X_t - \mu = \rho(X_{t-1} - \mu) + \varepsilon_t$$

for $t \geq 1$, where $\rho : L^2[0,1] \to L^2[0,1]$ is an integral operator defined by

$$\rho f(\cdot) = \int_0^1 \mathcal{K}(\cdot, \sigma) f(\sigma) \mathrm{d}\sigma$$

for $f \in L^2[0,1]$ with a kernel $\mathcal{K} \in L^2([0,1] \times [0,1])$ and iid errors $\{\varepsilon_t\}_{t \in \mathbb{Z}}$.

Functional time series under the alternative hypothesis

We consider four different FAR(1) models where the errors are either Brownian motions or Brownian bridges and the kernel of the integral operator is either the Gaussian kernel

$$\mathcal{K}_G(\tau, \sigma) = c_G \exp\left(\frac{\tau^2 + \sigma^2}{2}\right)$$

or the Wiener kernel

$$\mathcal{K}_W(\tau, \sigma) = c_W \min(\tau, \sigma).$$

Empirical rejection probabilities under the alternative

ε_t	Brownian motions					
$\mathcal{K}$	Gaussian			Wiener		
T	10%	5%	1%	10%	5%	1%
128	82.6 (86.1)	80.7 (83.7)	65.9 (58.5)	87.6 (89.9)	82.4 (83.1)	66.9 (59.7)
256	99.0 (99.6)	98.2 (99.2)	98.2 (99.0)	99.4 (99.9)	98.3 (99.5)	94.2 (98.6)
512	99.8 (99.7)	99.6 (99.5)	99.6 (99.0)	99.9 (99.9)	99.9 (99.8)	99.6 (99.1)

The numbers in brackets give the corresponding results of the test of Zhang (2016)

Empirical rejection probabilities under the alternative

ε_t	Brownian bridges					
$\mathcal{K}$	Gaussian			Wiener		
T	10%	5%	1%	10%	5%	1%
128	80.1 (79.2)	77.4 (68.3)	60.1 (54.4)	87.6 (80.2)	79.9 (65.8)	61.2 (58.1)
256	100.0 (100.0)	97.0 (98.2)	95.5 (97.2)	99.9 (100.0)	98.3 (99.1)	98.1 (98.8)
512	100.0 (100.0)	99.3 (98.7)	99.3 (98.1)	100.0 (100.0)	100.0 (100.0)	98.8 (99.1)

The numbers in brackets give the corresponding results of the test of Zhang (2016)

Summary

- We propose a frequency-domain test for white noise (non-correlation) in functional time series.
- Our test requires neither selection of some parameters nor bootstrap to obtain critical values.
- The finite sample performance in testing for white noise is very similar to that of Zhang (2016).