Testing for white noise in functional time series

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Background and motivation

Currently available tests

Gabrys and Kokoszka [2007]

- A time-domain based portmanteau test of independence for functional observations.
- Based on the Karhunen–Loéve expansion.
- The number of principal components and the lag parameter need to be chosen.
- Extended by Gabrys et al. [2010] to test for independence of the errors of a functional linear model.

Horváth, Hušková, and Rice [2013]

- A test of independence that is based on the sum of the L^2 norms of the empirical correlation functions.
- There is no need to fix the number of principal components and the lag parameter goes to infinity with a certain rate as the sample size increases.

Zhang [2016]

- Test based on the L^2 norm of the periodogram function.
- Does not involve the choices of the functional principal components nor the lag truncation number.
- The approach is robust to dependence within white noise.
- The limiting distribution of the test statistic is non-pivotal and a bootstrap procedure is needed to obtain the critical values.

Test that we propose

- A frequency-domain based test with a simple asymptotic distribution.
- We do not need bootstrap nor do we need to choose any regularisation parameters.
- Our test is a generalisation of the test proposed by Dette, Kinsvater, and Vetter [2011].

Notations and definitions

$\{X_t\}_{t\in\mathbb{Z}}$ are *strictly* stationary $L^2([0,1],\mathbb{R})$-valued random elements. The *mean curve* is denoted by

$$\mu(\tau) = \mathrm{E}\, X_0(\tau)$$

provided that $\mathrm{E}\,\|X_0\|_2 < \infty$ and the *autocovariance kernel* at lag $t \in \mathbb{Z}$ is denoted by

$$r_t(\tau,\sigma) = \mathrm{E}[(X_t(\tau) - \mu(\tau))(X_0(\sigma) - \mu(\sigma))]$$

for $\tau, \sigma \in [0,1]$ provided that $\mathrm{E}\,\|X_0\|_2^2 < \infty$. The *spectral density kernel* is defined as

$$f_\omega = \frac{1}{2\pi}\sum_{t\in\mathbb{Z}} \exp(-i\omega t) r_t$$

for $\omega \in [-\pi,\pi]$ provided that $\sum_{t\in\mathbb{Z}}\|r_t\|_2 < \infty$, where $\|\cdot\|_2$ is the norm of $L^2([0,1]^2,\mathbb{R})$.

Distance from white noise

We measure the distance between f_ω, $\omega \in [-\pi,\pi]$, and a spectral density function $f \in L^2([0,1]^2,\mathbb{C})$ corresponding to white noise using the distance

$$M^2(f) = \int_{-\pi}^{\pi} \|f_\omega - f\|_2^2 d\omega.$$

It is possible to minimise the distance function M and obtain an explicit expression of the minimum. The minimum distance is given by

$$m^2 = \min_f M^2(f) = \int_{-\pi}^{\pi} \|f_\omega - \tilde{f}\|_2^2 d\omega,$$

where

$$\tilde{f}(\tau,\sigma) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\omega(\tau,\sigma) d\omega = \frac{1}{2\pi} r_0(\tau,\sigma)$$

for each $\tau,\sigma \in [0,1]$. Alternatively,

$$m^2 = \int_{-\pi}^{\pi} \|f_\omega\|_2^2 d\omega - 2\pi\|\tilde{f}\|_2^2$$

and

$$m^2 = \frac{1}{2\pi}\sum_{t\neq 0}\|r_t\|_2^2,$$

which clearly shows that the minimum distance is equal to 0 if and only if the time series $\{X_t\}_{t\in\mathbb{Z}}$ is uncorrelated.

Estimator of minimum distance

To estimate the minimum distance, we avoid a direct estimation of the spectral density kernel and propose to use sums of periodograms.
The *functional discrete Fourier transform (fDFT)* is defined as

$$\widetilde{X}_\omega^{(T)} = \frac{1}{\sqrt{2\pi T}}\sum_{t=0}^{T-1} \exp(-i\omega t) X_t$$

for $\omega \in [-\pi,\pi]$ and $T \geq 1$. The *periodogram kernel* is then defined as

$$p_\omega^{(T)}(\tau,\sigma) = [\widetilde{X}_\omega^{(T)}(\tau)][\overline{\widetilde{X}_\omega^{(T)}(\sigma)}]$$

for each $\tau,\sigma \in [0,1]$, where $\bar{x}$ denotes the complex conjugate of $x \in \mathbb{C}$.
The estimator of the minimum distance is defined as

$$\hat{m}_T = 2\pi\Big[\frac{2}{T}\sum_{k=2}^{\lfloor T/2\rfloor}\langle p_{\omega_k}^{(T)}, p_{\omega_{k-1}}^{(T)}\rangle - \Big\|\frac{1}{T}\sum_{k=1}^{\lfloor T/2\rfloor}[p_{\omega_k}^{(T)} + \bar{p}_{\omega_k}^{(T)}]\Big\|_2^2\Big],$$

where ω_k are the canonical frequencies defined by $\omega_k = 2\pi k/T$ for $1 \leq k \leq \lfloor T/2\rfloor$ and $T \geq 1$.

Hypothesis

We want to test the hypothesis

$$H_0 : m^2 = 0 \quad \text{versus} \quad H_1 : m^2 > 0.$$

To perform the test, we establish asymptotic distribution of the estimator $\hat{m}_T$.

Asymptotic distribution of estimator

Suppose that $\{X_t\}_{t\in\mathbb{Z}}$ is strictly stationary sequence of $L^2([0,1],\mathbb{R})$-valued random elements with moments of all orders, the integral

$$\int_0^1\int_0^1 \sum_{t_1,t_2,t_3\in\mathbb{Z}} |\,\mathrm{E}[X_{t_1}(\tau)X_{t_2}(\sigma)X_{t_3}(\tau)X_0(\sigma)]|d\tau d\sigma$$

is finite and the series

$$\sum_{t_1,\dots,t_{k-1}\in\mathbb{Z}} (1+|t_j|)\|\operatorname{cum}(X_{t_1},\dots,X_{t_{k-1}},X_0)\|_2$$

converges for $j = 1,2,\dots,k-1$ and all $k \geq 1$. Then

$$\sqrt{T}(\hat{m}_T - m^2) \xrightarrow{d} N(0,v^2) \quad \text{as} \quad T \to \infty,$$

where v^2 is the asymptotic variance. Under the null hypothesis, v^2 is given by $v_{H_0}^2 = 8\pi^2\|f_0\|_2^4$.

Rejection rule

The null hypothesis is rejected if

$$\hat{m}_T > \frac{\widehat{v_{H_0}}}{\sqrt{T}} z_{1-\alpha},$$

where $\widehat{v_{H_0}}$ is a consistent estimator of the asymptotic standard deviation given by

$$\widehat{v_{H_0}} = \frac{4\pi}{T}\sum_{k=2}^{\lfloor T/2\rfloor}\langle p_{\omega_k}^{(T)}, p_{\omega_{k-1}}^{(T)}\rangle$$

for $T \geq 1$ and $z_{1-\alpha}$ is the $(1-\alpha)$-quantile of the standard normal distribution.

Finite sample performance

Under the null hypothesis, we simulate i.i.d. standard Brownian motions and uncorrelated but dependent values of FARCH(1) process.

Table: Empirical rejection probabilities (in percentage) under the null hypothesis. The numbers in brackets give the corresponding results of the test of Zhang [2016].

	Brownian Motion			FARCH(1)		
T	10%	5%	1%	10%	5%	1%
128	9.5 (11.0)	4.8 (4.2)	1.1 (0.8)	11.1 (10.7)	5.7 (5.9)	0.8 (0.9)

Under the alternative hypothesis, we consider the values of FAR(1) process with two different integral operators (Gaussian or Wiener) and i.i.d. standard Wiener processes as errors.

Table: Empirical rejection probabilities (in percentage) under the alternative hypothesis. The numbers in brackets give the corresponding results of the test of Zhang [2016].

Kernel	Gaussian			Wiener		
T	10%	5%	1%	10%	5%	1%
128	82.6 (86.1)	80.7 (83.7)	65.9 (58.5)	87.6 (89.9)	82.4 (83.1)	66.9 (59.7)
256	99.0 (99.6)	98.2 (99.2)	98.2 (99.0)	99.4 (99.9)	98.3 (99.5)	94.2 (98.6)

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