The maximum of the periodogram of a sequence of functional data

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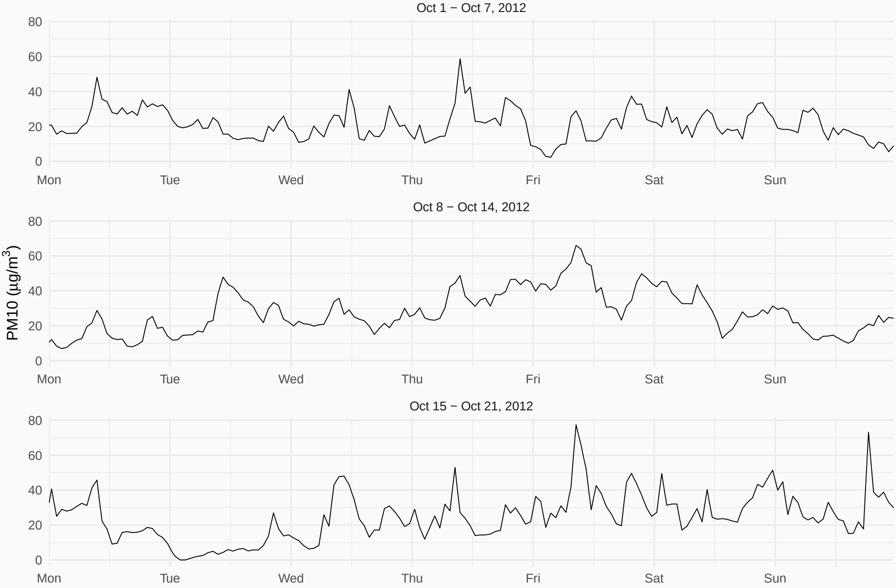
Motivation and problem

PM10 data

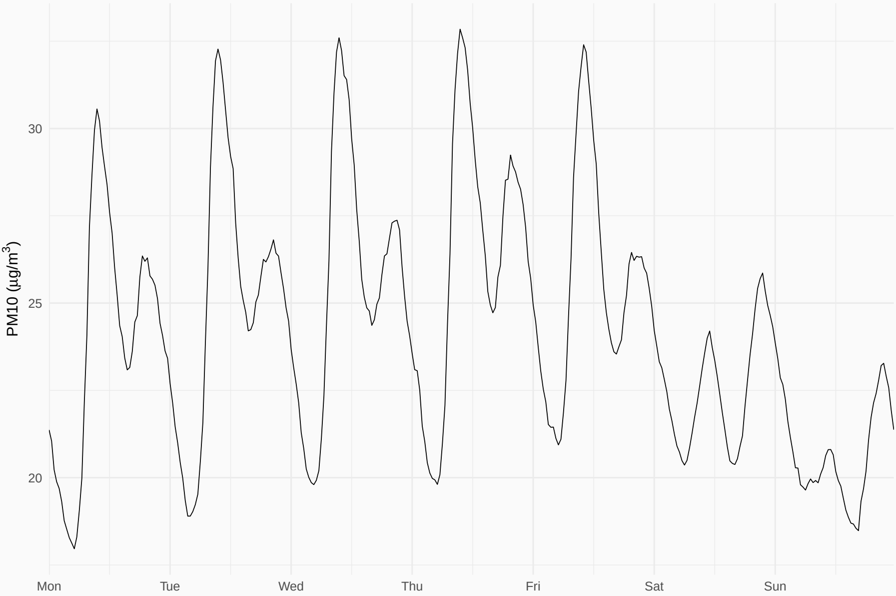
- Air quality data from Graz, Austria.
- The amount of particulate matter with a diameter of 10 µm or less (PM10) is measured.
- PM10 can settle in the bronchi and lungs and cause health problems.
- Starting on February 18, 2010, the amount of PM10 in $\mu g/m^3$ is recorded every 30 minutes resulting in 48 observations per day.

Raw data

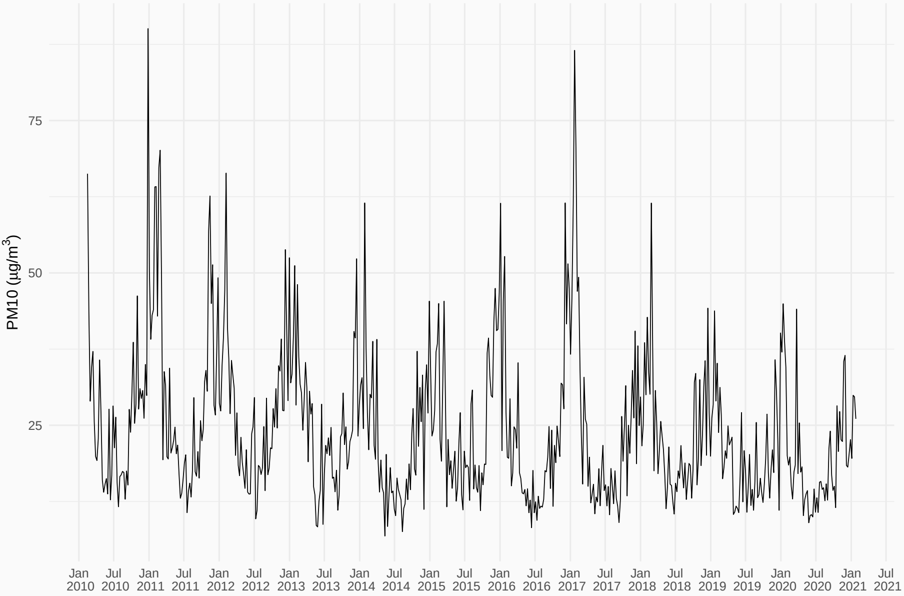


Oct 1 − Oct 7, 2012
Oct 8 − Oct 14, 2012
Oct 15 − Oct 21, 2012
PM10 (μg/m3)
0
20
40
60
80
Mon
Tue
Wed
Thu
Fri
Sat
Sun

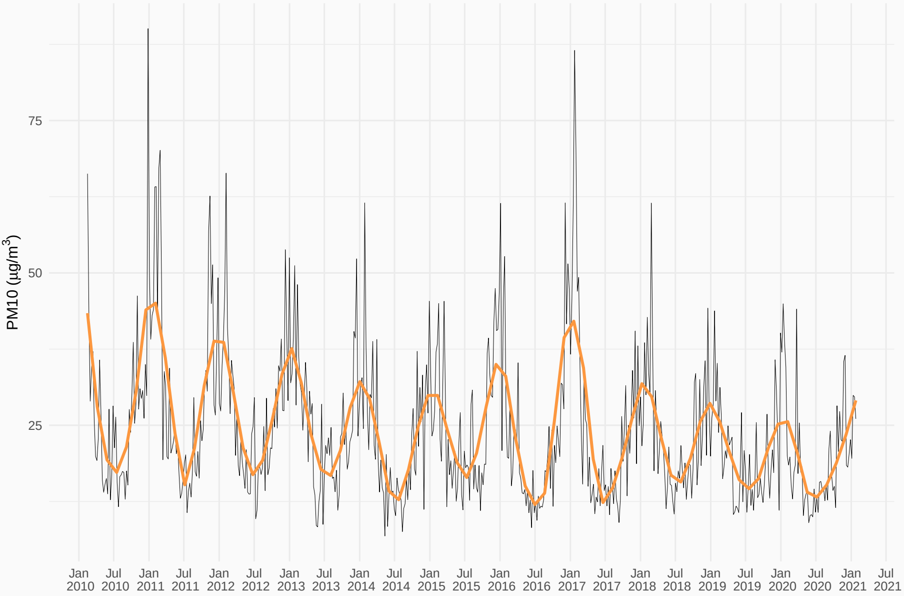
Weekly mean curve



Weekly averages



Weekly averages



Functional time series

- A functional time series is a sequence $\{X_t\}_{t\in\mathbb{Z}}$ such that each X_t is a curve $\{X_t(u)\}_{u\in[0,1]}$.
- We separate a continuous time process $\{\xi(u)\}_{u\in\mathbb{R}}$ using natural consecutive intervals, i.e.

  $$X_t(u) = \xi(t+u)$$

  for $u \in [0,1]$ and $t \in \mathbb{Z}$.
- Such segmentation accounts for a periodic structure in the underlying continuous time process.
- There might still remain a periodic signal with respect to the discrete time parameter $t \in \mathbb{Z}$.

Model

$\{X_t\}_{t\in\mathbb{Z}}$ is a time series with values in a real separable Hilbert space $\mathbb{H}$ (e.g. $L^2[0,1]$) defined by

$$X_t = \mu + s_t + Y_t$$

for each $t \in \mathbb{Z}$, where

- $\mu \in \mathbb{H}$;
- $\{s_t\}_{t\in\mathbb{Z}} \subset \mathbb{H}$ is a deterministic sequence such that

  $$s_t = s_{t+T} \quad \text{and} \quad \sum_{t=1}^{T} s_t = 0$$

  for all $t \in \mathbb{Z}$ with some $T \geq 2$;
- $\{Y_t\}_{t\in\mathbb{Z}}$ is a stationary sequence of zero mean random elements with values in $\mathbb{H}$.

Hypothesis testing

We develop a methodology to test

$$H_0 : X_t = \mu + Y_t \quad \text{versus} \quad H_1 : X_t = \mu + s_t + Y_t$$

with an unknown $T \geq 2$.

Main results

Frequency domain approach

Our methodology is based on the frequency domain approach to the analysis of functional time series.

DFT

Definition

The discrete Fourier transform (DFT) of $X_1, \ldots, X_n$ is defined by

$$\mathcal{X}_n(\omega_j) = n^{-1/2} \sum_{t=1}^{n} X_t e^{-it\omega_j}$$

for $n \geq 1$, where

i) $\omega_j = 2\pi j/n$ with $j = -\lfloor (n-1)/2 \rfloor, \ldots, \lfloor n/2 \rfloor$ are the Fourier frequencies;

ii) $i = \sqrt{-1}$.

Maximum of periodogram

The test statistic is given by

$$M_n = \max_{1 \leq j \leq q} \|\mathcal{X}_n(\omega_j)\|^2$$

for $n > 2$, where

i) $\omega_j = 2\pi j/n$ with $1 \leq j \leq q = \lfloor n/2 \rfloor$;

ii) $\| \cdot \|$ is the norm of the complexification of $\mathbb{H}$.

Maximum of periodogram

The test statistic is given by

$$M_n = \max_{1 \le j \le q} \|\mathcal{X}_n(\omega_j)\|^2$$

for $n > 2$.

- Small values of M_n indicate that there is no periodic component.
- Large values of M_n indicate that there is a periodic component.
- We need a criterion to decide when M_n is small and when M_n is large.

Linear processes

Suppose that $\{Y_t\}_{t\in\mathbb{Z}}$ is a linear process with values in $\mathbb{H}$ given by

$$Y_t = \sum_{k=-\infty}^{\infty} a_k(\varepsilon_{t-k})$$

for each $t \in \mathbb{Z}$, where

- $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ are iid zero mean random elements with values in $\mathbb{H}$;
- $\{a_k\}_{k\in\mathbb{Z}} \subset L(\mathbb{H})$.

Assumptions

Assumption 1

i) $E\|\varepsilon_0\|^r < \infty$ where $r > 2$ if $\dim \mathbb{H} < \infty$ and $r \geq 4$ otherwise;

ii) the eigenvalues λ_k of $E[\varepsilon_0 \otimes \varepsilon_0]$ are distinct and the sequence $\{k\lambda_k\}_{k\geq 1}$ is ultimately non-increasing;

iii) some technical conditions on the decay rate of $\{\lambda_k\}_{k\geq 1}$.

Assumption 2

i) $\sum_{k\neq 0} \log(|k|)\|a_k\| < \infty$;

ii) $A^{-1}(\omega)$ exists for each $\omega \in [-\pi, \pi]$, where $A(\omega) = \sum_{k=-\infty}^{\infty} a_k e^{-ik\omega}$ with $\omega \in [-\pi, \pi]$ is the transfer function;

iii) $\sup_{\omega\in[0,\pi]} \|A^{-1}(\omega)\| < \infty$.

Theorem

Under H_0 and Assumptions 1 and 2, we have that

$$\lambda_1^{-1}\Big(\max_{1\leq j\leq q}\|A^{-1}(\omega_j)\mathcal{X}_n(\omega_j)\|^2 - b_n\Big) \xrightarrow{d} G \quad \text{as} \quad n\to\infty,$$

where

- $A(\omega_j) = \sum_{k=-\infty}^{\infty} a_k e^{-ik\omega_j}$ with $j = 1,\ldots,q$;
- $b_n = \lambda_1 \log q - \lambda_1 \sum_{j=2}^{\infty} \log(1-\lambda_j/\lambda_1)$;
- G is the standard Gumbel distribution with the CDF given by $F(x) = \exp\{-\exp\{-x\}\}$ for $x\in\mathbb{R}$.

FAR(1)

$\{Y_t\}_{t\in\mathbb{Z}}$ is an FAR(1) model given by

$$Y_t = \rho(Y_{t-1}) + \varepsilon_t = \sum_{j=0}^{\infty} \rho^j(\varepsilon_{t-j})$$

for $t \in \mathbb{Z}$ with $\rho \in L(\mathbb{H})$.

Assumption 3

i) There is an $n_0 \geq 1$ such that $\|\rho^{n_0}\| < 1$;

ii) $\hat{\rho}$ is an estimator of ρ such that

$$\|\hat{\rho} - \rho\|_{op} = o_p(1/\tau_n')$$

as $n \to \infty$ with $\tau_n' \geq \log n$.

The transfer function, residuals and their eigenvalues

- $\{\hat{\varepsilon}_k\}_{2\leq k\leq n}$ are the residuals given by

$$\hat{\varepsilon}_k = X_k - \hat{\rho}\,(X_{k-1})$$

  for $k = 2, \ldots, n$.
- $\{\hat{\lambda}_j\}_{j\geq 1}$ are the eigenvalues of

$$\frac{1}{n-1}\sum_{k=2}^{n} \hat{\varepsilon}_k \otimes \hat{\varepsilon}_k.$$

- The transfer function $A(\omega) = (I - e^{-i\omega}\rho)^{-1}$ and hence $A^{-1}(\omega) = I - e^{-i\omega}\rho$ for $\omega \in [-\pi, \pi]$.

Test statistic

Theorem

Under H_0 and Assumptions 1 and 3,

$$G_n := \hat{\lambda}_1^{-1} \max_{1 \le j \le q} \|(I - e^{-i\omega_j}\hat{\rho})(\mathcal{X}_n(\omega_j))\|^2 - \log q + \max\left\{ \sum_{j=2}^{\tau_n} \log(1 - \hat{\lambda}_j/\hat{\lambda}_1), c_n \right\} \xrightarrow{d} \mathcal{G}$$

as $n \to \infty$, where $\{\tau_n\}_{n\ge1} \subset \mathbb{N}$ and $\{c_n\}_{n\ge1} \subset \mathbb{R}$ are sequences that satisfy certain technical conditions.

Consistency

Theorem

Under H_1,

$$G_n/\ell_n \xrightarrow{p} \infty \quad \text{as} \quad n \to \infty$$

for any positive sequence $\ell_n = o(n)$ as $n \to \infty$ provided certain technical conditions are satisfied.

Empirical study

PM10 time series

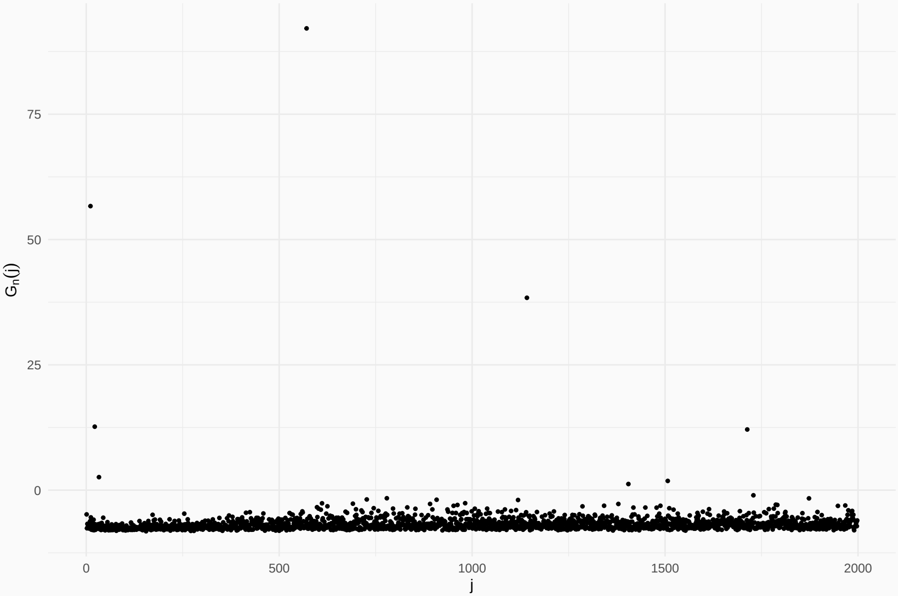
- We plot the points $(j, G_n(j))$ with $j = 1, \ldots, q = 1998$ and

$$G_n(j) := \lambda_1^{-1} \|(I - e^{-i\omega_j} \hat{\rho})(\mathcal{X}_n(\omega_j))\|^2 - \log q + \max\Big\{ \sum_{j=2}^{\tau_n} \log(1 - \hat{\lambda}_j / \hat{\lambda}_1), c_n \Big\},$$

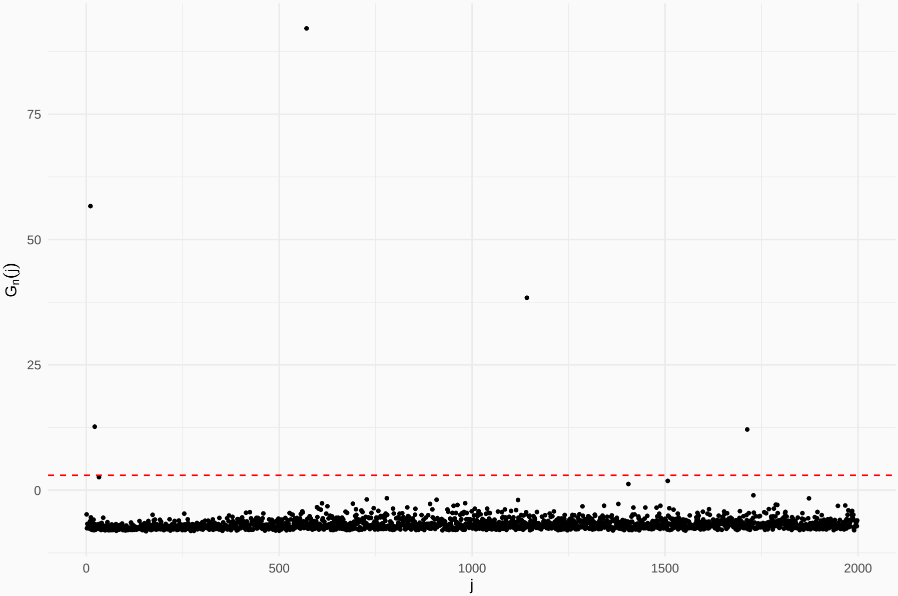
  where $n = 3997$.
- Observe that

$$G_n = \max_{1 \le j \le q} G_n(j).$$

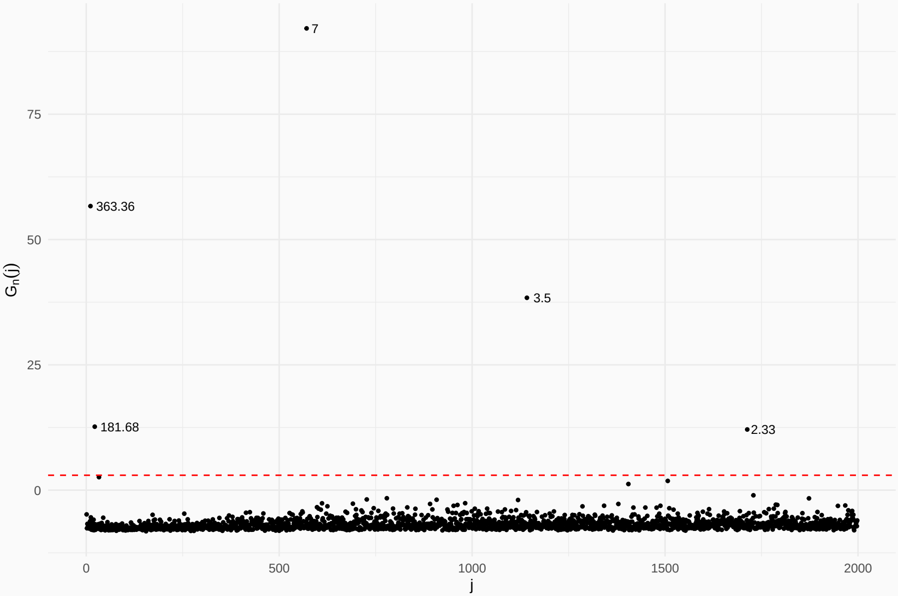
PM10 time series



PM10 time series



PM10 time series



Representation of periodic signals

Lemma

Suppose that $\{s_t\}_{t\in\mathbb{Z}}$ *is a deterministic sequence with values in* $\mathbb{H}$ *such that*

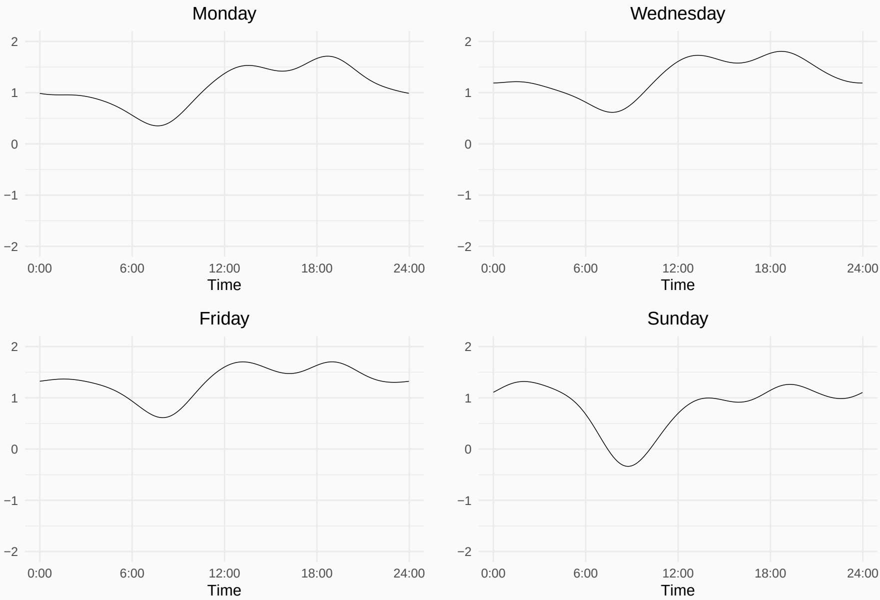
$$s_t = s_{t+T} \quad \text{and} \quad \sum_{t=1}^{T} s_t = 0$$

for all $t \in \mathbb{Z}$ *with some* $T \geq 2$*. Then there exist* $w_{11}, \ldots, w_{1\lfloor T/2\rfloor} \in \mathbb{H}$ *and* $w_{21}, \ldots, w_{2\lfloor T/2\rfloor} \in \mathbb{H}$ *such that*

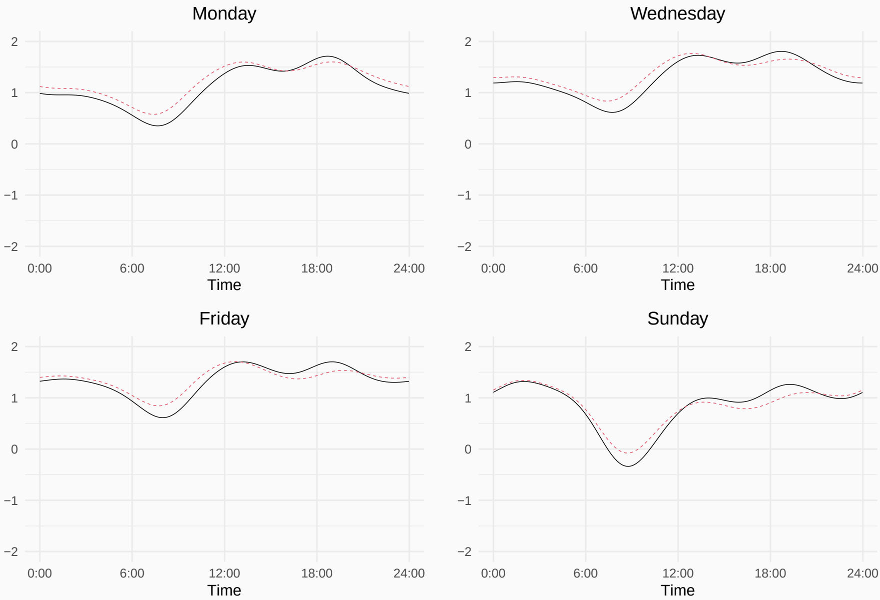
$$s_t = \sum_{k=1}^{\lfloor T/2\rfloor} \left[\cos(2\pi kt/T) w_{1k} + \sin(2\pi kt/T) w_{2k}\right]$$

for all $t \in \mathbb{Z}$.

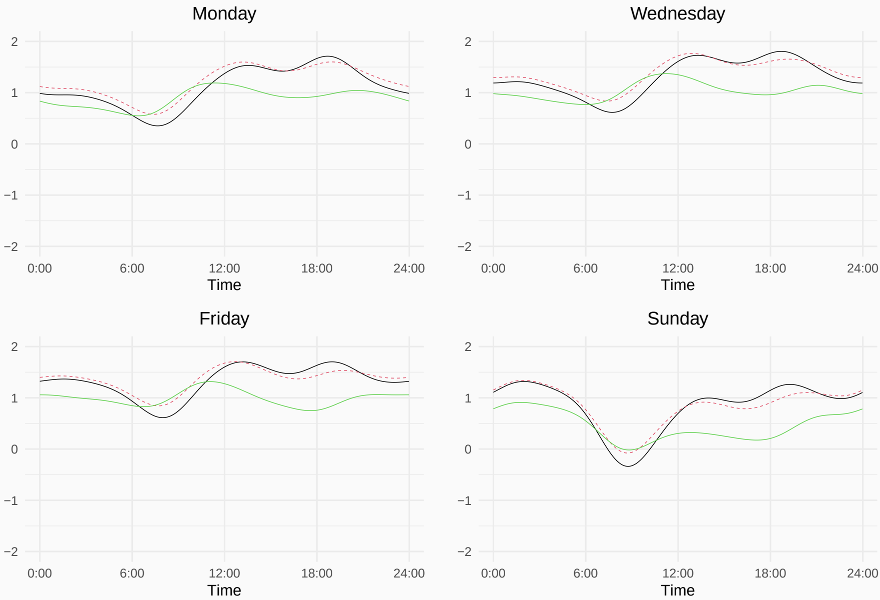
Yearly periodic component



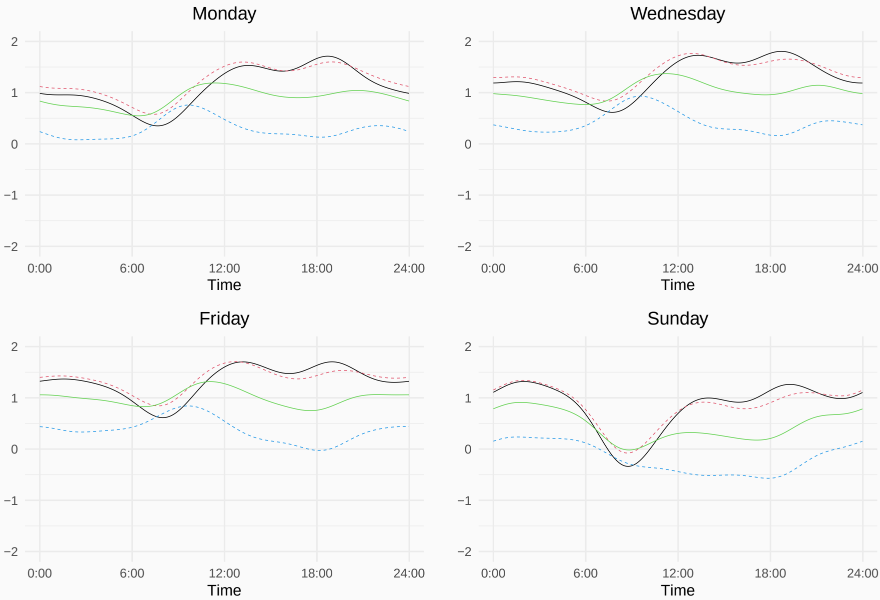
Periodic component



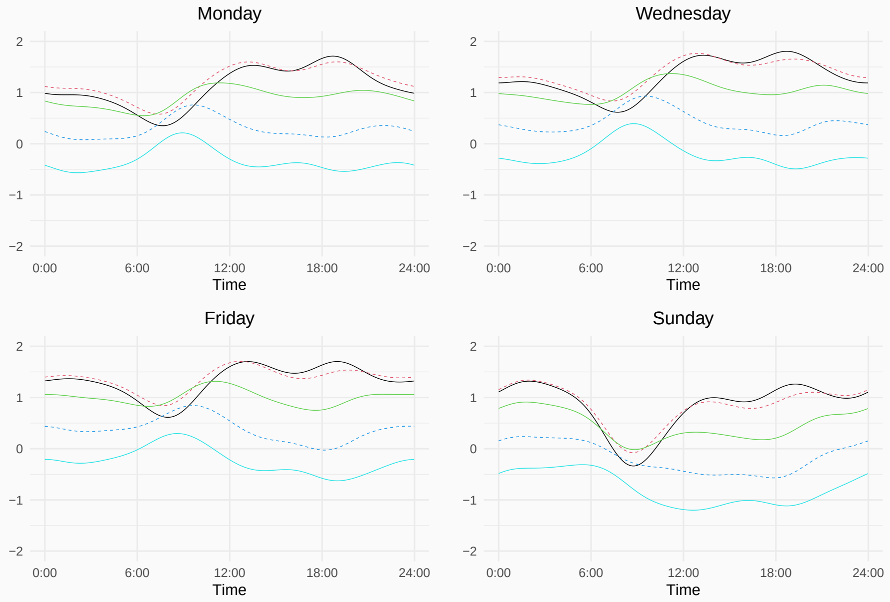
Periodic component



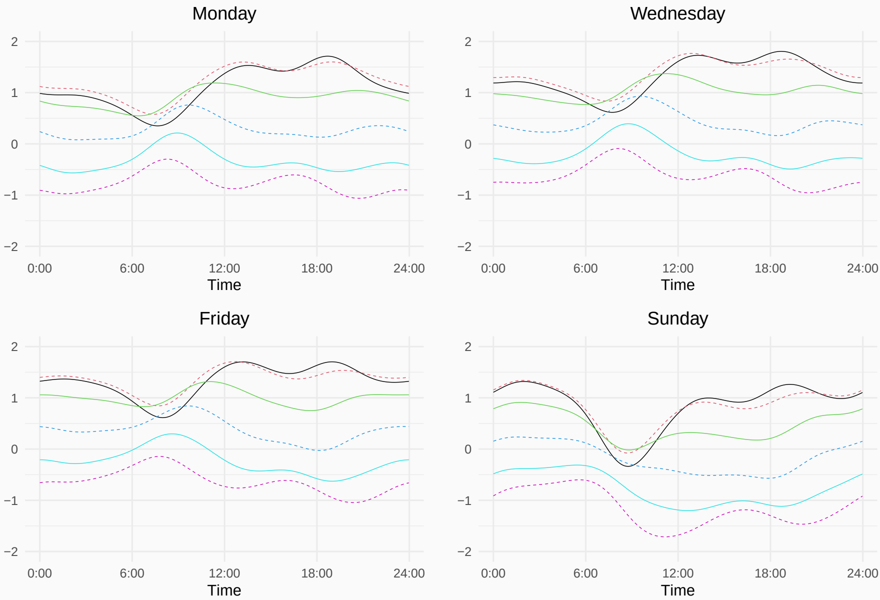
Periodic component



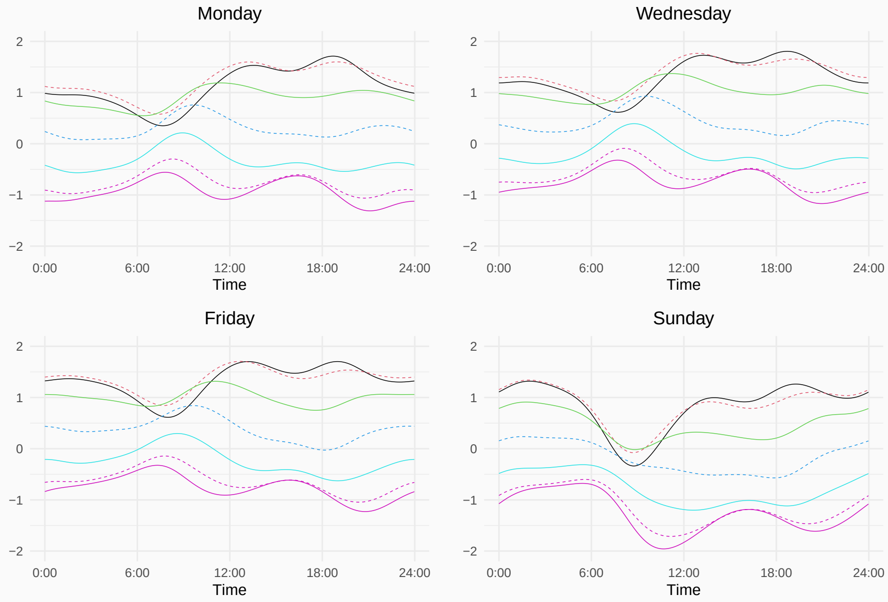
Periodic component



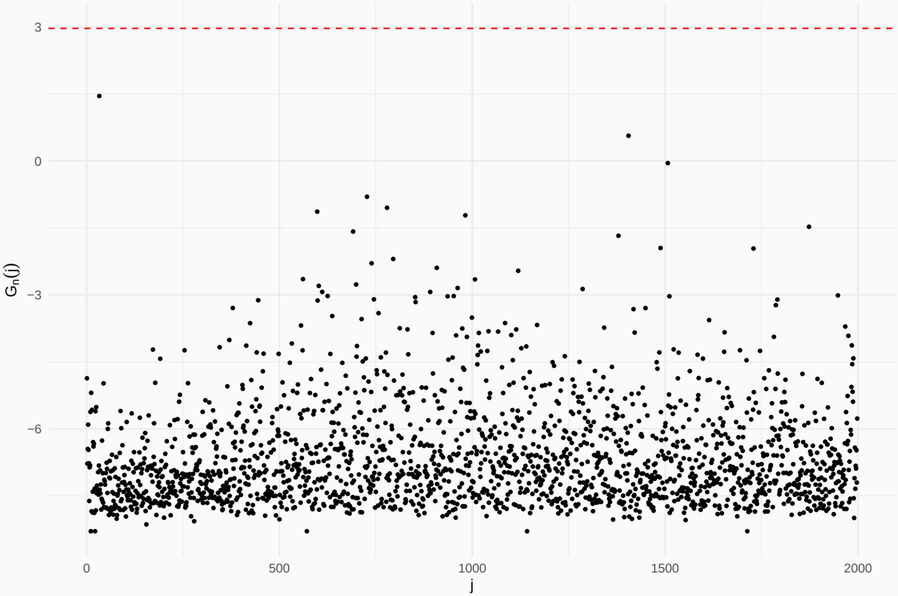
Periodic component



Periodic component



Deseasonalized data



Summary

Summary

- A general test for periodic signals in Hilbert space valued time series when the length of the period is unknown.
- The appropriately standardized maximum of the periodogram converges in distribution to the standard Gumbel distribution.
- A weekly as well as a yearly periodic components are detected in the PM10 data.
- The periodic signals in the PM10 data are not pure sinusoids but are actually driven by several sinusoids.

`https://imada.sdu.dk/u/characiejus/`