Testing for hidden periodicities in functional time series

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Outline

Functional time series with periodicities

Asymptotic results

Summary

Time series with values in $\mathbb{H}$

$\{X_t\}_{t \geq 1}$ is a time series with values in a real separable Hilbert space $\mathbb{H}$ ($\mathbb{R}^d$ with $d \geq 1$, ℓ^2, $L^2[0,1]$, etc.).

Simple model with periodic signal

$\{X_t\}_{t\geq 1}$ is defined by

$$X_t = a\cos(\theta t)\omega + Z_t$$

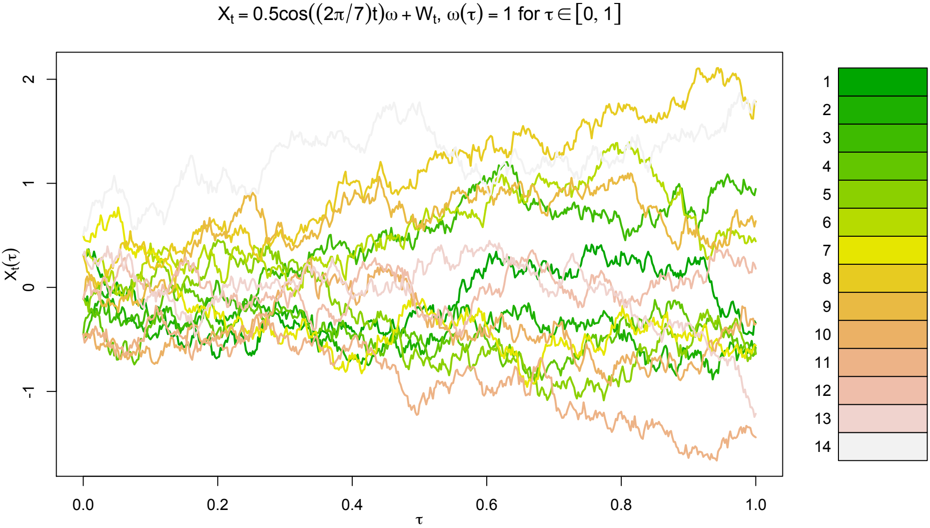
for $t \geq 1$, where

(i) $a \in \mathbb{R}$, $\theta \in [-\pi, \pi]$ and $\omega \in \mathbb{H}$ are non-random;

(ii) $\{Z_t\}_{t\geq 1}$ are iid $\mathbb{H}$-valued random elements with zero means.

The period of $\{X_t\}_{t\geq 1}$ is equal to $d = 2\pi/\theta$.

Simulated example



Hypothesis

The hypothesis that we want to test is as follows

$$H_0 \ : \ a\omega = 0 \quad \text{versus} \quad H_1 \ : \ a\omega \neq 0.$$

To detect periodicities in the data, we propose to use the periodogram.

DFT and periodogram

Definition

The DFT of $\{X_t\}_{1\leq t\leq n}$ is defined by

$$\mathcal{X}_n(\omega) = n^{-1/2}\sum_{t=1}^{n} X_t e^{-it\omega}$$

with $i=\sqrt{-1}$ for $\omega \in [-\pi, \pi]$ and $n \geq 1$.

Definition

The periodogram of $\{X_t\}_{1\leq t\leq n}$ is defined by

$$I_n(\omega) = \mathcal{X}_n(\omega) \otimes \mathcal{X}_n(\omega) = \langle \cdot, \mathcal{X}_n(\omega)\rangle \mathcal{X}_n(\omega)$$

for $\omega \in [-\pi, \pi]$ and $n \geq 1$.

Maximum of periodogram

The test statistic is given by

$$M_n = \max_{1 \leq j \leq q} |||I_n(\omega_j)|||_2 = \max_{1 \leq j \leq q} \|\mathcal{X}_n(\omega_j)\|^2$$

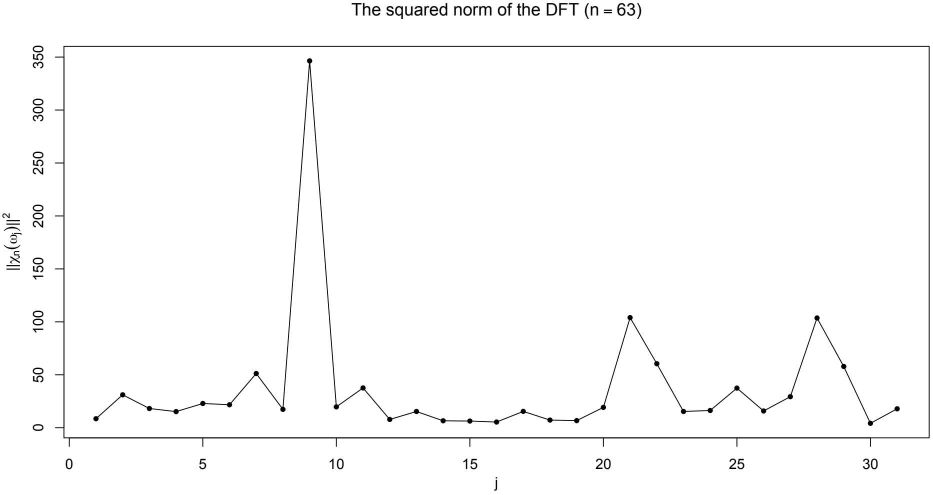
for $n \geq 1$, where

(i) $\omega_j = 2\pi j/n$ are the Fourier frequencies with $1 \leq j \leq q$;

(ii) $|||\cdot|||_2$ is the Hilbert-Schmidt norm;

(iii) $q = \lfloor (n-1)/2 \rfloor$.

Simulated example (cont.)



$\mathbb{H} = \mathbb{R}$

Theorem (Davis and Mikosch (1999))

Let us suppose that $\{Z_t\}_{t \geq 1}$ are iid random variables such that $\mathrm{E}\, Z_1 = 0$, $\mathrm{E}\,|Z_1|^2 = 1$ and $\mathrm{E}\,|Z_1|^s < \infty$ with $s > 2$. Then

$$\max_{1 \leq j \leq q} I_n(\theta_j) - \log q \xrightarrow{d} \mathcal{G} \quad \text{as} \quad n \to \infty,$$

where $q = \lfloor (n-1)/2 \rfloor$ and $\mathcal{G}$ is the standard Gumbel distribution with the CDF given by $F(x) = \exp\{-\exp\{-x\}\}$ for $x \in \mathbb{R}$.

Assumptions

Assumption 1

$\{X_t\}_{t\geq 1}$ are iid zero mean random elements with values in $\mathbb{H}$.

The eigenvectors of $\mathrm{E}[X_1 \otimes X_1]$ are denoted by $\{v_k\}_{k\geq 1}$ and their corresponding eigenvalues are denoted by $\{\lambda_k\}_{k\geq 1}$.

Assumption 2

$\lambda_k > \lambda_{k+1}$ for each $k \geq 1$.

Finite dimensional approximation

Since $\{v_k\}_{k\geq 1}$ is an ONB of $\mathbb{H}$, we have that

$$X_t = \sum_{k=1}^{\infty} \langle X_t, v_k \rangle v_k$$

for $t \geq 1$.

We denote

$$X_t^p = \sum_{k=1}^{p} \langle X_t, v_k \rangle v_k, \quad \mathcal{X}_n^p(\omega) = n^{-1/2} \sum_{t=1}^{n} X_t^p e^{-it\omega}$$

for $t \geq 1$ and $\omega \in [-\pi, \pi]$. We also denote

$$M_n^p = \max_{1 \leq j \leq q} \|\mathcal{X}_n^p(\omega_j)\|^2$$

for $n \geq 1$.

Asymptotic distribution when p is fixed

Theorem 1

If $\mathsf{E}\,\|X_1\|^s < \infty$ *with* $s > 2$ *and* $p \geq 1$ *is fixed, then*

$$a_n^{-1}(M_n^p - b_n^p) \xrightarrow{d} \mathcal{G} \quad \text{as} \quad n \to \infty,$$

where

(i) $a_n = \lambda_1$;

(ii) $b_n^p = \lambda_1 \log(q\alpha_{1,p})$ *with* $q = \lfloor (n-1)/2 \rfloor$ *and*

$$\alpha_{1,p} = \prod_{j=2}^{p} (1 - \lambda_j/\lambda_1)^{-1}.$$

Asymptotic distribution when $p = p_n \to \infty$

Theorem 2

Suppose that $\mathrm{E}\,\|X_1\|^4 < \infty$ *and* $p = p_n \to \infty$ *as* $n \to \infty$. *Then*

$$a_n^{-1}(M_n^{p_n} - b_n^{p_n}) \xrightarrow{d} \mathcal{G} \quad \text{as} \quad n \to \infty$$

provided that

(i) $\{k\lambda_k\}_{k\geq 1}$ *is eventually monotonic;*

(ii) $p_n^3/\lambda_n^{1/2} = o(n^{1/6}/\log^{7/6} n)$ *as* $n \to \infty$;

(iii) $p_n = O(n^{\gamma_0})$ *as* $n \to \infty$ *with*

$$\gamma_0 < \min\{\min_{k\geq 2}\{k^{-1}(\lambda_1/\lambda_k - 1)\}, 1\}.$$

Asymptotic distribution of M_n

Theorem 3

Suppose that $\mathrm{E}\,\|X_1\|^s < \infty$ *with* $s \geq 4$ *and that the assumptions of Theorem 2 are satisfied. Then*

$$a_n^{-1}(M_n - b_n) \xrightarrow{d} \mathcal{G} \quad as \quad n \to \infty$$

provided that there exists a positive sequence $\{\ell_k\}_{k\geq 1}$ *such that* $\sum_{k\geq 1} \ell_k = 1$ *and*

(i) $\sum_{k>p_n} \ell_k^{-s/2}\,\mathrm{E}\,|\langle X_1, v_k\rangle|^s = o(n^{s/2-2})$ *as* $n \to \infty$*;*

(ii) $\sum_{k>p_n} (\lambda_k/\ell_k)^{s/2} = o(n^{-1})$ *as* $n \to \infty$*.*

Summary

- Hidden periodicities in functional time series;
- Asymptotic distribution of the maximum of the periodogram;
- Generalisation of the result of Davis and Mikosch (1999);
- Asymptotic distribution when the dimension of the subspace is fixed or grows to infinity;
- Results hold when the eigenvalues are decaying exponentially or polynomially.