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L^2-Invariants for Quantum Groups

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To my grandfather

Preface

The present text constitutes my dissertation for the PhD degree in mathematics and consists of a detailed treatment of my work done during the period from May 2005 to April 2008. The guiding question throughout my time as a PhD student has been the following.

How can one extend the theory of L^2-invariants from the category of groups to the category of quantum groups?

The thesis suggests one possible answer to this question and discusses its consequences by doing concrete computations and extending classical results from the group case to the quantum group case.

Thanks are due to many people; first of all to my adviser Ryszard Nest for guiding, and to a great extend conducting, my mathematical education during the last five years and for sharing with me his points of view on mathematics in general. Secondly, I thank my fellow students in Copenhagen for keeping me mathematical, as well as non-mathematical, company and the *SNF center in non-commutative geometry* for the financial support during my first two years as a PhD student. Finally, I thank E. Blanchard, A. Thom and S. Vaes for their instructive comments and suggestions in connection with my work.

Copenhagen, April 2008
David Kyed

About this text

The main new results in the thesis all stem from the articles L^2*-homology for compact quantum groups* [Kye06] and L^2*-Betti numbers of coamenable quantum groups* [Kye07], which are included at the end of the thesis. These articles are addressed to people working with L^2-invariants and/or quantum groups, but the present text aims at a broader audience — for instance students working in the field of operator algebras. The thesis is therefore structured such that Chapter 1, 2 and 3 consist, to a great extend, of background material which, to the author's opinion, makes the primary contents of the thesis, contained in Chapter 4 and Chapter 5, easier to digest. Almost no proofs are given throughout the text, but any result stated without proof is followed by a reference — if possible to the original source. The classical theory surveyed in Chapter 1, 2 and 3 deals with the theory of L^2-invariants, the theory of compact quantum groups and the theory of fusion algebras, but is by no means a comprehensive treatment of any of these subjects. The selection of the results presented is mainly based on what is needed in order to understand the contents of Chapter 4 and Chapter 5, which contain a presentation of the author's results from the articles [Kye06] and [Kye07]. To compensate for the sparse treatment of the above mentioned theories, a couple of general references to sources treating the relevant subject in details will be given in the beginning of each chapter.

The article [Kye06] is already accepted for publication in its current form and therefore only minor changes will occur before publication. The article [Kye07] is only submitted for publication and may therefore change dramatically before publication.

Contents

Chapter 1

Introduction to L^2-invariants

The subject of L^2-invariants deals, very roughly, with measuring the size of homological objects relative to an action of a tracial von Neumann algebra. The philosophy is that although these homology modules may be very big and perhaps difficult to compute, it might still be possible to make statements about their sizes by having some suitable notion of dimension to replace the linear dimension in case this is infinite. In this chapter we shall introduce two examples of L^2-invariants, known as the L^2-Betti numbers and the Novikov-Shubin invariants respectively. A good general reference for the subject is W. Lück's book [Lüc02].

1.1 L^2-Betti numbers

The notion of L^2-Betti numbers was introduced by M. F. Atiyah in a differential geometrical setting, and for the sake of completeness we shall begin by introducing this classical approach. Consider a connected manifold X together with a free, proper and cocompact action of a discrete, countable group Γ, and assume moreover that X is endowed with a Γ-invariant Riemannian metric. The canonical example of such an action is the action by deck transformations of the fundamental group $\pi_1(N)$ of a compact, connected manifold N on the universal covering manifold $\tilde{N}$. Denote by Δ_p the closed, densely defined, unbounded Laplacian acting on the space of square integrable p-forms $L^2\Omega^p(X)$. The space $\mathcal{H}^p_{(2)}(X)$ of square integrable, harmonic p-forms on X (i.e. the kernel of Δ_p) becomes a finitely generated Hilbert module for the group von Neumann algebra $\mathscr{L}(\Gamma)$; i.e. there exists an $\mathscr{L}(\Gamma)$-equivariant embedding of $\mathcal{H}^p_{(2)}(X)$ into $\ell^2(\Gamma)^{\oplus n}$ for some $n \in \mathbb{N}$. This follows for instance from the L^2-Hodge-de Rham Theorem ([Lüc02, 1.59]). The p-th L^2-Betti number, $\beta^{(2)}_p(X,\Gamma)$, of the action is then defined as the unnormalized trace of the projection $q = (q_{ij})^n_{i,j=1}$ of $\ell^2(\Gamma)^{\oplus n}$ onto $\mathcal{H}^p_{(2)}(X)$. In symbols:

$$\beta^{(2)}_p(X,\Gamma) = \dim_{\mathscr{L}(\Gamma)} \mathcal{H}^p_{(2)}(X) = \sum_{i=1}^{n} \tau(q_{ii}).$$

Here $\tau\colon B(\ell^2(\Gamma)) \to \mathbb{C}$ is the tracial vector state corresponding to the identity in the group Γ and the dimension $\dim_{\mathscr{L}(\Gamma)}(\cdot)$ appearing in the definition is called the *Murray-von Neumann dimension* of the Hilbert module $\mathcal{H}^p_{(2)}(X)$. In this analytic setting one can derive formulas for computing the L^2-Betti numbers. Applying functional calculus to the self-adjoint, unbounded operator Δ_p we get, for each $t \geq 0$, another operator $e^{-t\Delta_p}$. To this operator an integral kernel

$$(x,y) \longmapsto k(x,y,t) \in \mathrm{Hom}_{\mathbb{R}}(\wedge^p T^*_y X, \wedge^p T^*_x X).$$

(called the heat kernel) is associated; i.e. this section has the property that for every $\omega \in L^2\Omega^p(X)$ and every $x \in X$

$$e^{-t\Delta_p}(\omega)(x) = \int_{\mathcal{F}} k(x,y,t)(\omega(y))\,\mathrm{dvol}(y).$$

Here $\mathcal{F}$ is a fundamental domain for the cocompact action of Γ and vol is the (lift to $\mathcal{F}$ of the) volume form on the compact manifold X/Γ. At each point on the diagonal of $X \times X$, the heat kernel is a self-adjoint endomorphism of a finite dimensional vector space and as such it has well defined (and real) trace, and one can therefore consider the function $\theta_p\colon [0,\infty[\to \mathbb{R}$ given by

$$\theta_p(t) = \int_{\mathcal{F}} \mathrm{Tr}(k(x,x,t))\,\mathrm{dvol}(x).$$

The p-th L^2-Betti number can then be computed ([Lüc02, 3.136]) by the following formula.

$$\beta^{(2)}_p(X,\Gamma) = \lim_{t\to\infty} \theta_p(t).$$

Although defined through analytic data, the L^2-Betti numbers turn out to be invariant under homotopy; so if M and N are homotopic, compact, connected manifolds and Γ denotes $\pi_1(M) = \pi_1(N)$ then

$$\beta^{(2)}_p(\tilde{M},\Gamma) = \beta^{(2)}_p(\tilde{N},\Gamma),$$

for all $p \geq 0$. See e.g. [Dod77] and [BMW97]. The L^2-Betti numbers are therefore to be considered as a rather stable invariant and the homotopy invariance, of course, also makes concrete computations easier. If one only allows contractible manifolds to appear in the definition above, the L^2-Betti numbers become independent of the choice of X and are therefore just invariants of the group Γ. In case such a contractible, Riemannian manifold X exists, the L^2-Betti numbers are referred to as the L^2-Betti numbers of Γ; denoted $\beta^{(2)}_p(\Gamma)$. However, not every discrete group can act freely, properly and cocompactly on a contractible, Riemannian manifold and hence the above definition does not cover all discrete groups. For instance, if Γ acts freely, cocompactly and properly on a contractible manifold X this forces X/Γ to be an Eilenberg-MacLane space for Γ. Since X/Γ is also a manifold this, in turn, forces Γ to be torsion free ([Hat02, 2.45]). Historically, this problem was first circumvented by J. Cheeger and M. Gromov in [CG86], by taking an inverse limit of certain free, cocompact, simplical Γ-complexes. We shall, however, use another method due to W. Lück to bypass the problem and will therefore not elaborate further on the construction of Cheeger and Gromov. The reader is referred to the original article [CG86] and to [Lüc02] for more details. The idea of Lück is to transport the notion of Murray-von Neumann dimension to an algebraic setting and thereafter extend the domain of definition to cover all (algebraic) $\mathscr{L}(\Gamma)$-modules. The details are carried out in the following subsection.

1.1.1 The extended dimension function

Consider a finite von Neumann algebra M endowed with a distinguished, faithful, normal, tracial state τ. Any finitely generated, projective (algebraic) M-module P has the form $M^n A$ for some idempotent matrix $A = (a_{ij})_{i,j=1}^n$ in $\mathbb{M}_n(M)$, and A may actually be chosen self-adjoint. The Murray-von Neumann dimension of P is then defined as

$$\dim_M(P) = \sum_{i=1}^{n} \tau(a_{ii}) \in [0, \infty[.$$

This dimension is independent of the chosen representation of P (as $M^n A$) and is a natural algebraic analogue of the classical notion of Murray-von Neumann dimension of finitely generated Hilbert M-modules. Actually, [Lüc97, 0.1] shows that there is an equivalence of categories

Finitely generated projective M-modules with M-linear homomorphisms	$\xrightarrow[\sim]{\nu}$	Finitely generated Hilbert M-modules with M-equivariant bounded operators

In this language, the above definition boils down to $\dim_M(P) = \dim_M(\nu(P))$. On the algebraic level, the dimension function is extended to arbitrary M-modules by setting

$$\dim'_M(X) = \sup\{\dim_M(P) \mid P \subseteq X, P \text{ finitely generated, projective}\}.$$

Clearly the dimension function $\dim'_M(\cdot)$ may attain the value infinity and, in contrast to $\dim_M(\cdot)$, it is not faithful. I.e. there may exist non-trivial M-modules with dimension zero. But, besides these problems the dimension function $\dim'_M(\cdot)$ is surprisingly well behaved, as the following theorem shows.

Theorem 1.1.1 ([Lüc98, 0.6])**.** *The dimension function* $\dim'_M(\cdot)$ *has the following properties.*

(i) *Extension property; for any finitely generated, projective M-module P we have* $\dim'_M(P) = \dim_M(P)$.

(ii) *Additivity; for any short exact sequence* $0 \to X_1 \to X_2 \to X_3 \to 0$ *of* M*-modules we have*

$$\dim'_M(X_2) = \dim'_M(X_1) + \dim'_M(X_3).$$

(iii) *Cofinality; for any* M*-module* X *and any directed family* $(X_i)_{i\in I}$ *of submodules with* $X = \cup_i X_i$ *we have*

$$\dim'_M(X) = \sup_i \dim'_M(X_i).$$

For a submodule Y *of a finitely generated* M*-module* X*, the algebraic closure of* Y *inside* X *is defined as*

$$\overline{Y}^{\mathrm{alg}} = \bigcap_{\substack{f\in \mathrm{Hom}_M(X,M)\\ Y\subseteq \ker(f)}} \ker(f).$$

The algebraic closure has the following properties.

(iv) *Continuity; if* Y *is a submodule of a finitely generated* M*-module* X *then* $\dim'_M(Y) = \dim'_M(\overline{Y}^{\mathrm{alg}})$.

(v) *If* Y *is a submodule of a finitely generated* M*-module* X *then* $\overline{Y}^{\mathrm{alg}}$ *is a direct summand in* X *and* $X/\overline{Y}^{\mathrm{alg}}$ *is finitely generated and projective.*

(vi) *For a finitely generated module* X*, one defines*

$$T(X) = \overline{\{0\}}^{\mathrm{alg}} \quad \textit{and} \quad P(X) = X/T(X).$$

Then $P(X)$ *is finitely generated and projective,* X *splits as* $P(X) \oplus T(X)$ *and*

$$\dim'_M(X) = \dim_M(P(X)) \quad \textit{and} \quad \dim'_M(T(X)) = 0.$$

Because of part (i) in the above theorem we will henceforth suppress the distinction between $\dim_M(\cdot)$ and $\dim'_M(\cdot)$ and simply denote both dimension functions by $\dim_M(\cdot)$. As the following theorem shows, the extended dimension function is also well behaved with respect to induction.

Theorem 1.1.2 ([Lüc98, 3.3],[Sau02, 3.18])**.** *Let* M *and* N *be finite von Neumann algebras endowed with faithful, normal, tracial states and assume that* $\varphi\colon N \to M$ *is a trace preserving* $*$*-homomorphism. Then for any* N*-module* X *we have*

$$\dim_M(M \odot_N X) = \dim_N(X).$$

Here, and in what follows, the symbol $\odot$ denotes algebraic tensor product.

A finite von Neumann algebra M has the important ring-theoretical property of being semihereditary, i.e. any finitely generated (left) ideal in M is projective considered as an M-module. Nevertheless, from the point of view of homological algebra, there is ring with even better properties containing M, namely the $*$-algebra $\mathscr{U}(M)$ of operators affiliated with M. If M acts on the Hilbert space H then a closed, (possibly unbounded), densely defined operator D on H is said to be *affiliated* with M if the equation

$$U^*DU = D,$$

holds — as an equation of unbounded operators — for all unitaries U in the commutant M'. Because M is assumed to be finite, the set $\mathscr{U}(M)$ becomes a $*$-algebra ([MVN36]) and M is naturally included in $\mathscr{U}(M)$ as its bounded elements. From a ring theoretical point of view, the ring $\mathscr{U}(M)$ can be considered as the Ore localization of M with respect to the set of all non-zerodivisors. It therefore has fine ring theoretical properties, of which the most important is probably that it is *von Neumann regular.* I.e. every $\mathscr{U}(M)$-module is automatically flat. Moreover, as shown by H. Reich, it is possible to obtain a dimension theory for $\mathscr{U}(M)$-modules with the following properties.

Theorem 1.1.3 ([Rei01, 3.11])**.** *There exists a dimension function* $\dim_{\mathscr{U}(M)}(\cdot)$ *on the category of arbitrary $\mathscr{U}(M)$-modules satisfying additivity and cofinality (see Theorem 1.1.1). Moreover, the functor $\mathscr{U}(M)\odot_M -$ is exact and dimension preserving; i.e. for any M-module X we have*

$$\dim_{\mathscr{U}(M)}(\mathscr{U}(M)\odot_M X) = \dim_M(X).$$

With the machinery introduced so far, we may, in particular, consider the dimension of an arbitrary module over a group von Neumann algebra, which leads to the following definition.

Definition 1.1.4 ([Lüc98, 4.1])**.** *For a discrete group Γ its p-th L^2-homology is defined as*

$$H_p^{(2)}(\Gamma) = \mathrm{Tor}_p^{\mathbb{C}\Gamma}(\mathscr{L}(\Gamma), \mathbb{C}),$$

and its p-th L^2-Betti number as

$$\beta_p^{(2)}(\Gamma) = \dim_{\mathscr{L}(\Gamma)} H_p^{(2)}(\Gamma).$$

The above definition requires a bit of explanation; there we consider $\mathbb{C}$ as a left $\mathbb{C}\Gamma$-module via the trivial representation and $\mathscr{L}(\Gamma)$ as a right $\mathbb{C}\Gamma$-module via the left regular representation. Each of the abelian groups $\mathrm{Tor}_p^{\mathbb{C}\Gamma}(\mathscr{L}(\Gamma), \mathbb{C})$ carries a natural left $\mathscr{L}(\Gamma)$-module structure, arising from multiplication from the left, and it is with respect to this action, and the canonical trace, the dimension is calculated.

Remark 1.1.5 (Comparison with the analytic approach)**.** Assume that Γ acts on a contractible, Riemannian manifold X in a free, proper and cocompact way.

In this case we have ([Lüc02, 1.59, 6.53]) an isomorphism of finitely generated Hilbert $\mathscr{L}(\Gamma)$-modules

$$\mathcal{H}^p_{(2)}(X) \simeq H^p_{\mathrm{sing}}(X, \mathscr{L}(\Gamma)),$$

where the right hand side is the singular cohomology of X with coefficients in $\mathscr{L}(\Gamma)$. More precisely, the action of Γ on X induces an action on the singular cochains $C^*_{\mathrm{sing}}(X, \mathbb{Z})$ and $H^p_{\mathrm{sing}}(X, \mathscr{L}(\Gamma))$ is then defined as the cohomology of the complex

$$(\mathscr{L}(\Gamma) \underset{\mathbb{Z}\Gamma}{\odot} C^*_{\mathrm{sing}}(X), \mathrm{id} \otimes \partial^*_{\mathrm{sing}}).$$

Note that the tensor product is over the group ring $\mathbb{Z}\Gamma$ and not the integers. The cohomology groups $H^p_{\mathrm{sing}}(X, \mathscr{L}(\Gamma))$ carry a natural action of $\mathscr{L}(\Gamma)$ inherited from multiplication (from the left) on the first tensor factor. Since X is assumed to be contractible, the singular cohomology complex becomes a free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$, and we therefore have isomorphisms of $\mathscr{L}(\Gamma)$-modules

$$H^p_{\mathrm{sing}}(X, \mathscr{L}(\Gamma)) \simeq \mathrm{Tor}_p^{\mathbb{Z}\Gamma}(\mathscr{L}(\Gamma), \mathbb{Z}) \simeq \mathrm{Tor}_p^{\mathbb{C}\Gamma}(\mathscr{L}(\Gamma), \mathbb{C}),$$

proving that the homological algebraic definition of $H_p^{(2)}(\Gamma)$ and $\beta_p^{(2)}(\Gamma)$ extends the classical analytic definition.

Another, essentially equivalent, approach to the generalization of L^2-invariants from the analytic setting to a more algebraic one, was suggested by M.S. Faber in [Far96]. We have chosen to present the approach of W. Lück since this seems to be the one most used among people working within the area of L^2-invariants.

1.2 Novikov-Shubin invariants

In this section we introduce the so-called Novikov-Shubin invariants. As with L^2-Betti numbers, these were first defined ([NS86]) in a differential geometrical setting and later ([Lüc97],[LRS99]) transported and generalized to the setting of algebraic modules over a tracial von Neumann algebra M. According to Theorem 1.1.1, every finitely generated M-module X splits as $P(X) \oplus T(X)$ and the dimension function $\dim_M(\cdot)$ measures the size of $P(X)$. The Novikov-Shubin invariants are complementary, in the sense that they are concocted to measure the size of $T(X)$. We here first sketch the classical analytic approach and then proceed with a more detailed treatment of the algebraic one.

1.2.1 The analytic approach

As we saw in the previous section, the p-th L^2-Betti number of a free, proper, co-compact, Riemannian Γ-manifold X measures the size of the kernel of the Laplacian Δ_p acting on $L^2\Omega^p(X)$. Or, in other words, the size of the range of its spectral projection at zero. The Novikov-Shubin invariants, which we are about to introduce, measures how fast the spectral projections approaches the zeroth one. To make this statement precise we need a few definitions. The *spectral density function* $F_p\colon [0,\infty[\to [0,\infty]$ for Δ_p is defined by

$$F_p(\lambda) = \dim_{\mathscr{L}(\Gamma)} \operatorname{rg}(\chi_{[0,\lambda^2]}(\Delta_p^*\Delta_p)).$$

This function turns out to be non-decreasing and right continuous, and in case $F_p(\lambda) > F_p(0)$ for all $\lambda > 0$, the p-th *Novikov-Shubin invariant* of the pair (X,Γ) is defined as

$$\alpha_p(X,\Gamma) = \liminf_{\lambda\searrow 0} \frac{\log(F_p(\lambda)-F_p(0))}{\log(\lambda)} = \sup_{\mu>0}\inf_{\mu\geq\lambda>0}\left\{\frac{\log(F_p(\lambda)-F_p(0))}{\log(\lambda)}\right\} \in [0,\infty].$$

The p-th L^2-Betti number can also be expressed using the spectral density function by means of the following simple formula

$$\beta_p^{(2)}(X,\Gamma) = F_p(0).$$

This formula, together with the definition of the Novikov-Shubin invariants, makes precise what is meant by the introductory statement about the L^2-Betti number measuring the size of the spectral projection *at* zero and the Novikov-Shubin invariant measuring how fast the spectral projections approaches the zeroth one. As with the L^2-Betti numbers, Novikov-Shubin invariants turn out to be invariant under homotopy ([Lüc02, 2.68],[GS91]) which is far from obvious and makes them interesting invariants of the manifold.

As we saw in the previous section, the theory of L^2-Betti numbers has been put into a purely algebraic framework by W. Lück, and a similar procedure has been carried out by W. Lück, H. Reich and T. Schick for the theory of Novikov-Shubin invariants — this is done using the language of capacities, which are essentially inverses of Novikov-Shubin invariants.

1.2.2 Capacities

Let again M be a finite von Neumann algebra with a distinguished, normal, faithful, tracial state τ and consider an operator $T \in M$. Its *spectral density function* $F_T\colon [0,\infty[\to [0,1]$ is defined by

$$F_T(\lambda) = \tau(\chi_{[0,\lambda^2]}(T^*T)),$$

and this function turns out to be non-decreasing and right continuous. The *Novikov-Shubin invariant* of T is then defined as

$$\alpha(T) = \liminf_{\lambda\searrow 0} \frac{\log(F_T(\lambda)-F_T(0))}{\log(\lambda)} = \sup_{\mu>0}\inf_{\mu\geq\lambda>0}\left\{\frac{\log(F_T(\lambda)-F_T(0))}{\log(\lambda)}\right\},$$

if $F_T(\lambda) > F_T(0)$ for all $\lambda > 0$. Otherwise we put $\alpha(T) = \infty^+$, where ∞^+ is a new formal symbol. We order the set $[0,\infty]\cup\{\infty+\}$ by the standard ordering on $[0,\infty]$

and the convention that $t < \infty^+$ for all $t \in [0, \infty]$. It will also be convenient to introduce a formal multiplicative inverse of ∞^+, called 0^-, and extend the order to $[0, \infty]^+_- = [0, \infty] \cup \{0^-, \infty^+\}$ by setting $0^- < t$ for all $t \in [0, \infty] \cup \{\infty^+\}$. Moreover we will extend the usual rules of addition and multiplication to $[0, \infty]^+_-$ by declaring

$$\begin{aligned} t + \infty^+ = &\quad \infty^+ \quad \text{for all } t \in [0,\infty]^+_-, \\ t + 0^- = &\quad t \quad\quad \text{for all } t \in [0,\infty]^+_-, \end{aligned}$$

and

$$\frac{1}{0} = \infty, \ \frac{1}{\infty} = 0, \ \frac{1}{0^-} = \infty^+ \ \text{ and } \ \frac{1}{\infty^+} = 0^-.$$

The number $c(T) = \frac{1}{\alpha(T)} \in [0, \infty]^+_-$ is called the *capacity* of T. Consider now a finitely presented, zero-dimensional M-module Z. Then there exists ([Lüc97, 3.4]) a short exact sequence of the form

$$0 \longrightarrow M^n \xrightarrow{f} M^n \longrightarrow Z \longrightarrow 0,$$

where f is multiplication from the right with some matrix $T \in \mathbb{M}_n(M)$. The Novikov-Shubin invariant (respectively capacity) of Z is defined as $\alpha(Z) = \alpha(T)$ (respectively $c(Z) = c(T)$), computed relative to the von Neumann algebra $\mathbb{M}_n(M)$ with trace state

$$\tau_n((a_{ij})_{i,j=1}^n) = \frac{1}{n} \sum_{i=1}^n \tau(a_{ii}).$$

If the von Neumann algebra, with respect to which the capacity is computed, is not clear from the context we shall decorate the capacity function suitably; i.e. write $c_M(Z)$ or $c_{M,\tau}(Z)$ instead of just $c(Z)$. The values of $\alpha(Z)$ and $c(Z)$ are independent of the choice of short exact sequence, as can be seen from [Lüc97, 3.6,3.9]. Following the approach in [LRS99], we will now describe how the notion of capacity can be extended to the category of all M-modules. In order to do this, some definitions will be convenient.

Definition 1.2.1 ([LRS99, 2.1]). *Denote by* $\mathbf{FP}_0(M)$ *the category of all finitely presented, zero-dimensional* M*-modules considered as a full subcategory of the category of all* M*-modules. An* M*-module* X *is said to be*

- *measurable, if it is a quotient of some module in* $\mathbf{FP}_0(M)$.
- *cofinal measurable, if every finitely generated submodule of* X *is measurable.*

It is not difficult to see that a measurable M-module is also cofinal measurable. This is due to the fact that the category of finitely presented M-modules is abelian ([Lüc97, 0.2]), which is true because M is semihereditary. Now two (a priori different) capacity functions are defined in the following way.

Definition 1.2.2 ([LRS99, 2.2]). *For a measurable M-module X its capacity is defined as*

$$c'(X) = \inf\{c(Z) \mid Z \in \mathbf{FP}_0(M) \text{ and } X \text{ is a quotient of } Z\}.$$

For an arbitrary module X its capacity is defined as

$$c''(X) = \sup\{c'(Y) \mid Y \text{ measurable and } Y \subseteq X\}.$$

These definitions are justified by the following theorem.

Theorem 1.2.3 ([LRS99, 2.2,2.4]). *The capacity functions c, c' and c'' have the following properties.*

(i) *If $Z \in \mathbf{FP}_0(M)$ then $c'(Z) = c(Z)$ and if X is a measurable module then $c'(X) = c''(X)$.*

(ii) *If $0 \to X_1 \to X_2 \to X_3 \to 0$ is a short exact sequence of M-modules then*

- $c''(X_1) \leq c''(X_2)$;
- $c''(X_3) \leq c''(X_2)$ *if X_2 is cofinal measurable;*
- $c''(X_2) \leq c''(X_1) + c''(X_3)$ *if* $\dim_M(X_2) = 0$.

(iii) *If X is a directed union of submodules $(X_i)_{i\in I}$ then $c''(X) = \sup_{i\in I} c''(X_i)$.*

(iv) *If $(X_i)_{i\in I}$ is any family of modules then $c''(\oplus_{i\in I} X_i) = \sup_{i\in I} c''(X_i)$.*

(v) *If X is finitely presented then $c''(P(X)) = 0^-$ and $c''(X) = c''(T(X))$.*

Part (v) is not mentioned explicitly in [LRS99], but follows easily: since $X = P(X) \oplus T(X)$ and $P(X)$ is finitely generated and projective (Theorem 1.1.1) we have $c''(P(X)) = 0^-$. Part (iv) then gives

$$c''(X) = \max\{c''(P(X)), c''(T(X))\} = \max\{0^-, c''(T(X))\} = c''(T(X)).$$

Because of part (i) in the above theorem, we will henceforth omit the primes on the different capacity functions and simply write $c(X)$ for an arbitrary M-module X. Comparing part (vi) in Theorem 1.1.1 with part (v) in Theorem 1.2.3 one sees that, for a finitely generated M-module X, the dimension function measures the size of the projective part $P(X)$ while the capacity measures the size of its complement $T(X)$. Having extended the notion of capacity to arbitrary modules the following definition is natural.

Definition 1.2.4 ([LRS99, 3.1]). *For a discrete group Γ its p-th capacity is defined as*

$$c_p(\Gamma) = c(H_p^{(2)}(\Gamma)),$$

where the capacity on the right hand side is calculated with respect to the natural action of $\mathscr{L}(\Gamma)$ on $H_p^{(2)}(\Gamma)$.

Due to work of W. Lück, H. Reich, T. Schick in [LRS99], R. Sauer in [Sau02] and L. Vaš in [Vaš05] it is also known that the capacity function is well behaved with respect to induction.

Theorem 1.2.5 ([Vaš05, 7.1])**.** *If N and M are finite von Neumann algebras with distinguished, faithful, normal, tracial states and $\varphi\colon N \to M$ is a trace preserving $*$-homomorphism, then for any N-module X*

$$c_M(M \odot_N X) = c_N(X).$$

In general, computations of capacities are difficult. We quote here a computational result about the zeroth capacity, in order for the reader to get a feeling for the behavior of capacities.

Theorem 1.2.6 ([LRS99, 3.2])**.** *Let Γ be a finitely generated, discrete group. Then*

$$c_0(\Gamma) = \begin{cases} 0^- & \text{if } \Gamma \text{ is finite or non-amenable;} \\ \frac{1}{n} & \text{if } \Gamma \text{ has polynomial growth of degree } n; \\ 0 & \text{if } \Gamma \text{ is infinite and amenable but not virtually nilpotent.} \end{cases}$$

Since the growth rate of $\mathbb{Z}^n$ is exactly n, the above theorem in particular gives $c_0(\mathbb{Z}^n) = \frac{1}{n}$. We shall later prove that this result has a natural generalization to the context of abelian, compact quantum groups. See e.g. Theorem 5.1.6 and the remarks preceding it.

1.3 L^2-homology for tracial algebras

In [CS05], A. Connes and D. Shlyakhtenko developed a theory of L^2-homology for certain subalgebras of finite von Neumann algebras. We recapitulate their theory and its main properties in this section.

Consider a finite von Neumann algebra M endowed with a faithful, normal tracial state τ and let A be a weakly dense, unital $*$-subalgebra of M. Then the *p-th L^2-homology* of the pair (A,τ) is defined as

$$H_p^{(2)}(A,\tau) = \operatorname{Tor}_p^{A\odot A^{\mathrm{op}}}(M\bar{\otimes}M^{\mathrm{op}}, A).$$

Here A^{op} denotes the opposite algebra and A is considered a left $A\odot A^{\mathrm{op}}$-module via the action $(a\otimes b^{\mathrm{op}})\cdot x = axb$. The symbol $M\bar{\otimes}M^{\mathrm{op}}$ denotes the von Neumann algebraic tensor product of M and M^{op}; i.e. the weak operator closure of $M\odot M^{\mathrm{op}}$ acting on the tensor product of the representation spaces of M and M^{op} respectively. The *p-th L^2-Betti number* of the pair (A,τ) is then defined as

$$\beta_p^{(2)}(A,\tau) = \dim_{M\bar{\otimes}M^{\mathrm{op}}} H_p^{(2)}(A,\tau),$$

where the dimension is computed with respect to the tracial state $\tau\bar{\otimes}\tau^{\mathrm{op}}$ on $M\bar{\otimes}M^{\mathrm{op}}$. This extends the definition for groups by means of the following.

Proposition 1.3.1 ([CS05, 2.3]). *For a discrete group Γ we have*

$$\beta_p^{(2)}(\mathbb{C}\Gamma,\tau)=\beta_p^{(2)}(\Gamma),$$

when $\mathbb{C}\Gamma$ is considered as a $*$-subalgebra of $\mathscr{L}(\Gamma)$ with its natural trace τ.

One can of course hope for the relation $\beta_p^{(2)}(\mathscr{L}(\Gamma),\tau)=\beta_p^{(2)}(\Gamma)$, but at this moment this is still unproven and seems out of reach with the current methods. If this relation holds true, one would obtain a solution to the famous non-isomorphism conjecture for the free group factors $\mathscr{L}(\mathbb{F}_n)$, stating that $\mathscr{L}(\mathbb{F}_n)\not\simeq\mathscr{L}(\mathbb{F}_m)$ when $n\neq m$. This is due to the fact that $\beta_1^{(2)}(\mathbb{F}_n)=n-1$. As an intermediate result between the relation $\beta_1^{(2)}(\mathbb{C}\Gamma,\tau)=\beta_1^{(2)}(\Gamma)$ and the desired relation $\beta_1^{(2)}(\mathscr{L}(\Gamma),\tau)=\beta_1^{(2)}(\Gamma)$, A. Thom ([Tho06, 4.6]) proved that whenever A is a weakly dense C^*-algebra inside a tracial von Neumann algebra (M,τ) then $\beta_1^{(2)}(A,\tau)=\beta_1^{(2)}(M,\tau)$.

The main properties of the Connes-Shlyakhtenko L^2-Betti numbers are gathered in the following theorem.

Theorem 1.3.2 ([CS05, 2.4,2.5,2.6]). *The L^2-Betti numbers satisfy the following properties.*

(i) *For $i=1,\dots,n$ let $A_i\subseteq(M_i,\tau_i)$ be a weakly dense $*$-algebra and endow $A=\oplus_{i=1}^n A_i$ and $M=\oplus_{i=1}^n M_i$ with the trace*

$$\tau(x_1,\dots,x_n)=\sum_{i=1}^n s_i\tau_i(x_i),$$

for some $s_1,\dots,s_n>0$ with $\sum_i s_i=1$. Then

$$\beta_p^{(2)}(A,\tau)=\sum_{i=1}^n s_i^2\beta_p^{(2)}(A_i,\tau_i).$$

(ii) *If M is a factor and $e\in M$ is a projection with $\tau(e)=\lambda>0$ then*

$$\beta_p^{(2)}(eMe,\tfrac{1}{\lambda}\tau\big|_{eMe})=\frac{1}{\lambda^2}\beta_p^{(2)}(M,\tau).$$

(iii) *If M is a $\mathbf{II}_1$ factor then $H_0^{(2)}(M,\tau)\neq 0$ if and only if M is the hyper finite factor.*

In general, these L^2-Betti numbers are very hard to compute and (to the best of the author's knowledge) so far no complete calculation of the L^2-Betti numbers of any $\mathbf{II}_1$ factor has been conducted. Partial results can be obtained for instance for the hyper finite factor $\mathscr{R}$ using the compression formula (ii) from Theorem 1.3.2.

More precisely, since the fundamental group of $\mathscr{R}$ is all of $\mathbb{R}_+^\times$, we can choose a projection $e \in \mathscr{R}$ of trace $\frac{1}{2}$ such that $e\mathscr{R}e \simeq \mathscr{R}$, and then the compression formula (ii) implies that $\beta_p^{(2)}(\mathscr{R}, \tau) \in \{0, \infty\}$. The exact values of the L^2-Betti numbers of the hyper finite factor are still unknown.

If the center of the von Neumann algebra is sufficiently big, the L^2-Betti numbers can also be computed, as the following theorem shows.

Theorem 1.3.3 ([Tho06, 2.2])**.** *If (M, τ) is a tracial von Neumann algebra with diffuse center then $\beta_p^{(2)}(M, \tau) = 0$ for all $p \geq 0$.*

Chapter 2

Compact quantum groups

In this chapter we review Woronowicz's theory of compact quantum groups. There are many good expositions of the basics of the theory; here we choose the original source [Wor98] and the lecture notes [MVD98],[KT99] as standard references. Throughout the chapter, the symbol $\otimes$ will be used to denote the minimal (spatial) tensor product of C^*-algebras.

2.1 Definitions

We take here Woronowicz's original definition of a compact quantum group.

Definition 2.1.1 ([Wor98, 1.1])**.** *A compact quantum group $\mathbb{G}$ is a pair (A, Δ) where A is a separable, unital C^*-algebra and $\Delta\colon A \longrightarrow A \otimes A$ is a $*$-homomorphism satisfying*

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta \qquad \text{(coassociativity)}$$
$$[\Delta(A)(1 \otimes A)] = [\Delta(A)(A \otimes 1)] = A \otimes A \qquad \text{(non-degeneracy)}$$

Here, for a subset S of a normed space, $[S]$ denotes the closed linear span of the elements in S. The basic example of a compact quantum group, on which the above definition is based, is the following: Let G be a compact, second countable, Hausdorff, topological group and consider its Gelfand dual $C(G)$ of continuous functions on G. Define $\Delta_c\colon C(G) \to C(G) \otimes C(G) = C(G \times G)$ by

$$\Delta_c(f)(s, t) = f(st).$$

Then coassociativity of Δ_c is equivalent to associativity of the multiplication in G. The more subtle non-degeneracy condition is a noncommutative analogue of the existence of inverse elements in a group. In the case when $A = C(G)$ the non-degeneracy condition is equivalent to the fact that the group multiplication in G has the cancellation property. Since it is well-known that a compact semigroup with cancellation is actually a group, the following result is to be expected.

Proposition 2.1.2 ([Wor98, Rem. 3]). *If $\mathbb{G} = (A, \Delta)$ is a compact quantum group and A is an abelian C^*-algebra then there exists a compact, second countable, Hausdorff topological group G and a $*$-isomorphism $\alpha\colon A \to C(G)$ such that $\Delta_c = (\alpha \otimes \alpha)\Delta\alpha^{-1}$.*

Example 2.1.3. If Γ is a discrete, countable group its reduced C^*-algebra $C^*_{\mathrm{red}}(\Gamma)$ becomes a compact quantum group when endowed with comultiplication given by

$$\Delta_{\mathrm{red}}(\lambda_\gamma) = \lambda_\gamma \otimes \lambda_\gamma.$$

Here $\lambda\colon \Gamma \to B(\ell^2(\Gamma))$ denotes the left regular representation.

Definition 2.1.4. *A compact quantum group $\mathbb{G} = (A, \Delta)$ is called cocommutative if $\sigma \circ \Delta = \Delta$, where $\sigma\colon A \otimes A \to A \otimes A$ is the flip-automorphism.*

Note that $(C^*_{\mathrm{red}}(\Gamma), \Delta_{\mathrm{red}})$ is an example of a cocommutative, compact quantum group, and one can prove ([KT99, p. 53]) that every compact, cocommutative quantum group (whose Haar state is faithful (see Theorem 2.1.7)) has this form. As an example of a compact quantum group that is neither of the form $(C(G), \Delta_c)$ nor of the form $(C^*_{\mathrm{red}}(\Gamma), \Delta_{\mathrm{red}})$ we now present Woronowicz's quantum $SU(2)$ introduced and studied in [Wor87b]. This is probably the most classical among such examples.

Example 2.1.5 (Quantum $SU(2)$). Let $q \in [-1, 1] \setminus \{0\}$ be given and denote by A_0 the universal, unital $*$-algebra generated by the symbols α and γ subject to the following relations.

$$\begin{aligned} \alpha^*\alpha + \gamma^*\gamma &= 1 & \qquad \alpha\gamma - q\gamma\alpha &= 0 \\ \alpha\alpha^* + q^2\gamma\gamma^* &= 1 & \alpha\gamma^* - q\gamma^*\alpha &= 0 \\ \gamma\gamma^* - \gamma^*\gamma &= 0 \end{aligned}$$

These relations are easily seen to be equivalent to the requirement that the 2×2 matrix

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is unitary. This unital $*$-algebra can be endowed with comultiplication Δ, antipode S and counit ε which on the generators are given by the formulas

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha - q\gamma^* \otimes \gamma & \qquad S(\gamma) &= -q\gamma \\ \Delta(\gamma) &= \gamma \otimes \alpha + \alpha^* \otimes \gamma & S(\gamma^*) &= -\tfrac{1}{q}\gamma \\ S(\alpha) &= \alpha^* & \varepsilon(\alpha) &= 1 \\ S(\alpha^*) &= \alpha & \varepsilon(\gamma) &= 0 \end{aligned}$$

It is not difficult to prove ([KT99, p. 37]) that (A_0, Δ) becomes a Hopf $*$-algebra with counit ε and antipode S. Using the defining relations, one sees that the norm

$$\|a\|_u = \sup\{\|\pi(a)\| \mid \pi \text{ unital } *\text{-representation of } A_0\}$$

is finite. Moreover, one can explicitly construct ([KT99, p. 39]) a faithful representation of A_0 and therefore $\|\cdot\|_u$ becomes a C^*-norm. By construction of the norm, the comultiplication Δ extends to the $\|\cdot\|_u$-completion A of A_0, turning it into a compact quantum group. This compact quantum group is Woronowicz's famous quantum $SU(2)$, denoted $SU_q(2)$ in the following. The reason for the name is that $SU_1(2)$ equals $(C(SU(2)), \Delta_c)$.

Similar to the construction of $SU_q(2)$, one can also deform other compact Lie groups to obtain noncommutative, compact quantum groups. Another way to obtain examples of compact quantum groups is by construction of the *free versions* of the orthogonal groups $O(n)$ and unitary groups $U(n)$, which will be discussed in the following example. These were first introduced by S. Wang and A. van Daele in [Wan95], [VDW96] and studied intensively by T. Banica in [Ban96] and [Ban97].

Example 2.1.6. Let $n \geq 2$ and denote by A_0 the universal, unital $*$-algebra generated by n^2 symbols $\{v_{ij} \mid 1 \leq i, j \leq n\}$ subject to the relations making the matrix $v = (v_{ij})$ orthogonal. I.e. $v_{ij} = v_{ij}^*$ and the transpose v^t is the inverse of v. We endow A_0 with a comultiplication $\Delta\colon A_0 \to A_0 \odot A_0$ by setting

$$\Delta(v_{ij}) = \sum_{k=1}^{n} v_{ik} \otimes v_{kj},$$

and defining $S\colon A_0 \to A_0$ and $\varepsilon\colon A_0 \to \mathbb{C}$ by

$$S(v_{ij}) = v_{ji}^* = v_{ji} \quad \text{and} \quad \varepsilon(v_{ij}) = \delta_{ij},$$

turns $(A_0, \Delta, S, \varepsilon)$ into a Hopf $*$-algebra. By considering the matrix products $v^t v$ and vv^t, one sees that this $*$-algebra admits a universal enveloping C^*-algebra A, and by construction Δ extends to A yielding a compact quantum group. This quantum group is called the free orthogonal quantum group, denoted $A_o(n)$. The reason for the name is that the abelianization of A, together with the induced comultiplication, is $(C(O(n)), \Delta_c)$.

If, instead, we begin with the the universal, unital $*$-algebra generated by n^2 symbols $\{u_{ij} \mid 1 \leq i, j \leq n\}$ subject to the relations making the matrix $u = (u_{ij})$ as well as $\bar{u} = (u_{ij}^*)$ unitary, the same formulas — with v_{ij} replaced with u_{ij} — yield a comultiplication, antipode and counit. The universal enveloping C^*-algebra exists and the comultiplication extends boundedly, giving rise to the so-called free unitary quantum group $A_u(n)$. Again, the reason for the name is that the abelianization of $A_u(n)$ equals $(C(U(n)), \Delta_c)$.

In the theory of compact groups, the existence and uniqueness of the Haar measure is a deep result with many important implications. The following theorem is to be considered as a quantum analogue of this result.

Theorem 2.1.7 ([Wor98, 1.3]). *Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group. There exists a unique state $h\colon A \to \mathbb{C}$ which is both left and right invariant in the sense that*

$$(\mathrm{id} \otimes h)\Delta(a) = h(a)1_A = (h \otimes \mathrm{id})\Delta(a),$$

for all $a \in A$. The state h is called the Haar state of $\mathbb{G}$.

As it should be expected, when $\mathbb{G} = (C(G), \Delta_c)$ the Haar state is given by integration with respect to the unique Haar probability measure on G.

2.2 Theory of corepresentations

As with compact groups, it turns out that compact quantum groups have a rich (co)representation theory, of which we will survey the highlights in the following. A *corepresentation* of a compact quantum group $\mathbb{G} = (A, \Delta)$ on a finite dimensional (complex) Hilbert space H is an invertible element $v \in B(H) \otimes A$ satisfying

$$(\mathrm{id} \otimes \Delta)v = v_{(12)}v_{(13)}. \tag{2.1}$$

Here we use the so-called *leg-numbering convention*; if $v = x \otimes y$ then for instance $v_{(12)} = x \otimes y \otimes 1$ and $v_{(13)} = x \otimes 1 \otimes y$. By choosing a basis $e_1, \ldots, e_n$ for H we get an identification of $B(H) \otimes A$ with $\mathbb{M}_n(A)$. If, under this identification, v becomes the matrix (v_{ij}) the defining property (2.1) becomes

$$\forall i, j \in \{1, \ldots, n\}: \ \Delta v_{ij} = \sum_{k=1}^{n} v_{ik} \otimes v_{kj}.$$

The elements $v_{ij} \in A$ are called the *matrix coefficients* of v with respect to the basis $e_1, \ldots, e_n$. Another choice of basis yields a different system of matrix coefficients, but two such sets will always generate the same linear subspace of A. The corepresentation v is called *unitary* if v is a unitary in $B(H) \otimes A$.

Example 2.2.1. Assume that $\mathbb{G} = (C(G), \Delta_c)$ for some compact group G and let $v\colon G \to \mathrm{GL}_n(\mathbb{C})$ be a finite dimensional representation. Then the matrix coefficients v_{ij} are in $C(G)$ and the matrix $(v_{ij}) \in \mathbb{M}_n(C(G))$ is a corepresentation of $\mathbb{G}$. One easily checks that every finite dimensional corepresentation of $\mathbb{G}$ is obtained in this way. Note also that the matrices (v_{ij}) and (u_{ij}), used to define $A_o(n)$ and $A_u(n)$ respectively, are corepresentations by construction.

Given two finite dimensional corepresentations $v = \sum_i T_i \otimes a_i \in B(H) \otimes A$ and $w = \sum_j S_j \otimes b_j \in B(K) \otimes A$ their *direct sum* $v \oplus w \in B(H \oplus K) \otimes A$ is defined as

$$v \oplus w = \sum_i \begin{pmatrix} T_i & 0 \\ 0 & 0 \end{pmatrix} \otimes a_i + \sum_j \begin{pmatrix} 0 & 0 \\ 0 & S_j \end{pmatrix} \otimes b_j.$$

Their *tensor product* $v \circledT w$ in $B(H \otimes K) \otimes A$ is defined as

$$v \circledT w = v_{(13)} w_{(23)} = \sum_{i,j} T_i \otimes S_j \otimes a_i b_j.$$

An element $X \in B(H, K)$ is called a *morphism* (or *intertwiner*) from v to w if $(X \otimes 1)v = w(X \otimes 1)$. We denote the set of morphisms from v to w by $\mathrm{Mor}(v, w)$ and v and w are called *equivalent* if $\mathrm{Mor}(v, w)$ contains an invertible operator. A subspace H_0 of H is called *v-invariant* if

$$\Big\{ \sum_i T_i(\xi) \otimes a_i \mid \xi \in H_0 \Big\} \subseteq H_0 \otimes A,$$

and v is called *irreducible* if it allows only the trivial invariant subspaces. The following quantum version of Schur's lemma is of great importance in the theory of corepresentations.

Lemma 2.2.2 ([MVD98, 6.6]). *If u and v are unitary, irreducible corepresentations then* $\mathrm{Mor}(u, v) = \{0\}$ *if u and v are not equivalent and* $\dim_{\mathbb{C}} \mathrm{Mor}(u, v) = 1$ *if u and v are equivalent.*

Just as with classical representations of compact groups, the corepresentations of compact quantum groups have the following decomposability property.

Theorem 2.2.3 ([Wor98, 3.4],[MVD98, 6.4]). *Every corepresentation is equivalent to a direct sum of irreducible, unitary corepresentations. In particular, every irreducible corepresentation is equivalent to a unitary one.*

Consider now a complete set $\{u^\alpha\}_{\alpha \in I}$ of representatives for the equivalence classes of irreducible, unitary corepresentations. That is, the elements in $\{u^\alpha\}_{\alpha \in I}$ are pairwise inequivalent and every irreducible corepresentation is equivalent to (exactly) one of the u^α's. Denote by H_α the representation space of u^α and by n_α its dimension, and choose a basis for H_α to identify $B(H_\alpha) \otimes A$ with $\mathbb{M}_{n_\alpha}(A)$. Then the following quantum version of the Peter-Weyl orthogonality relations holds.

Theorem 2.2.4 ([Wor98, Section 6]). *For each $\alpha \in I$ there exists a unique, positive, invertible matrix $F^\alpha \in \mathbb{M}_{n_\alpha}(\mathbb{C})$ with the properties that* $\mathrm{Tr}(F^\alpha) = \mathrm{Tr}((F^\alpha)^{-1})$ *and*

$$h(u_{ij}^{\alpha *} u_{kl}^{\beta}) = \frac{1}{\mathrm{Tr}(F^\alpha)} \delta_{\alpha,\beta} \delta_{j,l} (F^\alpha)_{ik},$$
$$h(u_{ij}^{\alpha} u_{kl}^{\beta *}) = \frac{1}{\mathrm{Tr}(F^\alpha)} \delta_{\alpha,\beta} \delta_{i,k} ((F^\alpha)^{-1})_{jl}.$$

If h is a trace, Theorem 2.2.4 implies that the set

$$\{ \sqrt{n_\alpha} u_{ij}^\alpha \mid \alpha \in I, 1 \le i, j \le n_\alpha \}$$

is an orthonormal basis for the GNS space $L^2(A, h)$.

Remark 2.2.5. For the sake of simplicity, we have restricted our attention to finite dimensional corepresentations in the above. However, there is a well developed theory of infinite dimensional corepresentations; these are invertible elements in the multiplier algebra of $\mathcal{K}(H) \otimes A$ for an infinite dimensional Hilbert space H satisfying (2.1). By [Wor98], every irreducible unitary corepresentation is finite dimensional and Theorem 2.2.3 holds true also in the infinite dimensional case. Considering only finite dimensional corepresentations is therefore not very restrictive.

For a compact quantum group $\mathbb{G} = (A, \Delta)$, we will denote by A_0 the unital $*$-algebra generated by the matrix coefficients arising from a complete set of representatives $(u^\alpha)_{\alpha \in I}$ for the equivalence classes of irreducible, unitary corepresentations of $\mathbb{G}$. One easily checks that if u and v are equivalent, n-dimensional, unitary corepresentations then

$$\operatorname{span}_{\mathbb{C}}\{u_{ij} \mid 1 \leq i, j \leq n\} = \operatorname{span}_{\mathbb{C}}\{v_{ij} \mid 1 \leq i, j \leq n\},$$

and the definition of A_0 is therefore independent of the choice of representatives. The set

$$\{u_{ij}^\alpha \mid \alpha \in I, 1 \leq i, j \leq n_\alpha\}$$

turns out to be a a linear basis for A_0 ([MVD98, 7.3]) and we can therefore define linear maps $S\colon A_0 \to A_0$ and $\varepsilon\colon A_0 \to \mathbb{C}$ by

$$S(u_{ij}^\alpha) = u_{ji}^{\alpha *} \qquad \text{and} \qquad \varepsilon(u_{ij}^\alpha) = \delta_{ij}.$$

With the above notation the following holds.

Theorem 2.2.6 ([Wor98, 1.2])**.** *The quadruple $(A_0, \Delta, S, \varepsilon)$ is a Hopf $*$-algebra and A_0 is dense in A.*

The reader is referred to [KS97] for a definition of Hopf algebras.

Definition 2.2.7. *If $\mathbb{G}$ allows a unitary corepresentation $u \in \mathbb{M}_n(A)$ such that the matrix coefficients of u generate A_0 as a $*$-algebra then $\mathbb{G}$ is called a compact matrix quantum group and u is called a fundamental corepresentation.*

Remark 2.2.8. Consider again the commutative example $(C(G), \Delta_c)$. Since representations of G is the same as corepresentations of $C(G)$, the algebra $C(G)_0$ is nothing but the algebra of representative functions. The antipode becomes $S(f)(g) = f(g^{-1})$ and the counit is given by $\varepsilon(f) = f(e)$. It is not difficult to see that $(C(G), \Delta_c)$ is a compact matrix quantum group exactly when G is a Lie group. For the cocommutative example $(C^*_{\text{red}}(\Gamma), \Delta_{\text{red}})$ it is clear that each λ_γ is a one-dimensional, unitary corepresentation and we will prove later (Proposition 5.1.1) that there are no other irreducible ones — up to equivalence. So, in this case we get $C^*_{\text{red}}(\Gamma)_0 = \lambda(\mathbb{C}\Gamma)$ with (the linear extension of) inversion as antipode and (the linear extension of) the trivial representation as counit. Moreover, $(C^*_{\text{red}}(\Gamma), \Delta_{\text{red}})$ is seen to be a compact matrix quantum group exactly when Γ is finitely generated.

Since it will be of importance to us later on, we end this section by introducing the so-called contragredient corepresentation. Consider again a finite dimensional, unitary corepresentation $u \in B(H) \otimes A$ and define

$$u^c = ((-)') \otimes S)u,$$

where, for an operator $S \in B(H)$, $S' \in B(H')$ denotes the dual operator on the dual Hilbert space H'. This is still a corepresentation, but it is in general not unitary — although equivalent to a unitary corepresentation, according to Theorem 2.2.3. Assume that an orthonormal basis $e_1, \ldots, e_n$ has been chosen for H in which u is identified with the matrix $(u_{ij}) \in \mathbb{M}_n(A)$. By endowing the dual space H' with the dual basis $e_1', \ldots, e_n'$, the contragredient corepresentation u^c is identified with the matrix $\bar{u} = (u_{ij}^*)$.

2.3 Coamenability

Corresponding to the notion of amenability for groups we find, in the setting of quantum groups, the notion of coamenability which we shall study in this section. Amenability and coamenability of quantum groups have been studied by many different authors in different settings — a number of references are [BMT01], [Voi79], [Rua96], [Ban99a], [Ban99b], [ES92] and [BS93]. Since we are dealing with *compact* quantum groups, the approach in [BMT01] is the most natural and we will follow this reference throughout this section.

Consider a compact quantum group $\mathbb{G} = (A, \Delta)$ and denote by h its Haar state with corresponding GNS representation π_h. The reduced C^*-algebra $A_{\text{red}} = \pi_h(A)$ inherits the structure of a compact quantum group from $\mathbb{G}$ ([BMT01, 2.1]) and we will denote this quantum group by $\mathbb{G}_{\text{red}} = (A_{\text{red}}, \Delta_{\text{red}})$. The state induced by h on A_{red} is the Haar state on $\mathbb{G}_{\text{red}}$ and is faithful by construction. Since h is faithful on A_0 ([MVD98]), π_h injects A_0 into A_{red} and in the following we will consider A_0 as a subalgebra of A_{red} via this embedding. Also the universal C^*-envelope A_u (see e.g. Example 2.1.5) of A_0 exists and inherits the structure of a compact quantum group from $\mathbb{G}$; we denote this quantum group by $\mathbb{G}_u = (A_u, \Delta_u)$. The reader is referred to [BMT01] for the details on these constructions.

Definition 2.3.1 ([BMT01])**.** *A compact quantum group $\mathbb{G} = (A, \Delta)$ is said to be coamenable if the counit $\varepsilon\colon A_0 \to \mathbb{C}$ extends boundedly to the reduced C^*-algebra A_{red}.*

Example 2.3.2. If $\mathbb{G}$ is abelian, i.e. of the form $(C(G), \Delta_c)$ for some compact group G, then $\mathbb{G}$ is coamenable since the counit is given by evaluation at the identity and hence globally defined and bounded. Note, however, that the coamenability of $\mathbb{G}$ has nothing to do with the amenability of the underlying compact group G.

Example 2.3.3. Since a discrete group is amenable if and only if its trivial representation factorizes through its left regular representation, we see that

$(C^*_{\mathrm{red}}(\Gamma), \Delta_{\mathrm{red}})$ is coamenable if and only if Γ is amenable. For readers not familiar with amenability for discrete groups, we refer to the remarks following Theorem 3.2.5 for the definition and to [Gre69] for a detailed treatment.

Example 2.3.4. The compact quantum groups $SU_q(2)$ defined in Example 2.1.5 are all coamenable. For $q = 1$ this is clear from Example 2.3.2 and for $q \neq 1$ this can be deduced using the results presented in Chapter 4, since the so-called corepresentation ring (see Section 3.1.1) of $SU_q(2)$ is identical to the representation ring of $SU(2)$. See e.g. [Wor88] for this fact. Another proof of the coamenability of $SU_q(2)$, using different methods, can be found in [KT99, 2.12].

The following example discusses the coamenability properties of the free unitary and orthogonal quantum groups introduced in Example 2.1.6.

Example 2.3.5. The free orthogonal quantum group $A_o(n)$ is coamenable only for $n = 2$. The proof of this relies on the fact that the so-called fusion rules for $A_o(n)$ (see e.g. Section 3.1.1) can be explicitly computed. See [Ban96] for the computation of the fusion rules and [Ban99a] for a proof of the claim about coamenability. The corresponding free unitary quantum group $A_u(n)$ is not coamenable for any n. This follows from the fact that its reduced C^*-algebra is simple ([Ban97]) and therefore, in particular, does not allow a nontrivial one-dimensional representation.

Remark 2.3.6. The choice of terminology may require a bit of explanation. Most of the quantum group notation is set up to have the following property: A compact group/space G has property $\boxed{?}$ if and only if $(C(G), \Delta_c)$ has property co-$\boxed{?}$. This is the case with multiplication versus comultiplication, inverses versus coinverse (antipode), unit versus counit, commutative versus cocommutative etc. However, *any* compact group is amenable so choosing the definition of coamenability for compact quantum groups according to this Gelfand duality principle does not make much sense. Instead focus is put on the natural compact quantum group $(C^*_{\mathrm{red}}(\Gamma), \Delta_{\mathrm{red}})$ associated to a discrete group Γ and the definition is chosen such that this quantum group is coamenable exactly when Γ is amenable. So, if we were to insist that the natural quantum group associated to a (locally) compact group G is $(C_0(G), \Delta_c)$, and not $(C^*_{\mathrm{red}}(G), \Delta_{\mathrm{red}})$, then the above defined notion of coamenability should have been called *amenability*. And *co*amenability should then have been defined by requiring the dual quantum group to be amenable. This convention is used, for instance, in the work of T. Banica ([Ban99b]). We have chosen to stick with the above definition of coamenability, partially since it seems that this terminology is the dominant one at the moment and partially to be notation-wise synchronized with our primary source [BMT01] on the subject.

Some of the main results from [BMT01] are summarized in the following theorem.

Theorem 2.3.7 ([BMT01, 2.2,2.4,3.6])**.** *Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group. Then the following are equivalent.*

(i) $\mathbb{G}$ *is coamenable.*

(ii) *The Haar state h is faithful on A and the counit is bounded with respect to the norm on A.*

(iii) *The canonical surjection $A_u \longrightarrow A_{\text{red}}$ is an isomorphism.*

If $\mathbb{G}$ is a compact matrix quantum group and $u \in \mathbb{M}_n(A)$ is a fundamental corepresentation for $\mathbb{G}$ then the above statements are also equivalent to the following.

(iv) *The integer n belongs to $\sigma(\pi_h(\text{Re}(\chi(u))))$ where $\chi(u) = \sum_{i=1}^{n} u_{ii}$.*

Here, and in the sequel, $\sigma(T)$ denotes the spectrum of a given operator T. The condition (iv) above is G. Skandalis's quantum analogue of the so-called Kesten condition for groups (see [Kes59],[Ban99a]) which is proved by T. Banica in [Ban99b]. The next result is a generalization of the Kesten condition to the case where a fundamental corepresentation is not (necessarily) present.

Theorem 2.3.8 ([Kye07, 4.4])**.** *Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group. Then the following are equivalent.*

(i) $\mathbb{G}$ *is coamenable.*

(ii) *For any finite dimensional, unitary corepresentation $u \in \mathbb{M}_{n_u}(A)$ we have $n_u \in \sigma(\pi_h(\text{Re}(\chi(u))))$.*

The reason for our interest in the generalized Kesten condition is that it allows us to prove (see Chapter 4) that a compact quantum group is coamenable exactly when its so-called fusion algebra of irreducible corepresentations is amenable. This, in turn, provides us with a Følner condition for compact quantum groups which is needed in the study of L^2-Betti numbers for coamenable quantum groups.

Chapter 3

Fusion algebras

In this chapter we introduce the notion of fusion algebras and establish the basic properties of these objects. We then proceed by studying the concept of amenability within the class of fusion algebras. The meaning of the word "fusion algebra" is not completely settled in the literature, but the overall structure behind the different definitions is basically the same. Also the terminology is not completely settled, for instance in the work of V. Sunder ([Sun92]) the word hypergroup is used to describe the structure we will refer to as a fusion algebra. Here we take the approach introduced in the work of F. Hiai and M. Izumi ([HI98]), and further developed in [Yam99], since [HI98] will be our primary reference on the subject of amenability for fusion algebras.

3.1 Definitions and basic properties

We first give the definition of a fusion algebra in the sense of [HI98]. Here, and in what follows, the set of non-negative integers is denoted $\mathbb{N}_0$.

Definition 3.1.1 ([HI98, 1.1]). *A fusion algebra is a unital ring R which is free as a $\mathbb{Z}$-module, of at most countable dimension, together with a distinguished $\mathbb{Z}$-basis I such that the following holds.*

- *The unit e of $R = \mathbb{Z}[I]$ is an element of I.*
- *For all $\xi, \eta \in I$ the structure constants $(N_{\xi\eta}^{\alpha})_{\alpha \in I}$ satisfying*

$$\xi \cdot \eta = \sum_{\alpha \in I} N_{\xi\eta}^{\alpha} \alpha,$$

  are non-negative. I.e. the monoid $\mathbb{N}_0[I] \subseteq \mathbb{Z}[I]$ is closed under multiplication.
- *There exists an anti-multiplicative $\mathbb{Z}$-linear map $R \ni r \mapsto \bar{r} \in R$ of period two, called the involution (or conjugation) of R, preserving the basis I globally.*

- *The structure constants satisfy Frobenius reciprocity, i.e. for all* $\alpha, \xi, \eta \in I$ *we have*
$$N_{\xi\eta}^{\alpha} = N_{\bar{\xi}\alpha}^{\eta} = N_{\alpha\bar{\eta}}^{\xi}.$$
- *There exists a ring homomorphism* $d\colon R \to \mathbb{R}$, *called the dimension function of* R, *such that* $d(\xi) > 0$ *and* $d(\xi) = d(\bar{\xi})$ *for all* $\xi \in I$.

Note that both the choice of basis, conjugation and dimension function are parts of the data constituting the fusion algebra. The reason for requiring the basis I to be at most countable is mainly to avoid unnecessary technicalities, since we will not encounter examples without this property. Our main example of interest will be the corepresentation ring of a compact quantum group (described in detail in Section 3.1.1 below), and since we require a compact quantum group to have a separable underlying C^*-algebra, its corepresentation ring will always have an at most countable basis.

Fusion algebras have the following basic properties.

(i) Each fusion algebra $R = \mathbb{Z}[I]$ comes with a canonical tracial state given by
$$\tau(\sum_{\xi \in I} k_\xi \xi) = k_e.$$

(ii) Note also that $N_{\xi\eta}^{e} = \delta_{\bar{\xi},\eta}$ because Frobenius reciprocity yields
$$N_{\xi\eta}^{e} = N_{\bar{\xi}e}^{\eta} = \delta_{\bar{\xi},\eta}.$$

(iii) From the multiplicativity of d it follows that $d(e) = 1$ and (ii) implies that $e = \bar{e}$. Moreover, since $N_{\xi\bar{\xi}}^{e} = 1$ we get that
$$1 \leq d(\xi\bar{\xi}) = d(\xi)^2,$$
so that $d(\xi) \geq 1$ for all $\xi \in I$.

(iv) From the anti-multiplicativity of the conjugation it follows that $N_{\xi\eta}^{\alpha} = N_{\bar{\eta}\bar{\xi}}^{\bar{\alpha}}$.

(v) The requirement that the conjugation is an involution (i.e. that $\bar{\bar{r}} = r$) is actually redundant since Frobenius reciprocity implies that
$$N_{\bar{\xi}\bar{\bar{\xi}}}^{e} = N_{e\bar{\xi}}^{\bar{\xi}} = 1,$$
and by (ii) we therefore get $\bar{\bar{\xi}} = \xi$ for all $\xi \in I$.

(vi) We shall often pass to the complexified fusion algebra $\mathbb{C}[I] = \mathbb{C} \odot_{\mathbb{Z}} \mathbb{Z}[I]$, and $\mathbb{C}[I]$ will always be considered with the induced $*$-ring structure, dimension function and trace.

Example 3.1.2. For any discrete, countable group Γ the integral group ring $\mathbb{Z}\Gamma$ becomes a fusion algebra when endowed with (the $\mathbb{Z}$-linear extension of) inversion as involution and trivial dimension function given by $d(\gamma) = 1$ for all $\gamma \in \Gamma$.

For a compact group G its irreducible representations constitute the basis in a fusion algebra where the tensor product of representations is the product. We shall not go into details with this construction since it will be contained in a more general example, presented in the following subsection.

3.1.1 The ring of corepresentations

If $\mathbb{G} = (A, \Delta)$ is a compact quantum group its irreducible corepresentations constitute the basis of a fusion algebra with tensor product as multiplication. Since this example will play a prominent role later, we shall now elaborate on the construction. Denote by $\mathrm{Irred}(\mathbb{G}) = (u^\alpha)_{\alpha\in I}$ a complete family of representatives for the equivalence classes of irreducible, unitary corepresentations of $\mathbb{G}$. As explained in Chapter 2, for all $u^\alpha, u^\beta \in \mathrm{Irred}(\mathbb{G})$ there exist a finite subset $I_0 \subseteq I$ and a family $(N_{\alpha\beta}^\gamma)_{\gamma\in I_0}$ of positive integers such that $u^\alpha ⓣ u^\beta$ is equivalent to

$$\bigoplus_{\gamma\in I_0} (u^\gamma)^{\oplus N_{\alpha\beta}^\gamma}.$$

Thus, a product can be defined on the free $\mathbb{Z}$-module $\mathbb{Z}[\mathrm{Irred}(\mathbb{G})]$ by setting

$$u^\alpha \cdot u^\beta = \sum_{\gamma\in I_0} N_{\alpha\beta}^\gamma u^\gamma,$$

and the trivial corepresentation $e = 1_A \in \mathrm{Irred}(\mathbb{G})$ is a unit for this product. If we denote by $u^{\bar{\alpha}} \in \mathrm{Irred}(\mathbb{G})$ the unique representative equivalent to the contragredient $(u^\alpha)^c$ then the map $u^\alpha \mapsto u^{\bar{\alpha}}$ extends to a conjugation on the ring $\mathbb{Z}[\mathrm{Irred}(\mathbb{G})]$, and since each u^α is an element of $\mathbb{M}_{n_\alpha}(A)$ for some $n_\alpha \in \mathbb{N}$ we can also define a dimension function $d\colon \mathbb{Z}[\mathrm{Irred}(\mathbb{G})] \to [1, \infty[$ by $d(u^\alpha) = n_\alpha$.

When endowed with this multiplication, conjugation and dimension function $\mathbb{Z}[\mathrm{Irred}(\mathbb{G})]$ becomes a fusion algebra. The only thing that is not clear at this moment is that Frobenius reciprocity holds. To see this, we first note that for any $\alpha \in I$ and any finite dimensional corepresentation v we have (by Schur's Lemma (2.2.2)) that u^α occurs exactly

$$\dim_{\mathbb{C}} \mathrm{Mor}(u^\alpha, v)$$

times in the decomposition of v. Moreover, we have for any two unitary corepresentations v and w that

$$\dim_{\mathbb{C}} \mathrm{Mor}(v, w) = \dim_{\mathbb{C}}((V_w \otimes V_v')^{w \, ⓣ v^c}) \tag{3.1}$$

$$\dim_{\mathbb{C}} \mathrm{Mor}(v^{cc}, w) = \dim_{\mathbb{C}}((V_v' \otimes V_w)^{v^c \, ⓣ w}) \tag{3.2}$$

Here the right hand side denotes the linear dimension of the space of invariant vectors under the relevant corepresentation; a vector $\xi \in H$ is said to be invariant under a corepresentation $u = \sum_i T_i \otimes a_i \in B(H) \otimes A$ if

$$\sum_i T_i(\xi) \otimes a_i = \xi \otimes 1.$$

These formulas are proved in [Wor87a, 3.4] for compact matrix quantum groups, but the same proof carries over to the case where the compact quantum group in question does not necessarily possess a fundamental corepresentation. Using the formula (3.1), we get for $\alpha, \beta, \gamma \in I$ that

$$\begin{aligned} N_{\alpha\beta}^{\gamma} &= \dim_{\mathbb{C}} \mathrm{Mor}(u^{\gamma}, u^{\alpha} \textcircled{\scriptsize T} u^{\beta}) \\ &= \dim_{\mathbb{C}} (V_{\alpha} \otimes V_{\beta} \otimes V'_{\gamma})^{u^{\alpha} \textcircled{\scriptsize T} u^{\beta} \textcircled{\scriptsize T} (u^{\gamma})^c} \\ &= \dim_{\mathbb{C}} (V_{\gamma} \otimes V'_{\beta} \otimes V'_{\alpha})^{u^{\gamma} \textcircled{\scriptsize T} (u^{\beta})^c \textcircled{\scriptsize T} (u^{\alpha})^c} \\ &= \dim_{\mathbb{C}} \mathrm{Mor}(u^{\alpha}, u^{\gamma} \textcircled{\scriptsize T} (u^{\beta})^c) \\ &= N_{\gamma\bar{\beta}}^{\alpha} \end{aligned}$$

The remaining identity in Frobenius reciprocity follows similarly using the formula (3.2). The fusion algebra $\mathbb{Z}[\mathrm{Irred}(\mathbb{G})]$ is called the corepresentation ring (or fusion ring) of $\mathbb{G}$ and is denoted $R(\mathbb{G})$. Abusing notation a bit, we shall sometimes also use the symbol $R(\mathbb{G})$ to denote the complexified fusion algebra $\mathbb{C}[\mathrm{Irred}(\mathbb{G})]$.

Recall that the character of a corepresentation $u \in \mathbb{M}_n(A)$ is defined as $\chi(u) = \sum_{i=1}^n u_{ii}$. It follows from the classical work of Woronowicz ([Wor87b]) that the $\mathbb{Z}$-linear extension

$$\chi \colon \mathbb{Z}[\mathrm{Irred}(\mathbb{G})] \longrightarrow A_0$$

is an injective homomorphism of $*$-rings. I.e. χ is additive and multiplicative with $\chi(u^{\bar{\alpha}}) = (\chi(u^{\alpha}))^*$. This gives a link between the two $*$-algebras $R(\mathbb{G})$ and A_0 which will be of importance later.

Remark 3.1.3. Other interesting examples of fusion algebras arise from inclusions $N \subseteq M$ of $\mathbf{II}_1$-factors of finite index by considering classes of certain irreducible Hilbert bimodules over the pairs (N, N) and (M, M). See [HI98] and [JS97] for more details.

Consider again an abstract fusion algebra $R = \mathbb{Z}[I]$. For $\xi, \eta \in I$ we define the (weighted) convolution of the corresponding Dirac measures, δ_ξ and δ_η, as

$$\delta_\xi * \delta_\eta = \sum_{\alpha \in I} \frac{d(\alpha)}{d(\xi)d(\eta)} N_{\xi\eta}^{\alpha} \delta_\alpha \in \mathbb{C}[I].$$

For $f \in \ell^\infty(I)$ and $\xi \in I$ we define $\lambda_\xi(f), \rho_\xi(f) \colon I \to \mathbb{C}$ by

$$\begin{aligned} \lambda_\xi(f)(\eta) &= \sum_{\alpha \in I} f(\alpha)(\delta_{\bar{\xi}} * \delta_\eta)(\alpha), \\ \rho_\xi(f)(\eta) &= \sum_{\alpha \in I} f(\alpha)(\delta_\eta * \delta_\xi)(\alpha). \end{aligned}$$

Denote by σ the counting measure on I scaled with d^2; that is $\sigma(\xi) = d(\xi)^2$. Combining Proposition 1.3, Remark 1.4 and Theorem 1.5 in [HI98] we get the following.

Proposition 3.1.4 ([HI98]). *For each $f \in \ell^\infty(I)$ we have $\lambda_\xi(f) \in \ell^\infty(I)$ and for each $p \in \mathbb{N} \cup \{\infty\}$ the map $\lambda_\xi \colon \ell^\infty(I) \to \ell^\infty(I)$ restricts to a bounded operator on $\ell^p(I, \sigma)$ denoted $\lambda_{p,\xi}$. By linear extension, we therefore obtain a map $\lambda_{p,-} \colon \mathbb{Z}[I] \to B(\ell^p(I, \sigma))$. The map $\lambda_{p,-}$ respects the weighted convolution product. Moreover, for $p = 2$ the operator $U \colon \ell^2(I) \to \ell^2(I, \sigma)$ given by $U(\delta_\eta) = \frac{1}{d(\eta)}\delta_\eta$ is unitary and intertwines $\lambda_{2,\xi}$ with the operator*

$$l_\xi \colon \delta_\eta \longmapsto \frac{1}{d(\xi)} \sum_\alpha N_{\xi\eta}^\alpha \delta_\alpha.$$

Remark 3.1.5. Under the natural identification of the GNS space $L^2(\mathbb{C}[I], \tau)$ with $\ell^2(I)$ we see that $\pi_\tau(\xi) = d(\xi) l_\xi$. In particular, the GNS representation consists of bounded operators. Here τ denotes the trace defined just after Definition 3.1.1.

3.2 Amenability for fusion algebras

In this section we introduce the notion of amenability for fusion algebras following the approach in [HI98]. An equivalent notion was treated by T. Hayashi and S. Yamagami ([HY00]) in the context of so-called C^*-tensor categories with Frobenius duality — these are categories that in a natural way give rise to fusion algebras. In [HI98] only finitely generated fusion algebras are considered; a fusion algebra $R = \mathbb{Z}[I]$ is called *finitely generated* if there exists a finitely supported probability measure μ on I such that

$$I = \bigcup_{n \in \mathbb{N}} \operatorname{supp}(\mu^{*n}) \quad \text{and} \quad \forall \xi \in I : \mu(\bar{\xi}) = \mu(\xi).$$

That is, if the union of the supports of all powers of μ, with respect to convolution, equals I and if μ is invariant under the involution. The first condition is referred to as *nondegeneracy* of μ and the second condition is referred to as *symmetry* of μ.

Definition 3.2.1 ([HI98, 4.3]). *A finitely generated fusion algebra $R = \mathbb{Z}[I]$ is called amenable if $\|\lambda_{p,\mu}\| = 1$ for some finitely supported, symmetric, nondegenerate probability measure μ on I and some $1 < p < \infty$.*

From [HI98, 4.1], it follows that this is independent of the choice of μ and p. The above property is in [HI98] referred to as *strong* amenability of the fusion algebra, in contrast to *weak* amenability which is defined by requiring the existence of a state m on $\ell^\infty(I)$ such that $m(\lambda_\xi(f)) = m(f)$ for all $f \in \ell^\infty(I)$ and all $\xi \in I$. However, since we will not need the notion of weak amenability we have chosen to suppress the adjective "strong". We will have to treat fusion algebras that are not necessarily finitely generated and, to accommodate this, the definition of amenability is modified in the following way.

Definition 3.2.2 ([Kye07, 3.1]). *A fusion algebra $R = \mathbb{Z}[I]$ is called amenable if $1 \in \sigma(\lambda_{2,\mu})$ for each finitely supported, symmetric probability measure μ on I.*

Here, and in what follows, $\sigma(T)$ denotes the spectrum of a given operator T. This definition is justified by the following result.

Proposition 3.2.3 ([Kye07]). *Definition 3.2.2 extends Definition 3.2.1*

This result was remarked briefly in [Kye07] and, for the convenience of the reader, we therefore include a proof.

Proof. A straight forward calculation reveals that for a finitely supported function $\varphi\colon I \to \mathbb{C}$ we have $\lambda_{2,\varphi}^* = \lambda_{2,\tilde{\varphi}}$, where $\tilde{\varphi}\colon I \to \mathbb{C}$ is given by

$$\tilde{\varphi}(\xi) = \overline{\varphi(\bar{\xi})}.$$

Therefore, if μ is a finitely supported, symmetric probability measure then $\lambda_{2,\mu}$ is self-adjoint and by [HI98, 1.3] we have $\|\lambda_{2,\mu}\| \leq \|\mu\|_1 = 1$. Assume now that R is finitely generated and satisfies the requirement in Definition 3.2.1. It then follows from [HI98, 4.1] that $1 \in \sigma(\lambda_{2,\mu})$ for every finitely supported, symmetric probability measure μ on I. Conversely, if R satisfies the condition in Definition 3.2.2 then, in particular, we have that $1 \in \sigma(\lambda_{2,\mu})$ for all finitely supported, symmetric, *nondegenerate* probability measures μ and since $\|\lambda_{2,\mu}\| \leq 1$ we conclude that $\|\lambda_{2,\mu}\| = 1$. □

Next we shall introduce Følner-type conditions for fusion algebras, and for that we will need the following notion of boundary.

Definition 3.2.4 ([Kye07, 3.2]). *Let $R = \mathbb{Z}[I]$ be a fusion algebra. For two finite subsets $S, F \subseteq I$ we define the boundary of F relative to S as the set*

$$\partial_S(F) = \{\alpha \in F \mid \exists\, \xi \in S : \operatorname{supp}(\alpha\xi) \nsubseteq F\} \cup \{\alpha \in F^c \mid \exists\, \xi \in S : \operatorname{supp}(\alpha\xi) \nsubseteq F^c\}.$$

Here, and in the sequel, F^c denotes the complement of F in I.

In [HI98], several equivalent conditions for amenability is given in Theorem 4.1 and Theorem 4.6, the latter including two Følner-type conditions for finitely generated fusion algebras. The following theorem generalizes Theorem 4.6, by removing the requirement that the fusion algebra be finitely generated and by adding a third type of Følner condition (FC3) which will turn out to be important in connection with deriving a Følner condition for coamenable, compact quantum groups.

Theorem 3.2.5 ([Kye07, 3.3]). *Let $R = \mathbb{Z}[I]$ be a fusion algebra with dimension function d. Then the following are equivalent.*

(A) *The fusion algebra is amenable.*

(FC1) *For every finitely supported, symmetric probability measure μ with e in* $\operatorname{supp}(\mu)$ *and every $\varepsilon > 0$ there exists a finite subset $F \subseteq I$ such that*

$$\sum_{\xi \in \operatorname{supp}(\chi_F * \mu)} d(\xi)^2 < (1+\varepsilon) \sum_{\xi \in F} d(\xi)^2.$$

(FC2) *For every finite, non-empty subset $S \subseteq I$ and every $\varepsilon > 0$ there exists a finite subset $F \subseteq I$ such that*

$$\forall\, \xi \in S: \ \|\rho_{1,\xi}(\chi_F) - \chi_F\|_{1,\sigma} < \varepsilon \|\chi_F\|_{1,\sigma}.$$

(FC3) *For every finite, non-empty subset $S \subseteq I$ and every $\varepsilon > 0$ there exists a finite subset $F \subseteq I$ such that*

$$\sum_{\xi \in \partial_S(F)} d(\xi)^2 < \varepsilon \sum_{\xi \in F} d(\xi)^2.$$

Classically, amenability is a property that can be possessed by a group, and we will now show how the definition of amenability for fusion algebras generalizes the notion of amenability for groups. Consider a discrete, countable group Γ and recall that Γ is said to be amenable if it admits a left invariant (equivalently bi-invariant) mean; i.e. a state $m\colon \ell^\infty(\Gamma) \to \mathbb{C}$ such that for every $f \in \ell^\infty(\Gamma)$ and every $\gamma \in \Gamma$ we have $m(\lambda_\gamma(f)) = m(f)$. A fundamental result due to E. Følner ([Føl55]) says that Γ is amenable if and only if it satisfies a certain geometric condition called the Følner condition. There are many equivalent formulations of this condition. Here we will present the following, found in [BP92, F.6].

Theorem 3.2.6 ([Føl55]). *The group Γ is amenable if and only if the following holds: For every $\varepsilon > 0$ and every finite, non-empty subset $S \subseteq \Gamma$ there exists a finite subset $F \subseteq \Gamma$ such that*

$$|\{a \in F \mid \exists s \in S : as \notin F\}| < \varepsilon |F|.$$

Here, and in what follows, $|F|$ denotes the cardinality of a given set F. An easy exercise shows that Γ satisfies the condition given by Theorem 3.2.6 exactly when the fusion algebra $\mathbb{Z}[\Gamma]$ satisfies (FC3) from Theorem 3.2.5. This means that Γ is amenable if and only if the fusion algebra $\mathbb{Z}[\Gamma]$ is amenable.

Chapter 4

A Følner condition for quantum groups

Følner's condition for discrete groups (Theorem 3.2.6) has turned out to be effective when proving statements about amenable groups; for instance it enters in the proof of the fact that the full and the reduced group C^*-algebra of an amenable group coincide, as well as in the proof of nuclearity of the reduced group C^*-algebra. It is therefore desirable to obtain a quantum analogue of this result. The aim of this short chapter is to introduce such a Følner condition for compact quantum groups and, of course, to relate it to the notion of coamenability. Before presenting the condition, we remind the reader that for a corepresentation u, of a compact quantum group $\mathbb{G}$, n_u denotes the matrix size of u.

Definition 4.1.7 ([Kye07, 4.9])**.** *A compact quantum group $\mathbb{G} = (A, \Delta)$ is said to satisfy Følner's condition if the following holds. For every finite, non-empty subset $S \subseteq \mathrm{Irred}(\mathbb{G})$ and every $\varepsilon > 0$ there exists a finite set $F \subseteq \mathrm{Irred}(\mathbb{G})$ such that*

$$\sum_{u \in \partial_S(F)} n_u^2 < \varepsilon \sum_{u \in F} n_u^2.$$

Here $\partial_S(F)$ is the boundary of F relative to S, as defined in Definition 3.2.4.

In other words, $\mathbb{G}$ satisfies Følner's condition exactly when the fusion algebra $R(\mathbb{G}) = \mathbb{Z}[\mathrm{Irred}(\mathbb{G})]$ satisfies (FC3) from Theorem 3.2.5, which in turn is equivalent to the fusion algebra $R(\mathbb{G})$ being amenable. The main result in this chapter is the following which was remarked, without proof, in [HI98] in the restricted context of matrix quantum groups with tracial Haar states.

Theorem 4.1.8 ([Kye07, 4.5])**.** *A compact quantum group $\mathbb{G} = (A, \Delta)$ is coamenable if and only if the fusion algebra $R(\mathbb{G})$ is amenable.*

The proof of Theorem 4.1.8 has two main ingredients; the first one is the quantum Kesten condition (Proposition 2.3.8) and the second is the following lemma which is of interest in itself. Before stating the lemma we introduce some notation.

Let $\mathbb{G}$ be any compact quantum group and consider the complex corepresentation ring $R(\mathbb{G}) = \mathbb{C}[\mathrm{Irred}(\mathbb{G})]$ with its natural trace τ defined in the remarks following Definition 3.1.1. By Remark 3.1.5, the GNS representation π_τ consists of bounded operators and we may therefore consider the enveloping C^*-algebra

$$C^*_{\mathrm{red}}(R(\mathbb{G})) = \overline{\pi_\tau(R(\mathbb{G}))}^{\|\cdot\|_\tau} \subseteq B(L^2(R(\mathbb{G}), \tau)).$$

The statement, which was mentioned without proof in [Ban99a], now is the following.

Lemma 4.1.9 ([Kye07, 4.6])**.** *The character map $\chi\colon R(\mathbb{G}) \longrightarrow A_0$ (described in Section 3.1.1) extends to a bounded, injective $*$-homomorphism*

$$\chi\colon C^*_{\mathrm{red}}(R(\mathbb{G})) \longrightarrow A_{\mathrm{red}}.$$

Having Theorem 4.1.8, we immediately obtain the following.

Corollary 4.1.10 ([Kye07, 4.10])**.** *A compact quantum group is coamenable if and only if it satisfies Følner's condition.*

An important consequence of Theorem 4.1.8 is that the answer to the question of whether a compact quantum group is coamenable or not, is only depending on its corepresentation theory (a fact already noted by T. Banica ([Ban99a]) in the case of compact matrix quantum groups). For instance, if the corepresentation theory of a compact quantum group coincides with the representation theory of a compact group, then the quantum group is automatically coamenable since this is the case for the Gelfand dual of the compact group. This is the case for Woronowicz's quantum $SU(2)$ groups discussed in Example 2.1.5. Formulated precisely, the following holds.

Corollary 4.1.11. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be compact quantum groups and assume that $\mathbb{G}_1$ is coamenable. If there exists a dimension preserving bijection $\varphi\colon \mathrm{Irred}(\mathbb{G}_1) \to \mathrm{Irred}(\mathbb{G}_2)$ whose $\mathbb{Z}$-linear extension $\varphi\colon R(\mathbb{G}_1) \to R(\mathbb{G}_2)$ is a unital $*$-ring isomorphism, then $\mathbb{G}_2$ is also coamenable.*

For another application of the quantum Følner condition, the reader is referred to Theorem 5.2.1, and the corollaries following it, whose proof depends heavily on Følner's condition and it being equivalent to coamenability.

Chapter 5

L^2-invariants for quantum groups

In this chapter we give an introduction to L^2-invariants for compact quantum groups, based on the author's work in the articles [Kye06] and [Kye07]. The aim is to introduce the theory and present the results in a unified way, and therefore very few proofs will be presented. The reader is referred to the articles [Kye06] and [Kye07], found at the end of the thesis, for proofs of the theorems presented.

5.1 Definitions and computational results

The results presented in this section all stem from the article [Kye06]. Although parts of the results, e.g. Theorem 5.1.7, will be contained in a more general result (Theorem 5.2.1) discussed at the end of the present chapter, we present it here as well for the sake of giving a comprehensive introduction to the article [Kye06].

Consider a compact quantum group $\mathbb{G} = (A, \Delta)$ with Haar state h. Denote by $(A_0, \Delta, S, \varepsilon)$ the Hopf $*$-algebra generated by the matrix coefficients arising from irreducible corepresentations of $\mathbb{G}$, and by H the GNS space $L^2(A, h)$ with corresponding GNS representation π_h. Let M denote the enveloping von Neumann algebra $\pi_h(A)'' \subseteq B(H)$. To motivate our definitions we first present the following well-known result.

Proposition 5.1.1. *Assume that* $\mathbb{G} = (C^*_{\mathrm{red}}(\Gamma), \Delta_{\mathrm{red}})$ *for a discrete, countable group* Γ *with unit* e *and left regular representation* λ*. We then have the following.*

- *The Haar state* h *is the natural trace* $\tau(x) = \langle x\delta_e | \delta_e \rangle$*.*
- *The Hopf* $*$*-algebra* A_0 *becomes* $\lambda(\mathbb{C}\Gamma)$*.*
- *The enveloping von Neumann algebra* M *becomes* $\mathscr{L}(\Gamma)$*.*
- *The counit* ε *is the linear extension of the trivial representation of* Γ*.*

Proposition 5.1.1 seems to be a folklore-type result and, for the convenience of the reader, we therefore include a proof.

Proof. For $\gamma \in \Gamma$ we have

$$\begin{aligned}(\tau \otimes \mathrm{id})\Delta_{\mathrm{red}}(\lambda_\gamma) &= (\tau \otimes \mathrm{id})(\lambda_\gamma \otimes \lambda_\gamma)\\ &= \delta_{e,\gamma}\lambda_\gamma\\ &= \delta_{e,\gamma}\lambda_e\\ &= \tau(\lambda_\gamma)1\\ &= (\mathrm{id} \otimes \tau)\Delta_{\mathrm{red}}(\lambda_\gamma).\end{aligned}$$

By linearity and continuity we see that τ satisfies the defining property for the Haar state; thus $h = \tau$. For the computation of A_0, first note that each λ_γ is a 1-dimensional, unitary corepresentation and therefore $\lambda(\mathbb{C}\Gamma) \subseteq A_0$. If u is an irreducible, unitary corepresentation not equivalent to any of the λ_γ's then, by Theorem 2.2.4, each of its matrix coefficients u_{ij} is orthogonal in $L^2(C^*_{\mathrm{red}}(\Gamma), h)$ to the dense set $\lambda(\mathbb{C}\Gamma)$. Hence $h(u_{ij}^* u_{ij}) = 0$ and since h is faithful on A_0 this forces u_{ij} to be zero — contradicting the fact that u is unitary. Thus $A_0 = \lambda(\mathbb{C}\Gamma)$. It is a routine to check that the map $\lambda_\gamma \mapsto \delta_\gamma$ extends to a unitary $U\colon L^2(C^*_{\mathrm{red}}(\Gamma), h) \to \ell^2(\Gamma)$ intertwining the GNS representation π_h with the left regular representation λ; hence $M = \mathscr{L}(\Gamma)$. □

The above proposition supplies us with the following guiding dictionary which we shall use to extend the notion of L^2-homology from groups to quantum groups.

Discrete Group Γ	Compact Quantum Group $\mathbb{G} = (A, \Delta)$
Trivial representation of Γ	Counit $\varepsilon\colon A_0 \to \mathbb{C}$
Von Neumann trace $\tau(x) = \langle x\delta_e \mid \delta_e\rangle$	Haar state h
Group algebra $\mathbb{C}\Gamma$	Algebra of matrix coefficients A_0
Reduced C^*-algebra $C^*_{\mathrm{red}}(\Gamma)$	Reduced C^*-algebra $A_{\mathrm{red}} = \pi_h(A)$
Group von Neumann algebra $\mathscr{L}(\Gamma)$	$M = A''_{\mathrm{red}} \subseteq B(L^2(A,h))$

Drawing inspiration from this analogy table , we now make the following definition.

Definition 5.1.2 ([Kye06, 1.1,1.2])**.** *The p-th L^2-homology of $\mathbb{G}$ is defined as*

$$H_p^{(2)}(\mathbb{G}) = \mathrm{Tor}_p^{A_0}(M, \mathbb{C}),$$

where $\mathbb{C}$ is considered a left A_0-module via the counit $\varepsilon\colon A_0 \to \mathbb{C}$ and M is considered a right A_0-module via the GNS representation π_h. If h is tracial it extends to a faithful, normal, tracial state on M and the p-th L^2-Betti number and p-th capacity of $\mathbb{G}$ is then defined as

$$\beta_p^{(2)}(\mathbb{G}) = \dim_M(H_p^{(2)}(\mathbb{G})) \qquad \text{and} \qquad c_p(\mathbb{G}) = c_M(H_p^{(2)}(\mathbb{G})),$$

respectively. Both the dimension and the capacity are calculated with respect to the extended trace-state $h\colon M \to \mathbb{C}$.

Note that both the commutative and cocommutative examples satisfy that the Haar state is tracial, and one can also prove that this is the case for the free orthogonal and unitary quantum groups $A_o(n)$ and $A_u(n)$ introduced in Example 2.1.6. However, there are also important examples of quantum groups with non-tracial Haar state; for instance $SU_q(2)$ for $q \neq \pm 1$.

As a direct consequence of Proposition 5.1.1 and Definition 1.1.4 we obtain the following.

Proposition 5.1.3 ([Kye06, 1.3]). *If* $\mathbb{G} = (C^*_{\text{red}}(\Gamma), \Delta_{\text{red}})$ *then for all* $p \geq 0$ *we have* $\beta_p^{(2)}(\mathbb{G}) = \beta_p^{(2)}(\Gamma)$ *and* $c_p(\mathbb{G}) = c_p(\Gamma)$.

Since computing L^2-invariants for groups is generally hard, computing L^2-invariants for quantum groups must also be expected to be difficult, but as in the classical case the zeroth invariants are the most manageable. In the case of a discrete group Γ, one has $\beta_0^{(2)}(\Gamma) = \frac{1}{|\Gamma|} \in [0, \infty[$ and the following two results can therefore be viewed as the quantum group analogue of this result.

Proposition 5.1.4 ([Kye06, 2.9]). *Let* $\mathbb{G} = (A, \Delta)$ *be a quantum group and assume that* A *has finite linear dimension* N*. Then* $\beta_0^{(2)}(\mathbb{G}) = \frac{1}{N}$ *and* $\beta_p^{(2)}(\mathbb{G}) = 0$ *for* $p \geq 1$*. Moreover,* $c_p(\mathbb{G}) = 0^-$ *for all* $p \geq 0$.

Note that, in the above proposition, we do not have to require that the Haar state is tracial since this is automatic ([VD97]) for finite quantum groups, i.e. quantum groups whose underlying C^*-algebra is finite dimensional. As the following proposition shows, the statement about the zeroth L^2-Betti number from Proposition 5.1.4 holds true also in the infinite case.

Proposition 5.1.5 ([Kye06, 2.2]). *Let* $\mathbb{G} = (A, \Delta)$ *be a compact quantum group with tracial Haar state. If* $\dim_{\mathbb{C}}(A) = \infty$ *then* $\beta_0^{(2)}(\mathbb{G}) = 0$.

Proposition 5.1.5 is proved in [Kye06, 2.2] in the case where the enveloping von Neumann algebra M is a factor. It was later pointed out by S. Vaes that the result remains true also without the factor assumption and we therefore include a proof of Proposition 5.1.5 here. The proof uses the duality theory of locally compact quantum groups, which is a more general quantum group theory than the one described in Chapter 2. One of its main advantages is that every locally compact group has a dual which is again a locally compact quantum group. We refer the reader to [KV03] for an introduction to this theory.

Proof of Proposition 5.1.5. First note that

$$\mathrm{Tor}_0^{A_0}(M, \mathbb{C}) \simeq M \underset{A_0}{\odot} \mathbb{C} \simeq M/J,$$

where J is the left ideal in M generated by $\pi_h(\ker(\varepsilon))$. Denote by $\bar{J}$ the strong operator closure of J and note that

$$J \subseteq \bar{J} \subseteq \overline{J}^{\text{alg}} = \bigcap_{\substack{f \in \mathrm{Hom}_M(M,M) \\ J \subseteq \ker(f)}} \ker(f).$$

Since M is finitely (actually singly) generated as an M-module, Theorem 1.1.1 implies that $\dim_M(J) = \dim_M(\overline{J}^{\mathrm{alg}})$ and thus

$$\beta_0^{(2)}(\mathbb{G}) = 1 - \dim_M(J) = 1 - \dim_M(\bar{J}).$$

Our aim now is to prove that $\bar{J} = M$. Assume, conversely, that that $\bar{J} \neq M$ and note that since J is convex, $\bar{J}$ is weak operator closed as well. Because $1 \notin \bar{J}$, the counit ε extends naturally to the weakly closed subspace

$$\mathbb{C} + \bar{J} = \{\lambda 1 + x \mid \lambda \in \mathbb{C}, x \in \bar{J}\} \subseteq M,$$

by setting $\varepsilon(\lambda 1 + x) = \lambda$. To see that this extends ε, just note that each element $a \in A_0$ can be written uniquely as the sum of a scalar and an element from J:

$$a = \varepsilon(a)1 + (a - \varepsilon(a)1).$$

The extension $\varepsilon\colon \mathbb{C} + \bar{J} \to \mathbb{C}$ is weakly continuous since its kernel $\bar{J}$ is weakly closed ([KR83, 1.2.5]). By the Hahn-Banach Theorem, we may therefore extend ε to a weakly continuous functional, also denoted ε, on $B(H)$ where H denotes the GNS space $L^2(A, h)$. In particular, ε is weakly continuous on the unit ball of $B(H)$ and thus $\varepsilon \in B(H)_*$. Denote by η the natural inclusion $A_0 \subseteq H$ and by $W \in B(H \bar{\otimes} H)$ the multiplicative unitary for (M, Δ) given by

$$W^*(\eta(x) \otimes \eta(y)) = (\eta \otimes \eta)(\Delta(y)(x \otimes 1)).$$

For any $\omega \in B(H)_*$ and any $x \in A_0$ we have

$$(\omega \otimes \mathrm{id})(W^*)(\eta(x)) = \eta((\omega \otimes \mathrm{id})\Delta(x)). \qquad (\dagger)$$

This can be proved by a direct calculation when ω has the form $T \mapsto \langle T\eta(a) | \eta(b) \rangle$ and the general case follows from this since $B(H)_*$ is the norm closure of the linear span of such functionals ([KR86, 7.4.4]). See e.g. [KV00, 2.10] for more details. Using the formula ($\dagger$) with $\omega = \varepsilon$ we therefore obtain

$$(\varepsilon \otimes \mathrm{id})(W^*) = 1.$$

Since ε is weakly continuous, $\varepsilon \otimes \mathrm{id}$ restricts to a $*$-homomorphism from $M \bar{\otimes} B(H)$ to $B(H)$ and since $W \in M \bar{\otimes} B(H)$ it follows that $(\varepsilon \otimes \mathrm{id})(W) = 1$. This implies that the C^*-algebra of the reduced, C^*-algebraic, dual quantum group $\hat{\mathbb{G}}$, given by

$$\hat{A} = C^*\{(\omega \otimes \mathrm{id})W \mid \omega \in B(H)_*\} \subseteq B(H),$$

is unital. Therefore $\hat{\mathbb{G}}$ is compact and $\mathbb{G}$ thus both discrete and compact. This forces A to be finite dimensional, contradicting the assumptions. $\square$

Recall from Theorem 1.2.6 that $c_0(\mathbb{Z}^n) = \frac{1}{n}$. Using Pontryagin duality, this translates into the following identity:

$$\frac{1}{n} = c_0(C^*_{\mathrm{red}}(\mathbb{Z}^n), \Delta_{\mathrm{red}}) = c_0(C(\mathbb{T}^n), \Delta_c),$$

where $\mathbb{T}^n$ denotes the n-torus. The following theorem generalizes this result from the class of connected, compact, abelian Lie groups (i.e. tori) to the class of all compact Lie groups.

Theorem 5.1.6 ([Kye06, 2.4]). *Let G be a compact Lie group of positive dimension with Haar probability measure μ and put $\mathbb{G} = (C(G), \Delta_c)$. Then $H_0^{(2)}(\mathbb{G})$ is a finitely presented, zero-dimensional $L^\infty(G,\mu)$-module and*

$$c_0(\mathbb{G}) = \frac{1}{\dim(G)}.$$

Here $\dim(G)$ *is the dimension of G considered as a manifold.*

Note that Theorem 5.1.6 implies that $\beta_0^{(2)}(C(G), \Delta_c) = 0$; a fact that also follows directly from Proposition 5.1.5. Of course the geometry of the underlying Lie group is used to prove Theorem 5.1.6, and if this geometry is exploited in another way we obtain the following vanishing result.

Theorem 5.1.7 ([Kye06, 3.3]). *If $\mathbb{G} = (C(G), \Delta_c)$ for a compact, connected Lie group G then for any $C(G)_0$-module Z and any $p \geq 1$ we have*

$$\dim_{L^\infty(G,\mu)} \operatorname{Tor}_p^{C(G)_0}(L^\infty(G,\mu), Z) = 0.$$

In particular, $\beta_p^{(2)}(\mathbb{G}) = 0$ for every $p \geq 1$.

As before, μ denotes the Haar probability measure on G and the dimension is computed with respect to the corresponding trace-state given by $f \mapsto \int_G f \, \mathrm{d}\mu$.

By construction, the theory of L^2-homology for compact quantum groups is linked to the corresponding theory for discrete groups via Proposition 5.1.3. It is of course also desirable to connect the theory to the approach of A. Connes and D. Shlyakhtenko explained in Section 1.3. This is done in the following proposition which should be considered as a quantum group analogue of Proposition 1.3.1.

Proposition 5.1.8 ([Kye06, 4.1]). *If $\mathbb{G} = (A, \Delta)$ is a compact quantum group with tracial Haar state h then $\beta_p^{(2)}(\mathbb{G}) = \beta_p^{(2)}(A_0, h)$, where the right hand side is the p-th Connes-Shlyakhtenko L^2-Betti number of the tracial $*$-algebra (A_0, h).*

The Connes-Shlyakhtenko L^2-Betti numbers have turned out to be very difficult to compute in general (see e.g. the discussion in Section 1.3) and because of that, one should probably consider the formula $\beta_p^{(2)}(\mathbb{G}) = \beta_p^{(2)}(A_0, h)$ as a formula to compute the Connes-Shlyakhtenko L^2-Betti numbers of certain Hopf $*$-algebras — rather than a formula to compute the L^2-Betti numbers of the corresponding compact quantum group. For instance, combining Theorem 5.1.7 and Proposition 5.1.8 we obtain the following.

Corollary 5.1.9. *If G is a compact, connected Lie group with Haar measure μ and A_0 denotes the algebra of representative functions on G then $\beta_p^{(2)}(A_0, \mathrm{d}\mu) = 0$ for all $p \geq 1$.*

Recall from Remark 2.2.8 that the algebra of representative functions on G is the same as the algebra of matrix coefficients of $(C(G), \Delta_c)$.

5.2 The coamenable case

In this final section, we present the main theorem in [Kye07] which served as the motivation for introducing the quantum Følner condition since this condition enters, in an essential way, in the proof.

Theorem 5.2.1 ([Kye07, 6.1]). *Let $\mathbb{G} = (A, \Delta)$ be a coamenable, compact quantum group with tracial Haar state. Then for any A_0-module Z and any $p \geq 1$ we have*

$$\dim_M \operatorname{Tor}_p^{A_0}(M, Z) = 0.$$

As usual, the dimension is computed with respect to the extension of the Haar state h to M. If M were flat as a module over A_0 we would have $\operatorname{Tor}_p^{A_0}(M, Z) = 0$ for any Z and any $p \geq 1$, and the property in Theorem 5.2.1 is therefore referred to as *dimension flatness of the von Neumann algebra over the algebra of matrix coefficients*. Theorem 5.2.1 is a quantum group analogue of the following theorem of W. Lück which we obtain as a corollary.

Corollary 5.2.2 ([Lüc98, 5.1]). *If Γ is an amenable, countable, discrete group then for any $\mathbb{C}\Gamma$-module Z and any $p \geq 1$ we have*

$$\dim_{\mathscr{L}(\Gamma)} \operatorname{Tor}_p^{\mathbb{C}\Gamma}(\mathscr{L}(\Gamma), Z) = 0.$$

Proof. Put $\mathbb{G} = (C^*_{\mathrm{red}}(\Gamma), \Delta_{\mathrm{red}})$. Then $\mathbb{G}$ is coamenable if and only if Γ is amenable and the result now follows from Theorem 5.2.1. □

In [CG86], J. Cheeger and M. Gromov prove that an amenable, discrete group has vanishing L^2-Betti numbers in all positive degrees; a result that also follows from Corollary 5.2.2 by setting $Z = \mathbb{C}$. Similarly, we obtain from Theorem 5.2.1 the following.

Corollary 5.2.3 ([Kye07, 6.2]). *If $\mathbb{G}$ is a coamenable, compact quantum group with tracial Haar state then $\beta_p^{(2)}(\mathbb{G}) = 0$ for all $p \geq 1$.*

Noting that abelian, compact quantum groups are coamenable, we obtain the following (slight) generalization of Theorem 5.1.7 as a corollary.

Corollary 5.2.4. *If $\mathbb{G} = (C(G), \Delta_c)$ for a compact group G then for any $C(G)_0$-module Z and any $p \geq 1$ we have*

$$\dim_{L^\infty(G,\mu)} \operatorname{Tor}_p^{C(G)_0}(L^\infty(G, \mu), Z) = 0.$$

In particular, $\beta_p^{(2)}(\mathbb{G}) = 0$ for every $p \geq 1$.

Here μ denotes the Haar probability measure on G and the dimension is computed with respect to the corresponding trace-state given by $f \mapsto \int_G f \, \mathrm{d}\mu$.

Combining Theorem 5.2.1 and Proposition 5.1.8 we get.

Corollary 5.2.5. *Let $\mathbb{G} = (A, \Delta)$ be a compact, coamenable quantum group with tracial Haar state h. Then $\beta_p^{(2)}(A_0, h) = 0$ for all $p \geq 1$, where $\beta_p^{(2)}(A_0, h)$ is the p-th Connes-Shlyakhtenko L^2-Betti number of the tracial $*$-algebra (A_0, h).*

The knowledge of dimension-flatness also gives genuine homological information about the ring extension $A_0 \subseteq M$. More precisely, the following holds.

Corollary 5.2.6. *If $\mathbb{G} = (A, \Delta)$ is compact and coamenable with tracial Haar state then the induction functor $M \odot_{A_0} -$ is an exact functor from the category of finitely generated, projective A_0-modules to the category of finitely generated, projective M-modules.*

Since this is not made explicit in [Kye07] we present the proof.

Proof. Let X and Y be finitely generated, projective A_0-modules and let $f \colon X \to Y$ be an injective homomorphism. Then

$$0 \longrightarrow X \xrightarrow{f} Y \longrightarrow Y/\mathrm{rg}(f) \longrightarrow 0,$$

is a projective resolution of $Y/\mathrm{rg}(f)$. Thus $\mathrm{Tor}_1^{A_0}(M, Y/\mathrm{rg}(f)) = \ker(\mathrm{id}_M \otimes f)$ and from Theorem 5.2.1 we conclude that $\dim_M(\ker(\mathrm{id}_M \otimes f)) = 0$. Because $\mathrm{id}_M \otimes f$ is a map of finitely generated projective M-modules, $\ker(\mathrm{id}_M \otimes f) = \overline{\ker(\mathrm{id}_M \otimes f)}^{\mathrm{alg}}$ and by Theorem 1.1.1 we conclude that $\ker(\mathrm{id}_M \otimes f)$ is finitely generated and projective. But, since the dimension function is faithful on the category of finitely generated, projective modules this forces $\ker(\mathrm{id}_M \otimes f) = \{0\}$ and the claim follows. □

Corollary 5.2.6 in particular implies the following result, which was pointed out to us by A. Thom.

Corollary 5.2.7. *If $\mathbb{G} = (A, \Delta)$ is compact and coamenable with tracial Haar state and $x \in A_0$ is a non-zerodivisor in A_0 then x is also a non-zerodivisor in M.*

Proof. This follows by using Corollary 5.2.6 on the injective map $a \longmapsto ax$. □

We end this section by discussing some open problems concerning the theory of L^2-invariants for quantum groups. First of all, the theory lacks examples in which the L^2-invariants can be explicitly computed. For instance, except for the trivial cases, i.e. where the quantum group is cocommutative or finite, there are no known examples of a compact quantum groups with a non-vanishing L^2-Betti number. As Corollary 5.2.3 shows, such an example has to be non-coamenable and natural candidates are therefore the free unitary quantum groups $A_u(n)$. However, so far no manageable projective resolution of $\mathbb{C}$, as a module over the Hopf $*$-algebra of matrix coefficients $A_u(n)_0$, is known. Unpublished work of B. Collins, J. Härtel and A. Thom shows that in the case of $A_o(n)$, the trivial module $\mathbb{C}$ allows a resolution of length 4 consisting of finitely generated, free modules. It is likely that their methods may be generalized to give a manageable resolution also in the case of $A_u(n)$. Using the above mentioned resolution, it is not difficult to see that $\beta_p^{(2)}(A_o(n)) = 0$ for $p \notin \{1,2\}$. The values of $\beta_1^{(2)}(A_o(n))$ and $\beta_2^{(2)}(A_o(n))$ are still unknown except in the case $n = 2$ where the coamenability of $A_o(n)$ together with Corollary 5.2.3 gives $\beta_1^{(2)}(A_o(2)) = \beta_2^{(2)}(A_o(2)) = 0$.

Secondly, it is a considerable drawback that the numerical L^2-invariants are only defined when the Haar state is a trace. There are many interesting examples where this is not the case — most famous among these is probably $SU_q(2)$ for $q \neq \pm 1$ — and it is therefore desirable to find a way of extending the definition to cover also the non-tracial case. At the moment, it is not clear how to obtain such an extension, but it might be that the trace on the corepresentation ring can be used to measure the sizes of some, perhaps modified, L^2-homology modules of a generic, compact quantum group. This, however, is only speculations. As an even more ambitious project, one could hope some day to find a unifying definition of L^2-invariants covering the whole class of locally compact quantum groups.

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Article I

L^2-Homology For Compact Quantum Groups

L^2-HOMOLOGY FOR COMPACT QUANTUM GROUPS

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Abstract
A notion of L^2-homology for compact quantum groups is introduced, generalizing the classical notion for countable, discrete groups. If the compact quantum group in question has tracial Haar state, it is possible to define its L^2-Betti numbers and Novikov-Shubin invariants/capacities. It is proved that these L^2-Betti numbers vanish for the Gelfand dual of a compact Lie group and that the zeroth Novikov-Shubin invariant equals the dimension of the underlying Lie group. Finally, we relate our approach to the approach of A. Connes and D. Shlyakhtenko by proving that the L^2-Betti numbers of a compact quantum group, with tracial Haar state, are equal to the Connes-Shlyakhtenko L^2-Betti numbers of its Hopf $*$-algebra of matrix coefficients.

1. Introduction and definitions

The notion of L^2-invariants was introduced by M. F. Atiyah in [1] in the setting of a Riemannian manifold endowed with a free, proper and cocompact action of a discrete, countable group. Later this notion was generalized by J. Cheeger and M. Gromov in [4] and by W. Lück in [16]. The latter of these generalizations makes it possible to define L^2-homology and L^2-Betti numbers of an arbitrary topological space equipped with an arbitrary action of a discrete, countable group Γ. In particular, the L^2-homology and L^2-Betti numbers of Γ, which are defined in [16] using the action of Γ on $E\Gamma$, make sense and can be expressed in the language of homological algebra as

$$H_n^{(2)}(\Gamma) = \operatorname{Tor}_n^{\mathbb{C}\Gamma}(\mathscr{L}(\Gamma), \mathbb{C}) \quad \text{and} \quad \beta_n^{(2)}(\Gamma) = \dim_{\mathscr{L}(\Gamma)} H_n^{(2)}(\Gamma),$$

where $\dim_{\mathscr{L}(\Gamma)}(\cdot)$ is W. Lück's generalized Murray-von Neumann dimension introduced in [16]. A detailed introduction to this dimension theory can be found in [17].

Consider now a compact quantum group $\mathbb{G} = (A, \Delta)$ in the sense of S. L. Woronowicz; i.e. A is a unital C^*-algebra and $\Delta\colon A \to A \otimes A$ is a unital, coassociative $*$-homomorphism satisfying a certain non-degeneracy condition. We shall not elaborate further on the notion of compact quantum groups, but refer the reader to the survey articles [19] and [13] for more details. Denote by h the Haar state on $\mathbb{G}$ and by $(A_0, \Delta, S, \varepsilon)$ the Hopf $*$-algebra of matrix coefficients arising from irreducible corepresentations of $\mathbb{G}$. We recall ([19, Prop. 7.8]) that h is faithful on A_0. Consider the GNS-representation π_h of A on $L^2(A, h)$ and denote by $\mathscr{M}$ the enveloping von Neumann algebra $\pi_h(A)''$. We then make the following definition:

Definition 1.1. The n-th L^2-homology of the compact quantum group $\mathbb{G} = (A, \Delta)$ is defined as

$$H_n^{(2)}(\mathbb{G}) = \operatorname{Tor}_n^{A_0}(\mathscr{M}, \mathbb{C}).$$

Here $\mathbb{C}$ is considered a left A_0-module via the counit ε and $\mathscr{M}$ is considered a right A_0-module via the natural inclusion $\pi_h\colon A_0 \to \mathscr{M}$. The groups $H_n^{(2)}(\mathbb{G})$ have a natural left $\mathscr{M}$-module structure and when the Haar state h is tracial we may therefore define the n-th L^2-Betti number of $\mathbb{G}$ as

$$\beta_n^{(2)}(\mathbb{G}) = \dim_{\mathscr{M}} H_n^{(2)}(\mathbb{G}),$$

where $\dim_{\mathscr{M}}(\cdot)$ is W. Lück's extended dimension function arising from the extension to $\mathscr{M}$ of the tracial Haar state h.

Similar to the algebraic extension of the notion of Murray-von Neumann dimension, the classical notion of Novikov-Shubin invariants was transported to an algebraic setting by W. Lück ([15]) using finitely presented modules, and generalized to arbitrary modules by W. Lück, H. Reich and T. Schick in [18]. This generalization was worked out using capacities which are essentially inverses of Novikov-Shubin invariants (cf. [18, Section 2]). In particular, they defined the n-th capacity of a discrete countable group Γ as $c_n(\Gamma) = c(H_n^{(2)}(\Gamma))$, the right-hand side being the capacity of the n-th L^2-homology of Γ considered as a left module over the group von Neumann algebra $\mathscr{L}(\Gamma)$. Following this approach we make the following definition:

Definition 1.2. If h is tracial we define the n-th capacity of $\mathbb{G}$ as $c_n(\mathbb{G}) = c(H_n^{(2)}(\mathbb{G}))$.

To justify Definition 1.1 and Definition 1.2 we prove the following:

Proposition 1.3. *Let Γ be a countable discrete group and consider the compact quantum group $\mathbb{G} = (C^*_{\mathrm{red}}(\Gamma), \Delta_{\mathrm{red}})$ where Δ_{red} is defined by $\Delta_{\mathrm{red}}\lambda_\gamma = \lambda_\gamma \otimes \lambda_\gamma$ and λ denotes the left regular representation of Γ. Then $H_n^{(2)}(\mathbb{G}) = H_n^{(2)}(\Gamma)$ and in particular*

$$\beta_n^{(2)}(\mathbb{G}) = \beta_n^{(2)}(\Gamma) \quad \text{and} \quad c_n(\mathbb{G}) = c_n(\Gamma),$$

for all $n \in \mathsf{N}_0$.

Here, and in what follows, N_0 denotes the set of non-negative integers.

Proof. Since the Haar state on $\mathbb{G}$ is the trace state $\tau(x) = \langle x\delta_e | \delta_e \rangle$, the GNS-action of $C^*_{\mathrm{red}}(\Gamma)$ on $L^2(C^*_{\mathrm{red}}(\Gamma), \tau)$ is naturally identified with the standard action of $C^*_{\mathrm{red}}(\Gamma)$ on $\ell^2(\Gamma)$. Note that each λ_γ is a one-dimensional (hence irreducible) corepresentation of $\mathbb{G}$ and that these span a dense subspace in $L^2(C^*_{\mathrm{red}}(\Gamma), h) \simeq \ell^2(\Gamma)$. It now follows from the quantum Peter-Weyl Theorem (cf. [13, Thm. 3.2.3]) that the Hopf $*$-algebra of matrix coefficients coincides with $\lambda(\mathsf{C}\Gamma)$ and from this we see that the counit coincides with the trivial representation of Γ. Thus

$$H_n^{(2)}(\mathbb{G}) = \mathrm{Tor}_n^{(C^*_{\mathrm{red}}(\Gamma))_0}(\mathscr{L}(\Gamma), \mathsf{C}) = \mathrm{Tor}_n^{\mathsf{C}\Gamma}(\mathscr{L}(\Gamma), \mathsf{C}) = H_n^{(2)}(\Gamma).$$

In the following sections we shall focus on computations of L^2-invariants for some concrete compact quantum groups. More precisely, the paper is organized as follows: In Section 2 we focus on the zeroth L^2-Betti number and capacity and prove that if the compact quantum group in question is the Gelfand dual $C(G)$ of a compact Lie group G with $\dim(G) \geq 1$, then the zeroth L^2-Betti number vanishes and the zeroth capacity equals the inverse of $\dim(G)$. Section 3 is devoted to proving that also the higher L^2-Betti numbers of the abelian quantum group $C(G)$ vanish in the case when G is a compact, connected Lie group. In Section 4 we prove that the L^2-Betti numbers of a compact quantum group $\mathbb{G} = (A, \Delta)$ with tracial Haar state h are equal to the Connes-Shlyakhtenko L^2-Betti numbers (see [6]) of the tracial $*$-algebra (A_0, h).

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Notation. All tensor products between C^*-algebras occurring in the following are assumed to be minimal/spatial. These will be denoted $\otimes$ while tensor products in the category of Hilbert spaces and the category of von Neumann algebras will be denoted $\bar{\otimes}$. Algebraic tensor products will be denoted $\odot$.

2. The zeroth L^2-invariants

In this section we focus on the zeroth L^2-Betti number and capacity. The first aim is to prove that the zeroth L^2-Betti number of a compact quantum group, whose enveloping von Neumann algebra is a finite factor, vanishes. After that we compute the zeroth L^2-Betti number and capacity for Gelfand duals of compact Lie groups and finally we study the L^2-invariants of finite dimensional quantum groups.

2.1. *The factor case*

In this subsection we investigate the case when the enveloping von Neumann algebra is a finite factor. First a small observation.

Observation 2.1. Let $\mathscr{M}$ be a von Neumann algebra and let A_0 be a strongly dense $*$-subalgebra of $\mathscr{M}$. Let J_0 be a two-sided ideal in A_0 and denote by J the left ideal in $\mathscr{M}$ generated by J_0. Then the strong operator closure $\bar{J}$ is a two-sided ideal in $\mathscr{M}$. Clearly $\bar{J}$ is a left ideal and because A_0 is dense in $\mathscr{M}$ we get that $xm \in \bar{J}$ whenever $x \in J$ and $m \in \mathscr{M}$. From this it follows that $\bar{J}$ is also a right ideal in $\mathscr{M}$.

The following proposition should be compared to [6, Cor. 2.8].

Proposition 2.2. *Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group with tracial Haar state h. Denote by π_h the GNS-representation of A on $L^2(A, h)$ and assume that $\mathscr{M} = \pi_h(A_0)''$ is a factor. If $A \neq \mathsf{C}$ then $\beta_0^{(2)}(\mathbb{G}) = 0$.*

Proof. First note that

$$H_0^{(2)}(\mathbb{G}) = \mathrm{Tor}_0^{A_0}(\mathscr{M}, \mathsf{C}) \simeq \mathscr{M} \underset{A_0}{\odot} \mathsf{C} \simeq \mathscr{M}/J,$$

where J is the left ideal in $\mathscr{M}$ generated by $\pi_h(\ker(\varepsilon))$. Since the counit $\varepsilon \colon A_0 \to \mathsf{C}$ is a $*$-homomorphism its kernel is a two-sided ideal in A_0, and by Observation 2.1 we conclude that the strong closure $\bar{J}$ is a two-sided ideal in $\mathscr{M}$. Since $A \neq \mathsf{C}$ the kernel of ε is non-trivial and hence $\bar{J}$ is nontrivial. Any finite factor is simple ([10, Cor. 6.8.4]) and therefore $\bar{J} = \mathscr{M}$. Now note that

$$J \subseteq \bar{J} \subseteq \overline{J}^{\mathrm{alg}} = \bigcap_{\substack{f \in \mathrm{Hom}_{\mathscr{M}}(\mathscr{M}, \mathscr{M}) \\ J \subseteq \ker(f)}} \ker(f).$$

Since $\mathscr{M}$ is finitely (singly) generated as an $\mathscr{M}$-module, [16, Thm. 0.6] implies that $\dim_{\mathscr{M}}(J) = \dim_{\mathscr{M}}(\overline{J}^{\mathrm{alg}})$ and thus

$$\dim_{\mathscr{M}}(J) = \dim_{\mathscr{M}}(\bar{J}) = \dim_{\mathscr{M}}(\mathscr{M}) = 1.$$

Additivity of the dimension function ([16, Thm. 0.6]) now yields the desired conclusion.

Denote by $A_o(n)$ the free orthogonal quantum group. The underlying C^*-algebra A is the universal, unital C^*-algebra generated by n^2 elements $\{u_{ij} \mid 1 \leq i, j \leq n\}$ subject to the relations making the matrix (u_{ij}) orthogonal. The comultiplication is defined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$$

and the antipode $S \colon A_0 \to A_0$ by $S(u_{ij}) = u_{ji}$. These quantum groups were discovered by S. Wang in [24] and studied further by T. Banica in [2]. See also [3] and [22].

Corollary 2.3. *For $n \geq 3$ we have $\beta_0^{(2)}(A_o(n)) = 0$.*

Proof. Denote by (u_{ij}) the fundamental corepresentation of $A_o(n)$. Since $S(u_{ij}) = u_{ji}$ we have $S^2 = \mathrm{id}_{A_0}$ and therefore the Haar state h is tracial ([11, p. 424]). By [22, Thm. 7.1] the enveloping von Neumann algebra $\pi_h(A_0)''$ is a $\mathbf{II}_1$-factor and Proposition 2.2 applies.

2.2. *The commutative case*
Next we want to investigate the commutative quantum groups. Consider a compact group G and the associated abelian, compact quantum group $\mathbb{G} = (C(G), \Delta_c)$. Recall that the comultiplication $\Delta_c\colon C(G) \to C(G) \otimes C(G) = C(G \times G)$ is defined by

$$\Delta_c(f)(s,t) = f(st),$$

and that the Haar state and counit are given, respectively, by integration against the Haar probability measure and by evaluation at the identity. In the case when G is a connected *abelian* Lie group then G is isomorphic to T^m for some $m \in \mathsf{N}$ ([12, Cor. 1.103]) and therefore the Pontryagin dual group is Z^m. Moreover, the Fourier transform is an isomorphism of quantum groups between $\mathbb{G} = (C(\mathsf{T}^m), \Delta_c)$ and $(C^*_{\mathrm{red}}(\mathsf{Z}^m), \Delta_{\mathrm{red}})$. In particular we have, by Proposition 1.3, that $\beta_0^{(2)}(\mathbb{G}) = \beta_0^{(2)}(\mathsf{Z}^m) = 0$ and

$$c_0(\mathbb{G}) = c_0(\mathsf{Z}^m) = \frac{1}{m} = \frac{1}{\dim(G)},$$

where the second equality follows from [18, Thm. 3.7]. This motivates the following result.

Theorem 2.4. *Let G be a compact Lie group with $\dim(G) \geq 1$ and Haar probability measure μ. Denote by $\mathbb{G}$ the corresponding compact quantum group $(C(G), \Delta_c)$. Then $H_0^{(2)}(\mathbb{G})$ is a finitely presented and zero-dimensional $L^\infty(G,\mu)$-module, in particular $\beta_0^{(2)}(\mathbb{G}) = 0$, and*

$$c_0(\mathbb{G}) = \frac{1}{\dim(G)}.$$

Here $\dim(G)$ is the dimension of G considered as a real manifold.

For the proof we will need a couple of lemmas/observations probably well known to most readers. The first one is a purely measure theoretic result.

Lemma 2.5. *Let (X,μ) be measure space and consider $[f_1], \dots, [f_n] \in L^\infty(X, \mathsf{R})$. If we denote by f the function*

$$X \ni x \longmapsto \sqrt{f_1(x)^2 + \cdots + f_n(x)^2} \in \mathsf{R},$$

then the ideal $\langle [f_1], \dots, [f_n] \rangle$ in $L^\infty(X, \mathsf{C})$ generated by the $[f_i]$'s is equal to the ideal $\langle [f] \rangle$ generated by $[f]$.

Here, and in the sequel, $[g]$ denotes the class in $L^\infty(X, \mathsf{C})$ of a given function g.

Proof. Consider the real-valued representatives $f_1, \dots, f_n$. Put $N_i = \{x \in X \mid f_i(x) = 0\}$ and $N = \cap_i N_i$. Note that N is exactly the set of zeros for f.

"$\subseteq$". Let $i \in \{1, \dots, n\}$ be given. We seek $[T] \in L^\infty(X, \mathsf{C})$ such that $[f_i] = [T][f]$. The set N may be disregarded since f_i is zero here. Outside of N we may write

$$f_i(x) = \frac{f_i(x)}{f(x)} f(x),$$

and we have $|\frac{f_i(x)}{f(x)}| = \sqrt{\frac{f_i(x)^2}{\sum_j f_j(x)^2}} \leq 1$. The function

$$T(x) = \begin{cases} 0 & \text{if } x \in N; \\ \frac{f_i(x)}{f(x)} & \text{if } x \in X \setminus N, \end{cases}$$

therefore defines a class $[T]$ in $L^\infty(X, \mathsf{C})$ with the required properties.

"$\supseteq$". We must find $[T_1], \dots, [T_n] \in L^\infty(X, \mathsf{C})$ such that

$$f(x) = T_1(x)f_1(x) + \cdots + T_n(x)f_n(x) \quad \text{for } \mu\text{-almost all } x \in X. \tag{1}$$

For any choice of $T_1, \dots, T_n$ both left- and right-hand side of (1) is zero when $x \in N$, and it is therefore sufficient to define $T_1, \dots, T_n$ outside of N. Choose a disjoint measurable partition of $X \setminus N$ into n sets $A_1, \dots, A_n$ such that

$$|f_k(x)| = \max_i |f_i(x)| > 0 \quad \text{when } x \in A_k.$$

Then $1 - \chi_N = \sum_{i=1}^n \chi_{A_i}$ and for $x \notin N$ we therefore have

$$f(x) = \sum_{i=1}^n \chi_{A_i}(x) f(x) = \sum_{i=1}^n \Big(\chi_{A_i}(x)\frac{f(x)}{f_i(x)}\Big) f_i(x),$$

and

$$|\chi_{A_i}(x)\frac{f(x)}{f_i(x)}| = \chi_{A_i}(x)\sqrt{\frac{\sum_j f_j(x)^2}{f_i(x)^2}} \leq \sqrt{n}.$$

Hence the functions $T_1, \dots, T_n$ defined by

$$T_i(x) = \begin{cases} 0 & \text{if } x \in N; \\ \chi_{A_i}(x)\frac{f(x)}{f_i(x)} & \text{if } x \in X \setminus N, \end{cases}$$

determines classes $[T_1], \dots, [T_n]$ in $L^\infty(X, \mathsf{C})$ with the required properties.

Observation 2.6. Every compact Lie group G has a faithful representation in $GL_n(\mathsf{C})$ for some $n \in \mathsf{N}$ and for such a representation π it holds that the algebra of all matrix coefficients $C(G)_0$ is generated by the real and imaginary parts of the matrix coefficients of π. The existence of a faithful representation π follows from [12, Cor. 4.22]. Denote by π_{kl} its complex matrix coefficients. The fact that $C(G)_0$ is generated by the set

$$\{\mathrm{Re}(\pi_{kl}), \mathrm{Im}(\pi_{kl}) \mid 1 \leq k, l \leq n\}$$

is the content of [5, VI, Prop. 3].

Observation 2.7. Let A be a unital C-algebra generated by elements $x_1, \dots, x_n$. If $\varepsilon\colon A \to \mathsf{C}$ is a unital algebra homomorphism then $\ker(\varepsilon)$ is the two-sided ideal generated by the elements $x_1 - \varepsilon(x_1), \dots, x_n - \varepsilon(x_n)$. This essentially follows from the formula

$$ab - \varepsilon(ab) = (a - \varepsilon(a))b + \varepsilon(a)(b - \varepsilon(b))$$

Observation 2.8. Denote by $\mathfrak{gl}_n(\mathsf{C}) = \mathbb{M}_n(\mathsf{C})$ the Lie algebra of $GL_n(\mathsf{C})$ and by exp the exponential function

$$\mathfrak{gl}_n(\mathsf{C}) \ni X \longmapsto \sum_{k=0}^\infty \frac{X^k}{k!} \in GL_n(\mathsf{C}),$$

and consider the map $f\colon \mathbb{M}_n(\mathsf{C}) \to \mathbb{M}_n(\mathsf{C})$ given by $f(X) = \exp(X) - 1$. For any norm $\|\cdot\|$ on $\mathbb{M}_n(\mathsf{C})$ there exist $r, R > 0$ and $\lambda_0 \in]0, \frac{1}{2}]$ such that the following holds: If $X \in \mathbb{M}_n(\mathsf{C})$ has $\|X\|_\infty \leq \frac{1}{2}$ then for all $\lambda \in [0, \lambda_0]$ we have

- $\|X\| \leq \lambda \Rightarrow \|f(X)\| \leq R\lambda$
- $\|f(X)\| \leq \lambda \Rightarrow \|X\| \leq r\lambda$

In the case when the norm in question is the operator norm $\|\cdot\|_\infty$ this is proven, with $\lambda_0 = \frac{1}{2}$ and $R = r = 2$, by considering the Taylor expansion around 0 for the scalar versions (i.e. $n = 1$) of f and f^{-1}. Since all norms on finite dimensional vector spaces are equivalent, the general statement follows from this.

We are now ready to give the proof of Theorem 2.4.

PROOF OF THEOREM 2.4. By Observation 2.6, we may assume that G is contained in $GL_n(\mathsf{C})$ so that each $g \in G$ can be written as $g = (x_{kl}(g) + iy_{kl}(g))_{kl} \in GL_n(\mathsf{C})$. Again by Observation 2.6, we have that $A_0 \subseteq A = C(G)$ is given by

$$A_0 = \mathrm{Alg}_{\mathsf{C}}(x_{kl}, y_{kl} \mid 1 \leq k, l \leq n),$$

where x_{kl} and y_{kl} are now considered as functions on G. Since $\varepsilon\colon A_0 \to \mathsf{C}$ is given by evaluation at the identity have

$$\begin{aligned} &\varepsilon(x_{kl}) = \varepsilon(y_{kl}) = 0 \quad \text{when} \quad k \neq l, \\ &\varepsilon(x_{kk}) = \varepsilon(1) = 1, \\ &\varepsilon(y_{kk}) = 0. \end{aligned}$$

From Observation 2.7 we now get

$$\ker(\varepsilon) = \langle x_{kl}, y_{kl}, x_{kk} - 1, y_{kk} \mid 1 \leq k, l \leq n, k \neq l \rangle \subseteq A_0$$

Thus

$$H_0^{(2)}(\mathbb{G}) = \mathrm{Tor}_0^{A_0}(L^\infty(G), \mathsf{C}) \simeq L^\infty(G) \underset{A_0}{\odot} \mathsf{C} \simeq L^\infty(G)/\langle \ker(\varepsilon) \rangle,$$

where $\langle \ker(\varepsilon) \rangle$ is the ideal in $L^\infty(G)$ generated by $\ker(\varepsilon) \subseteq A_0$. That is, the ideal

$$\langle x_{kl}, y_{kl}, 1 - x_{kk}, y_{kk} \mid 1 \leq k, l \leq n, k \neq l \rangle \subseteq L^\infty(G),$$

which by Lemma 2.5 is the principal ideal generated by the (class of the) function

$$f(g) = \sqrt{\sum_{k,l} (x_{kl}(g) - \delta_{kl})^2 + y_{kl}(g)^2}$$

Note that the zero-set for f consists only of the identity $1 \in G$ and is therefore a null-set with respect to the Haar measure. Hence we have a short exact sequence

$$0 \longrightarrow L^\infty(G) \xrightarrow{\cdot f} L^\infty(G) \longrightarrow H_0^{(2)}(\mathbb{G}) \longrightarrow 0. \tag{2}$$

By additivity of the dimension function ([16, Thm. 0.6]), this means that $\beta_0^{(2)}(\mathbb{G}) = 0$. Moreover, the short exact sequence (2) is a finite presentation of $H_0^{(2)}(\mathbb{G})$ and hence this module has a Novikov-Shubin invariant $\alpha(H_0^{(2)}(\mathbb{G}))$ (in the sense of [15, Section 3]) which can be computed using the spectral density function

$$\lambda \longmapsto h(\chi_{[0,\lambda^2]}(f^2)) = \mu(\{g \in G \mid f(g)^2 \leq \lambda^2\}).$$

Put $A_\lambda = \{g \in G \mid f(g)^2 \leq \lambda^2\}$. Since the zero-set for f is a μ-null-set we have

$$\alpha(H_0^{(2)}(\mathbb{G})) = \begin{cases} \liminf\limits_{\lambda \searrow 0} \dfrac{\log(\mu(A_\lambda))}{\log(\lambda)} & \text{if } \forall \lambda > 0 : \mu(A_\lambda) > 0; \\ \infty^+ & \text{otherwise.} \end{cases}$$

Put $m = \dim(G)$ and choose a linear identification of the Lie algebra $\mathfrak{g}$ of G with R^m. By [25, Thm. 3.31], we can choose neighborhoods $V \subseteq \mathfrak{g}$ and $U \subseteq G$, around 0 and 1 respectively, such that $\exp\colon V \to U$ is a diffeomorphism. This means that

$$\varphi = (\exp|_V)^{-1}\colon U \to V \subseteq \mathfrak{g} = \mathsf{R}^m,$$

constitutes a chart around $1 \in G$. Assume without loss of generality that

$$V \subseteq \mathfrak{g} \cap \{x \in \mathfrak{gl}_n(\mathsf{C}) \mid \|x\|_\infty \leq \tfrac{1}{2}\}.$$

For $g \in G$ we have

$$\begin{aligned} g \in A_\lambda &\Leftrightarrow \sum_{k,l} (x_{kl}(g) - \delta_{kl})^2 + y_{kl}(g)^2 \leq \lambda^2 \\ &\Leftrightarrow \|1 - g\|_2^2 \leq \lambda^2 \\ &\Leftrightarrow g \in B_\lambda(1), \end{aligned}$$

where $B_\lambda(1)$ is the closed λ-ball in $(\mathsf{R}^{2n^2}, \|\cdot\|_2)$ with center 1. Thus $A_\lambda = G \cap B_\lambda(1)$ and we can therefore choose $\lambda_0 \in]0, \frac{1}{2}]$ such that $A_{\lambda_0} \subseteq U$. Let ω denote the unique, positive, probability Haar volume form on G (see e.g. [12, Thm. 8.21, 8.23] or [14, Cor. 15.7]) and let $\lambda \in [0, \lambda_0]$. Then

$$\begin{aligned}\mu(A_\lambda) &= \int_G \chi_{A_\lambda}\, \mathrm{d}\mu \\ &= \int_U \chi_{A_\lambda}\omega \\ &= \int_V (\chi_{A_\lambda} \circ \varphi^{-1})(x_1, \ldots, x_m) F(x_1, \ldots, x_m)\, \mathrm{d}x_1 \cdots \mathrm{d}x_m \\ &= \int_{\varphi(A_\lambda)} F(x_1, \ldots, x_m)\, \mathrm{d}x_1 \cdots \mathrm{d}x_m,\end{aligned}$$

where $F\colon V \to \mathsf{R}$ is the unique positive function describing ω in the local coordinates (U, φ). By construction we have $F > 0$ on all of V and since $\varphi(A_{\lambda_0})$ is a compact set there exist $C, c > 0$ such that

$$c \leq F(x_1, \ldots, x_m) \leq C \quad \text{for all } (x_1, \ldots, x_m) \in \varphi(A_{\lambda_0})$$

For any $\lambda \in [0, \lambda_0]$ we therefore have

$$c\nu_m(\varphi(A_\lambda)) \leq \mu(A_\lambda) \leq C\nu_m(\varphi(A_\lambda)), \tag{3}$$

where ν_m denotes the Lebesgue measure in $\mathsf{R}^m = \mathfrak{g}$. Since $A_\lambda = G \cap B_\lambda(1)$ and φ is $(\exp|_V)^{-1}$, it follows from Observation 2.8 that there exist $d, D > 0$ and $\lambda_1 \in]0, \lambda_0]$ such that for all $\lambda \in [0, \lambda_1]$

$$B_{d\lambda}(0) \cap V \subseteq \varphi(A_\lambda) \subseteq B_{D\lambda}(0) \cap V.$$

Hence there exist $d', D' > 0$ such that for all $\lambda \in [0, \lambda_1]$

$$d'\lambda^m \leq \nu_m(\varphi(A_\lambda)) \leq D'\lambda^m. \tag{4}$$

From (3) and (4) we see that $\mu(A_\lambda) > 0$ for $\lambda \in]0, \lambda_1]$ and since

$$\lim_{\lambda \searrow 0} \frac{\log(d'\lambda^m)}{\log(\lambda)} = \lim_{\lambda \searrow 0} \frac{\log(D'\lambda^m)}{\log(\lambda)} = m,$$

we also conclude that

$$\alpha(H_0^{(2)}(\mathbb{G})) = \liminf_{\lambda \searrow 0} \frac{\log(\mu(A_\lambda))}{\log(\lambda)} = m = \dim(G).$$

By definition ([18, Def. 2.2]), the capacity of a finitely presented zero-dimensional module is the inverse of its Novikov-Shubin invariant and thus

$$c_0(\mathbb{G}) = c(H_0^{(2)}(\mathbb{G})) = \frac{1}{\dim(G)}.$$

2.3. *The finite dimensional case*

In Theorem 2.4 above we only considered compact Lie groups of positive dimension. What is left is the case when G is finite. When G is finite the algebra $C(G)$ is finite dimensional and we have $C(G)_0 = C(G) = L^\infty(G)$, which implies vanishing of $H_n^{(2)}(C(G), \Delta_c)$ for $n \geq 1$. For $n = 0$ we get

$$H_0^{(2)}(C(G), \Delta_c) = C(G) \underset{C(G)}{\odot} \mathsf{C} \simeq C(G)\delta_e.$$

This proves that $H_0^{(2)}(C(G), \Delta_c)$ is a finitely generated projective $C(G)$-module and hence

$$\beta_0^{(2)}(C(G), \Delta_c) = h(\delta_e) = \int_G \delta_e(g)\, \mathrm{d}\mu(g) = \frac{1}{|G|}.$$

Projectivity of $H_0^{(2)}(C(G),\Delta_c)$ implies (cf. [18]) that $c_0(C(G),\Delta_c) = 0^-$.

This argument generalizes in the following way.

PROPOSITION 2.9. *Let $\mathbb{G} = (A,\Delta)$ be a quantum group and assume that A has finite linear dimension N. Then*

$$\beta_0^{(2)}(\mathbb{G}) = \frac{1}{N},$$

and $\beta_n^{(2)}(\mathbb{G}) = 0$ for all $n \geq 1$. Moreover, $c_n(\mathbb{G}) = 0^-$ for all $n \in \mathsf{N}_0$.

PROOF. We first note that for a finite dimensional (hence compact) quantum group the Haar state is automatically tracial ([23, Thm. 2.2]), so that the numerical L^2-invariants make sense. The fact that the higher L^2-Betti numbers vanish is a trivial consequence of the fact that A is finite dimensional and therefore equal to both A_0 and its enveloping von Neumann algebra. To compute the zeroth L^2-Betti number we compute the zeroth L^2-homology as

$$H_0^{(2)}(\mathbb{G}) = \operatorname{Tor}_0^A(A,\mathsf{C}) \simeq A \underset{A}{\odot} \mathsf{C} = Ae,$$

where e is the projection in A projecting onto the C-summand $A/\ker(\varepsilon)$. Hence

$$\beta_0^{(2)}(\mathbb{G}) = \dim_A Ae = h(e) = \frac{1}{N},$$

where the last equality follows, for instance, from [26, A.2].
Each finite dimensional C^*-algebra is a semisimple ring and therefore all modules over it are projective. Hence all capacities of finite dimensional compact quantum groups are 0^-.

3. A vanishing result in the commutative case

Throughout this section, G denotes a compact, *connected* Lie group of dimension $m \geq 1$ and μ denotes the Haar probability measure on G. We will also use the following notation:

$$\begin{aligned}
\mathbb{G} &= (C(G),\Delta_c)\\
A &= C(G)\\
A_0 &= \text{The algebra of matrix coefficients}\\
\mathscr{A} &= L^\infty(G,\mu)\\
\mathscr{U} &= \text{The algebra of } \mu\text{-measurable functions on } G \text{ finite almost everywhere}
\end{aligned}$$

We aim to prove that $\beta_n^{(2)}(\mathbb{G}) = 0$ for all $n \geq 1$. Before doing this, a few comments on the objects defined above. We first note that $\mathscr{U}$ may be identified with the algebra of operators affiliated with $\mathscr{A}$ by [9, Thm. 5.6.4]. In [20] it is proved that there is a well defined dimension function $\dim_{\mathscr{U}}(\cdot)$ for modules over $\mathscr{U}$ satisfying properties similar to those enjoyed by $\dim_{\mathscr{M}}(\cdot)$ (cf. [16, Thm. 0.6]). Moreover, by [20, Thm. 3.1, Prop. 2.1] the functor $\mathscr{U} \odot_{\mathscr{A}} -$ is exact and dimension-preserving from the category of $\mathscr{A}$-modules to the category of $\mathscr{U}$-modules.
By [12, Cor. 4.22], we know that G can be faithfully represented in $GL_n(\mathsf{C})$ for some $n \in \mathsf{N}$. Since $GL_n(\mathsf{C})$ is a real analytic group (in the sense of [5]), this implies that G has a unique analytic structure making any faithful representation π analytic in the following sense: For any $g \in G$ and any function φ analytic around $\pi(g)$ the function $\varphi \circ \pi$ is analytic around g. This is the content of [5] Chapter IV, §XIV Proposition 1 and §XIII Proposition 1. We now choose some fixed faithful representation of G in $GL_n(\mathsf{C})$ which will be notationally suppressed in the following. That is, we consider G as an analytic subgroup of $GL_n(\mathsf{C})$. Denote by $\{x_{kl}, y_{kl} \mid 1 \leq k,l \leq n\}$

the natural $2n^2$ real functions on $GL_n(\mathsf{C})$ determining the analytic structure. As noted in Observation 2.6, the algebra A_0 is generated by the restriction of the functions x_{kl} and y_{kl} to G. Consider some polynomial in the variables x_{kl} and y_{kl}; this is clearly an analytic function on $GL_n(\mathsf{C})$ and it therefore defines an analytic function on G by restriction. Thus every function in A_0 is analytic.
The following result is probably well known to experts in Lie theory, but we were unable to find a suitable reference.

Proposition 3.1. *If $f \in A_0$ is not constantly zero then*

$$\mu(\{g \in G \mid f(g) = 0\}) = 0.$$

Hence f is invertible in $\mathscr{U}$.

For the proof we will need the following:

Observation 3.2. Let $V \subseteq \mathsf{R}^n$ be connected, convex and open and assume that $f\colon V \to \mathsf{R}$ is analytic. If f is not constantly zero on V then $N = \{x \in V \mid f(x) = 0\}$ is a set of Lebesgue measure 0. This is well known in the case $n = 1$, since in this case N is at most countable. The general case now follows from this by induction on n.

Proof of Proposition 3.1. Since $f(x) = 0$ iff $\mathrm{Re}(f(x)) = \mathrm{Im}(f(x)) = 0$ we may assume that f is real valued. Cover G with finitely many precompact, connected, analytic charts

$$(U_1, \varphi_1), \dots, (U_t, \varphi_t),$$

such that $\varphi(U_i) \subseteq \mathsf{R}^m$ is convex for each $i \in \{1, \dots, t\}$. Using the local coordinates and the Haar volume form on G, it is not hard to see that

$$\mu(\{g \in U_i \mid f(g) = 0\}) = 0 \;\Leftrightarrow \nu_m(\{x \in \varphi_i(U_i) \mid (f \circ \varphi_i^{-1})(x) = 0\}) = 0. \tag{5}$$

Here ν_m denotes the Lebesgue measure in R^m. Since $f \circ \varphi_i^{-1}$ is analytic it is (by Observation 3.2) sufficient to prove that f is not identically zero on any chart. Assume that f is constantly zero on some chart (U_{i_1}, φ_{i_1}). We then aim to show that f is zero on all of G, contradicting the assumption. If $G = U_{i_1}$ there is nothing to prove. If not, there exists $i_2 \neq i_1$ such that $U_{i_1} \cap U_{i_2} \neq \emptyset$, since otherwise we could split G as the union

$$U_{i_1} \cup (\bigcup_{i \neq i_1} U_i)$$

of to disjoint, non-empty, open sets, contradicting the fact that G is connected. Since the intersection $U_{i_1} \cap U_{i_2}$ is of positive measure and f is zero on it we conclude, by Observation 3.2 and (5), that f is zero on all of U_{i_2} . If $G = U_{i_1} \cup U_{i_2}$ we are done. If not, there exists $i_3 \notin \{i_1, i_2\}$ such that

$$U_{i_1} \cap U_{i_3} \neq \emptyset \qquad \text{or} \qquad U_{i_2} \cap U_{i_3} \neq \emptyset,$$

since otherwise G would be the union of two disjoint, non-empty, open sets. In either case we conclude that f is zero on all of U_{i_3}. Continuing in this way we conclude that f is zero on all of G since there are only finitely many charts.

The main result in this section is the following, which should be compared to [6, Thm. 5.1].

Theorem 3.3. *Let Z be any A_0-module. Then for all $n \geq 1$ we have*

$$\dim_{\mathscr{A}} \mathrm{Tor}_n^{A_0}(\mathscr{A}, Z) = 0.$$

Proof. As noted in the beginning of this section, we have

$$\begin{aligned} \dim_{\mathscr{A}} \mathrm{Tor}_n^{A_0}(\mathscr{A}, Z) &= \dim_{\mathscr{U}}(\mathscr{U} \underset{\mathscr{A}}{\otimes} \mathrm{Tor}_n^{A_0}(\mathscr{A}, Z)) \\ &= \dim_{\mathscr{U}} \mathrm{Tor}_n^{A_0}(\mathscr{U}, Z). \end{aligned}$$

We now aim to prove that $\operatorname{Tor}_n^{A_0}(\mathscr{U}, Z) = 0$. For this we first prove the following claim:

Each finitely generated A_0-submodule in $\mathscr{U}$ is contained in a finitely generated free A_0-submodule.

Let F be a non-trivial, finitely generated A_0-submodule in $\mathscr{U}$. We prove the claim by (strong) induction on the minimal number n of generators. If $n = 1$ then F is generated by a single element $\varphi \neq 0$, and since all elements in $A_0 \setminus \{0\}$ are invertible in $\mathscr{U}$ (Proposition 3.1) the function φ constitutes a basis for F. Hence F itself is free. Assume now that the result is true for all submodules that can be generated by n elements, and assume that F is a submodule with minimal number of generators equal to $n+1$. Choose such a minimal system of generators $\varphi_1, \dots, \varphi_{n+1}$. If these are linearly independent over A_0 there is nothing to prove. So assume that there exists a non-trivial tuple $(a_1, \dots, a_{n+1}) \in A_0^{n+1}$ such that

$$a_1\varphi_1 + \cdots + a_{n+1}\varphi_{n+1} = 0,$$

and assume, without loss of generality, that $a_1 \neq 0$. Define F_1 to be the A_0-submodule in $\mathscr{U}$ generated by

$$a_1^{-1}\varphi_2, \cdots, a_1^{-1}\varphi_{n+1}.$$

Then $F \subseteq F_1$ and the minimal number of generators for F_1 is a most n. By the induction hypothesis, there exists a finitely generated free submodule F_2 with $F_1 \subseteq F_2$ and in particular $F \subseteq F_2$. This proves the claim.

Denote by $(F_i)_{i\in I}$ the system of all finitely generated free A_0-submodules in $\mathscr{U}$. By the above claim, this set is directed with respect to inclusion. Since any module is the inductive limit of its finitely generated submodules, the claim also implies that $\mathscr{U}$, as an A_0-module, is the inductive limit of the system $(F_i)_{i\in I}$. But, since each F_i is free (in particular flat) and since Tor commutes with inductive limits we get

$$\operatorname{Tor}_n^{A_0}(\mathscr{U}, Z) = \varinjlim_{i} \operatorname{Tor}_n^{A_0}(F_i, Z) = 0,$$

for all $n \geq 1$.

Combining the results of Theorem 3.3 and Theorem 2.4 we get the following.

Corollary 3.4. *If G is a compact, non-trivial, connected Lie group then*

$$\beta_n^{(2)}(C(G), \Delta_c) = 0,$$

for all $n \in \mathsf{N}_0$.

4. Relation to the Connes-Shlyakhtenko approach

In [6], A. Connes and D. Shlyakhtenko introduced a notion of L^2-homology and L^2-Betti numbers in the setting of tracial $*$-algebras. More precisely, if A is a weakly dense $*$-subalgebra of a finite von Neumann algebra $\mathscr{M}$ with faithful, normal, trace-state τ, they defined ([6, Def. 2.1])

$$H_n^{(2)}(A) = \operatorname{Tor}_n^{A\odot A^{\mathrm{op}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, A) \quad \text{and} \quad \beta_n^{(2)}(A,\tau) = \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}} H_n^{(2)}(A).$$

This generalizes the notion of L^2-Betti numbers for groups in the sense that for a discrete group Γ we have $\beta_n^{(2)}(\mathsf{C}\Gamma, \tau) = \beta_n^{(2)}(\Gamma)$, as proven in [6, Prop. 2.3]. In this section we relate the notion of L^2-Betti numbers for quantum groups to the Connes-Shlyakhtenko approach. More precisely we prove the following:

THEOREM 4.1. *Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group with tracial Haar state h and algebra of matrix coefficients A_0. Then, for all $n \in \mathsf{N}_0$, we have $\beta_n^{(2)}(\mathbb{G}) = \beta_n^{(2)}(A_0, h)$, where the latter is the L^2-Betti numbers of the tracial $*$-algebra (A_0, h) in the sense of* [6].

For the proof of Theorem 4.1 we will need two small results. Denote by $S\colon A_0 \to A_0$ the antipode and by $\varepsilon\colon A_0 \to \mathsf{C}$ the counit. Recall ([11, p. 424]) that the trace property of h implies that $S^2 = \mathrm{id}_{A_0}$ and hence that S is a $*$-anti-isomorphism of A_0. Denote by $\mathscr{M}$ the enveloping von Neumann algebra $\pi_h(A_0)''$. In the following we suppress the GNS-representation π_h and put $\mathscr{H} = L^2(A, h)$. Denote by $\bar{\mathscr{H}}$ the conjugate Hilbert space, on which the opposite algebra A_0^{op} acts as $a^{\mathrm{op}} : \bar{\xi} \mapsto \overline{a^*\xi}$.

LEMMA 4.2. *There exists a unitary $V\colon \mathscr{H} \to \bar{\mathscr{H}}$ such that the map*

$$\mathscr{B}(\mathscr{H}) \supseteq A_0 \ni x \overset{\psi}{\longmapsto} (Sx)^{\mathrm{op}} \in A_0^{\mathrm{op}} \subseteq \mathscr{B}(\bar{\mathscr{H}})$$

takes the form $\psi(x) = VxV^*$. In particular, ψ extends to a normal $*$-isomorphism from $\mathscr{M}$ to $\mathscr{M}^{\mathrm{op}}$.

PROOF. Denote by η the inclusion $A_0 \subseteq \mathscr{H}$ and note that since A_0 is norm dense in A the set $\eta(A_0)$ is dense in $\mathscr{H}$. We now define the map V by

$$\eta(A_0) \ni \eta(x) \overset{V}{\longmapsto} \overline{\eta(Sx^*)} \in \overline{\eta(A_0)}.$$

It is easy to see that V is linear and

$$\begin{aligned} \|V\eta(x)\|_2^2 &= \|\overline{\eta(Sx^*)}\|_2^2 \\ &= \langle \eta(Sx^*) \,|\, \eta(Sx^*) \rangle \\ &= h((Sx^*)^* S(x^*)) \\ &= h(S(x^*x)) \\ &= h(x^*x) \\ &= \|\eta(x)\|_2^2, \end{aligned}$$

and hence V maps the dense subspace $\eta(A_0)$ isometrically onto the dense subspace $\overline{\eta(A_0)}$. Thus, V extends to a unitary which will also be denoted V. Clearly the adjoint of V is determined by

$$\overline{\eta(x)} \overset{V^*}{\longmapsto} \eta(Sx^*).$$

To see that V implements ψ we choose some $a \in A_0$ and calculate:

$$\begin{aligned} \overline{\eta(x)} &\overset{V^*}{\longmapsto} \eta(Sx^*) \\ &\overset{a}{\longmapsto} \eta(aS(x^*)) \\ &\overset{V}{\longmapsto} \overline{\eta(S(aS(x^*))^*)} \\ &= \overline{\eta((Sa^*)x)} \\ &= \psi(a)\overline{\eta(x)}. \end{aligned}$$

PROPOSITION 4.3. *The map $(\mathrm{id} \otimes \psi) \circ \Delta\colon A_0 \to A_0 \odot A_0^{\mathrm{op}}$ extends to a trace-preserving $*$-homomorphism $\varphi\colon \mathscr{M} \longrightarrow \mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$. Here ψ is the map constructed in Lemma 4.2 and $\mathscr{M} \bar{\otimes} \mathscr{M}^{\mathrm{op}}$ is endowed with the natural trace-state $h \otimes h^{\mathrm{op}}$.*

PROOF. The comultiplication is implemented by a multiplicative unitary $W \in \mathscr{B}(\mathscr{H} \bar{\otimes} \mathscr{H})$ in the sense that

$$\Delta(a) = W^*(1 \otimes a)W,$$

([13, page 60]) and it therefore extends to a normal $*$-homomorphism, also denoted Δ, from $\mathscr{M}$ to $\mathscr{M}\bar{\otimes}\mathscr{M}$. By Lemma 4.2, the map $\psi\colon \mathscr{M} \to \mathscr{M}^{\mathrm{op}}$ is normal and therefore $\varphi\colon \mathscr{M} \to \mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ is well defined and normal. Since φ is normal and A_0 is ultra-weakly dense in $\mathscr{M}$ it suffices to see that φ is trace-preserving on A_0. So, let $a \in A_0$ be given and write $\Delta a = \sum_i x_i \otimes y_i \in A_0 \odot A_0$. We then have

$$\begin{aligned}
(h\otimes h^{\mathrm{op}})\varphi(a) &= (h\otimes h^{\mathrm{op}})(1\otimes\psi)(\sum_i x_i\otimes y_i)\\
&= (h\otimes h^{\mathrm{op}})(\sum_i x_i\otimes (Sy_i)^{\mathrm{op}})\\
(h\circ S = h) \qquad &= \sum_i h(x_i)h(y_i)\\
&= h(h\otimes \mathrm{id})\Delta(a)\\
(\text{invariance of } h) \qquad &= h(h(a)1_A)\\
&= h(a).
\end{aligned}$$

We are now ready to give the proof of Theorem 4.1.

Proof of Theorem 4.1. By Proposition 4.3, we have that φ is a trace-preserving $*$-homomorphism from $\mathscr{M}$ to $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$. Via φ we can therefore consider $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ as a right $\mathscr{M}$-module and by [21, Thm. 1.48, 3.18] we have that the functor $(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}) \odot_{\mathscr{M}} -$ is exact and dimension-preserving from the category of $\mathscr{M}$-modules to the category of $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$-modules. Hence

$$\begin{aligned}
\beta_n^{(2)}(\mathbb{G}) &= \dim_{\mathscr{M}} \mathrm{Tor}_n^{A_0}(\mathscr{M}, \mathsf{C})\\
&= \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}} (\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}) \underset{\mathscr{M}}{\odot} \mathrm{Tor}_n^{A_0}(\mathscr{M}, \mathsf{C})\\
&= \dim_{\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}} \mathrm{Tor}_n^{A_0}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, \mathsf{C})
\end{aligned}$$

By [8, Prop. 2.3] (see also [7]), we have an isomorphism of vector spaces

$$\mathrm{Tor}_n^{A_0}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, \mathsf{C}) \simeq \mathrm{Tor}_n^{A_0\odot A_0^{\mathrm{op}}}(\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}, A_0), \tag{6}$$

where on the right-hand side $A_0 \odot A_0^{\mathrm{op}}$ acts on A_0 in the trivial way and on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ via the natural inclusion $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}} \supseteq A_0 \odot A_0^{\mathrm{op}}$. This isomorphism respects the natural left action of $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$, since on both sides of (6) only the multiplication from the right on $\mathscr{M}\bar{\otimes}\mathscr{M}^{\mathrm{op}}$ is used to compute the Tor-groups. The right-hand side of (6) is, by definition, equal to the L^2-homology of A_0 in the sense of [6] and the statement follows.

Corollary 4.4. *Let G be a non-trivial, compact, connected Lie group with Haar measure μ and denote by A_0 the algebra of matrix coefficients arising from irreducible representations of G. Then, for all $n \in \mathsf{N}_0$, we have $\beta_n^{(2)}(A_0, \mathrm{d}\mu) = 0$.*

Proof. This follows from Theorem 4.1 and Corollary 3.4 in conjunction

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Article II

L^2-Betti Numbers Of Coamenable Quantum Groups

L^2-BETTI NUMBERS OF COAMENABLE QUANTUM GROUPS

DAVID KYED

ABSTRACT. We prove that a compact quantum group is coamenable if and only if its corepresentation ring is amenable. We further propose a Følner condition for compact quantum groups and prove it to be equivalent to coamenability. Using this Følner condition, we prove that for a coamenable compact quantum group with tracial Haar state, the enveloping von Neumann algebra is dimension flat over the Hopf algebra of matrix coefficients. This generalizes a theorem of Lück from the group case to the quantum group case, and provides examples of compact quantum groups with vanishing L^2-Betti numbers.

0. INTRODUCTION

The theory of L^2-Betti numbers for discrete groups is originally due to Atiyah and dates back to the seventies [Ati76]. These L^2-Betti numbers are defined for those discrete groups that permit a free, proper and cocompact action on some contractible, Riemannian manifold X. If Γ is such a group, the space of square integrable p-forms on X becomes a finitely generated Hilbert module for the group von Neumann algebra $\mathscr{L}(\Gamma)$. As such it has a Murray-von Neumann dimension which turns out to be independent of the choice of X and is called the p-th L^2-Betti number of Γ, denoted $\beta_p^{(2)}(\Gamma)$. More recently, Lück ([Lüc97],[Lüc98a],[Lüc98b]) transported the notion of Murray-von Neumann dimension to the setting of finitely generated projective (algebraic) $\mathscr{L}(\Gamma)$-modules and extended thereafter the domain of definition to the class of all modules — accepting the possibility of having infinite dimension. With this extended dimension function, $\dim_{\mathscr{L}(\Gamma)}(\cdot)$, it is possible to extend the notion of L^2-Betti numbers to cover all discrete groups Γ by setting

$$\beta_p^{(2)}(\Gamma) = \dim_{\mathscr{L}(\Gamma)} \operatorname{Tor}_p^{\mathbb{C}\Gamma}(\mathscr{L}(\Gamma), \mathbb{C}).$$

For more details on the relations between the different definitions of L^2-Betti numbers and the extended dimension function we refer to Lück's book [Lüc02].

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All the ingredients in the homological algebraic definition above have fully developed analogues in the world of compact quantum groups, and using this dictionary the notion of L^2-Betti numbers was generalized to the quantum group setting in [Kye06]. Since this generalization will be used later in this paper, we shall now explain it in greater detail.

Consider a compact quantum group $\mathbb{G} = (A, \Delta)$ and assume that its Haar state h is a trace. If we denote by A_0 the unique dense Hopf $*$-algebra and by M the enveloping von Neumann algebra of A in the GNS representation arising from h, then the p-th L^2-Betti number of $\mathbb{G}$ is defined as

$$\beta_p^{(2)}(\mathbb{G}) = \dim_M \operatorname{Tor}_p^{A_0}(M, \mathbb{C}).$$

Here $\mathbb{C}$ is considered an A_0-module via the counit $\varepsilon\colon A_0 \to \mathbb{C}$ and $\dim_M(\cdot)$ is Lück's extended dimension function arising from (the extension of) the trace-state h. This definition extends the classical one ([Kye06, 4.2]) in the sense that

$$\beta_p^{(2)}(\mathbb{G}) = \beta_p^{(2)}(\Gamma),$$

when $\mathbb{G} = (C^*_{\mathrm{red}}(\Gamma), \Delta_{\mathrm{red}})$.

The aim of this paper is to investigate the L^2-Betti numbers of the class of coamenable, compact quantum groups. In the classical case we have that $\beta_p^{(2)}(\Gamma) = 0$ for all $p \geq 1$ whenever Γ is an amenable group. This can be seen as a special case of [Lüc98a, 5.1] where it is proved that the von Neumann algebra $\mathscr{L}(\Gamma)$ is *dimension flat* over $\mathbb{C}\Gamma$, meaning that

$$\dim_{\mathscr{L}(\Gamma)} \operatorname{Tor}_p^{\mathbb{C}\Gamma}(\mathscr{L}(\Gamma), Z) = 0 \qquad (p \geq 1)$$

for any $\mathbb{C}\Gamma$-module Z — provided, of course, that Γ is still assumed amenable. We generalize this result to the quantum group setting in Theorem 6.1. More precisely, we prove that if $\mathbb{G} = (A, \Delta)$ is a compact, coamenable quantum group with tracial Haar state and Z is any (left) module for the algebra of matrix coefficients A_0, then

$$\dim_M \operatorname{Tor}_p^{A_0}(M, Z) = 0. \qquad (p \geq 1)$$

Here M is again the enveloping von Neumann algebra in the GNS representation arising from the Haar state. In order to prove this result we need a *Følner condition* for compact quantum groups. The classical Følner condition for groups ([Føl55]) is a geometrical condition, on the action of the group on itself, which is equivalent to amenability of the group. In order to obtain a quantum analogue of Følner's condition a detailed study of the ring of corepresentations, associated to a compact quantum group, is needed. The ring of corepresentations is a special case of a so-called fusion algebra and we have therefore devoted a substantial part of this paper to the study of abstract fusion algebras and their amenability. Amenability for (finitely generated) fusion algebras was introduced by Hiai and Izumi in [HI98] where they also gave two equivalent Følner-type conditions for fusion algebras. We generalize their results to the non-finitely generated case and prove that a compact quantum group is coamenable if and only if its corepresentation ring is amenable. From this we obtain a Følner condition for compact quantum groups which is equivalent to coamenability. Using this Følner condition we prove our main result, Theorem 6.1, which implies that coamenable compact quantum groups have vanishing L^2-Betti numbers in all positive degrees.

Structure. The paper is organized as follows. In the first section we recapitulate (parts of) Woronowicz's theory of compact quantum groups. The second and third section is devoted to the study of abstract fusion algebras and amenability of such. In the fourth section we discuss coamenability of compact quantum groups and investigate the relation between coamenability of a compact quantum group and amenability of its corepresentation ring. The fifth section is an interlude in which the necessary notation concerning von Neumann algebraic compact quantum groups and their discrete duals is introduced. The sixth section is devoted to the proof of our main theorem (6.1) and the seventh, and final, section consists of examples.

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Notation. Throughout the paper, the symbol $\odot$ will be used to denote algebraic tensor products while the symbol $\bar{\otimes}$ will be used to denote tensor products in the category of Hilbert spaces or the category of von Neumann algebras. All tensor products between C^*-algebras are assumed minimal/spatial and these will be denoted by the symbol $\otimes$.

1. Preliminaries on compact quantum groups

In this section we briefly recall Woronowicz's theory of compact quantum groups. Detailed treatments, and proofs of the results stated, can be found in [Wor98], [MVD98] and [KT99].

A compact quantum group $\mathbb{G}$ is a pair (A, Δ) where A is a unital C^*-algebra and $\Delta\colon A \longrightarrow A \otimes A$ is a unital $*$-homomorphism from A to the minimal tensor product of A with itself satisfying:

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta \qquad \text{(coassociativity)}$$

$$\overline{\Delta(A)(1 \otimes A)} = \overline{\Delta(A)(A \otimes 1)} = A \otimes A \qquad \text{(non-degeneracy)}$$

For such a compact quantum group $\mathbb{G} = (A, \Delta)$, there exists a unique state $h\colon A \to \mathbb{C}$, called the Haar state, which is invariant in the sense that

$$(h \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes h)\Delta(a) = h(a)1,$$

for all $a \in A$. Let H be a Hilbert space and let $u \in M(\mathcal{K}(H) \otimes A)$ be an invertible multiplier. Then u is called a *corepresentation* if

$$(\mathrm{id} \otimes \Delta)u = u_{(12)}u_{(13)},$$

where we use the standard *leg numbering convention*; for instance $u_{(12)} = u \otimes 1$. *Intertwiners*, *direct sums* and *equivalences* between corepresentations as well as *irreducibility* are defined in a straight forward manner. See e.g. [MVD98] for details. We shall denote by $\mathrm{Mor}(u, v)$ the set of intertwiners from u to v. It is a fact that each irreducible corepresentation is finite dimensional and equivalent to a unitary corepresentation. Moreover, every unitary corepresentation is unitarily equivalent to a direct sum of irreducible corepresentations. For two finite dimensional unitary corepresentations u, v their *tensor product* is defined as

$$u \circledT v = u_{(13)}v_{(23)}.$$

This is again a unitary corepresentation of $\mathbb{G}$.

The algebra A_0 generated by all matrix coefficients arising from irreducible corepresentations becomes a Hopf $*$-algebra (with the restricted comultiplication) which is dense in A. We denote its antipode by S and its counit by ε. We also recall that the restriction of the Haar state to the $*$-algebra A_0 is always faithful. The quantum group $\mathbb{G}$ is called a compact *matrix* quantum group if there exists a *fundamental* unitary corepresentation; i.e. a finite dimensional, unitary corepresentation whose matrix coefficients generate A_0 as a $*$-algebra.

Each finite dimensional, unitary corepresentation u defines a *contragredient* corepresentation u^c on the dual Hilbert space H'. If $u \in B(H) \odot A_0$ for some finite dimensional Hilbert space H then $u^c \in B(H') \odot A_0$ is given by $u^c = ((\,\cdot\,)' \otimes S)u$, where for $T \in B(H)$ the operator $T' \in B(H')$ is the natural dual $(T'(y'))(x) = y'(Tx)$. In general u^c is not a unitary, but it is a corepresentation; i.e. it is invertible and satisfies $(\mathrm{id} \otimes \Delta)u^c = u^c_{(12)}u^c_{(13)}$ and is therefore equivalent to a unitary corepresentation. By choosing an orthonormal basis $e_1, \dots, e_n$ for H we get an identification of $B(H) \odot A_0$ and $\mathbb{M}_n(A_0)$. If, under this identification, u becomes the matrix (u_{ij}) then u^c is identified with the matrix $\bar{u} = (u^*_{ij})$, where we identify $B(H') \odot A_0$ with $\mathbb{M}_n(A_0)$ using the dual basis $e'_1, \dots, e'_n$. From this it follows that u^{cc} is equivalent to u. Note also that one has $(u \oplus v)^c = u^c \oplus v^c$ and $(u \circledT v)^c = v^c \circledT u^c$ for unitary corepresentations u and v (see e.g. [Wor87]). If $u \in B(H) \odot A_0$

is a finite dimensional corepresentation its *character* is defined as

$$\chi(u) = (\mathrm{Tr} \otimes \mathrm{id})u \in A_0,$$

where Tr is the unnormalized trace on $B(H)$. The character map has the following properties.

Proposition 1.1 ([Wor87]). *If u and v are finite dimensional, unitary corepresentations then*

$$\chi(u \textcircled{\scriptsize T} v) = \chi(u)\chi(v), \quad \chi(u \oplus v) = \chi(u) + \chi(v) \quad \text{and} \quad \chi(u^c) = \chi(u)^*.$$

Moreover, if u and v are equivalent then $\chi(u) = \chi(v)$.

We end this section with the two basic examples of compact quantum groups arising from actual groups.

Example 1.2. *If G is a compact, Hausdorff topological group then the Gelfand dual $C(G)$ becomes a compact quantum group with comultiplication $\Delta_c\colon C(G) \longrightarrow C(G) \otimes C(G) = C(G \times G)$ given by*

$$\Delta_c(f)(s,t) = f(st).$$

The Haar state is in this case given by integration against the Haar probability measure on G, and the finite dimensional unitary corepresentations of $C(G)$ are exactly the finite dimensional unitary representations of G.

Example 1.3. *If Γ is a discrete, countable group then the reduced group C^*-algebra $C^*_{\mathrm{red}}(\Gamma)$ becomes a compact quantum group when endowed with comultiplication given by*

$$\Delta_{\mathrm{red}}(\lambda_\gamma) = \lambda_\gamma \otimes \lambda_\gamma.$$

Here λ denotes the left regular representation of Γ. In this case, the Haar state is just the natural trace on $C^*_{\mathrm{red}}(\Gamma)$, and a complete family of irreducible, unitary corepresentations is given by the set $\{\lambda_\gamma \mid \gamma \in \Gamma\}$.

Remark 1.4. *All compact quantum groups to be considered in the following are assumed to have a separable underlying C^*-algebra. The quantum Peter-Weyl theorem* ([KT99, 3.2.3]) *then implies that the GNS space arising from the Haar state is separable and, in particular, that there are at most countable many (pairwise inequivalent) irreducible corepresentations.*

2. Fusion Algebras

In this section we introduce the notion of fusion algebras and amenability of such objects. This topic was treated by Hiai and Izumi in [HI98] and we will follow this reference closely throughout this section. Other references on the subject are [Yam99], [HY00] and [Sun92]. Throughout the section, $\mathbb{N}_0$ will denote the non-negative integers.

Definition 2.1 ([HI98])**.** *Let R be a unital ring and assume that R is free as $\mathbb{Z}$-module with basis I. Then R is called a fusion algebra if the unit e is an element of I and the following holds:*

(i) *The abelian monoid $\mathbb{N}_0[I]$ is stable under multiplication. That is; for all $\xi, \eta \in I$, the unique family $(N_{\xi,\eta}^{\alpha})_{\alpha \in I}$ of integers (only finitely many non-zero) satisfying*

$$\xi\eta = \sum_{\alpha \in I} N_{\xi,\eta}^{\alpha} \alpha,$$

consists of non-negative numbers.

(ii) *The ring R has a $\mathbb{Z}$-linear anti-multiplicative involution $x \mapsto \bar{x}$ preserving the basis I globally.*

(iii) *Frobenius reciprocity holds, that is for $\xi, \eta, \alpha \in I$ we have*

$$N_{\xi,\eta}^{\alpha} = N_{\bar{\xi},\alpha}^{\eta} = N_{\alpha,\bar{\eta}}^{\xi}.$$

(iv) *There exists a $\mathbb{Z}$-linear multiplicative function $d \colon R \to [1, \infty[$ such that $d(\xi) = d(\bar{\xi})$ for all $\xi \in I$. This function is called the dimension function.*

Note that both the distinguished basis, involution and dimension function are included in the data defining a fusion algebra. Each fusion algebra comes with a natural trace τ given by

$$\sum_{\alpha \in I} k_\alpha \alpha \overset{\tau}{\longmapsto} k_e.$$

We shall use this trace later to define a C^*-envelope of a fusion algebra. Note also that the multiplicativity of d implies

$$1 = \sum_{\alpha \in I} \frac{d(\alpha)}{d(\xi)d(\eta)} N_{\xi,\eta}^{\alpha},$$

for all $\xi, \eta \in I$. For an element $r = \sum_{\alpha \in I} k_\alpha \alpha \in R$, the set $\{\alpha \in I \mid k_\alpha \neq 0\}$ is called the support of r and denoted $\operatorname{supp}(r)$. We shall also consider the complexified fusion algebra $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}[I]$ which will be denoted $\mathbb{C}[I]$ in the following. Note that this becomes complex $*$-algebra with the induced algebraic structures.

Example 2.2. *For any discrete group Γ the integral group ring $\mathbb{Z}[\Gamma]$ becomes a fusion algebra when endowed with (the $\mathbb{Z}$-linear extension of) inversion as involution and trivial dimension function given by $d(\gamma) = 1$ for all $\gamma \in \Gamma$.*

For a compact group G its irreducible representations constitute the basis in a fusion algebra where the tensor product of representations is the product. We shall not go into details with this construction, since it will be contained in the following more general example.

Example 2.3. *If $\mathbb{G} = (A, \Delta)$ is a compact quantum group its irreducible corepresentations constitute the basis of a fusion algebra with tensor product as multiplication. Since this example will play a prominent role later, we shall now elaborate on the construction. Denote by* $\mathrm{Irred}(\mathbb{G}) = (u^\alpha)_{\alpha \in I}$ *a complete family of representatives for the equivalence classes of irreducible, unitary corepresentations of $\mathbb{G}$. As explained in Section 1, for all $u^\alpha, u^\beta \in \mathrm{Irred}(\mathbb{G})$ there exists a finite subset $I_0 \subseteq I$ and a family $(N^\gamma_{\alpha,\beta})_{\gamma \in I_0}$ of positive integers such that $u^\alpha \mathbin{\textcircled{T}} u^\beta$ is equivalent to*

$$\bigoplus_{\gamma \in I_0} (u^\gamma)^{\oplus N^\gamma_{\alpha,\beta}}.$$

Thus, a product can be defined on the free $\mathbb{Z}$-module $\mathbb{Z}[\mathrm{Irred}(\mathbb{G})]$ by setting

$$u^\alpha \cdot u^\beta = \sum_{\gamma \in I_0} N^\gamma_{\alpha,\beta} u^\gamma,$$

and the trivial corepresentation $e = 1_A \in \mathrm{Irred}(\mathbb{G})$ is a unit for this product. If we denote by $u^{\bar{\alpha}} \in \mathrm{Irred}(\mathbb{G})$ the unique representative equivalent to $(u^\alpha)^c$, then the map $u^\alpha \mapsto u^{\bar{\alpha}}$ extends to a conjugation on the ring $\mathbb{Z}[\mathrm{Irred}(\mathbb{G})]$ and since each u^α is an element of $\mathbb{M}_{n_\alpha}(A)$ for some $n_\alpha \in \mathbb{N}$ we can also define a dimension function $d\colon \mathbb{Z}[\mathrm{Irred}(\mathbb{G})] \to [1, \infty[$ by $d(u^\alpha) = n_\alpha$.

When endowed with this multiplication, conjugation and dimension function $\mathbb{Z}[\mathrm{Irred}(\mathbb{G})]$ becomes a fusion algebra. The only thing that is not clear at this moment is that Frobenius reciprocity holds. To see this, we first note that for for any $\alpha \in I$ and any finite dimensional corepresentation v we have (by Schur's Lemma [MVD98, 6.6]*) that u^α occurs exactly*

$$\dim_{\mathbb{C}} \mathrm{Mor}(u^\alpha, v)$$

times in the decomposition of v. Moreover, we have for any two unitary corepresentations v and w that

$$\dim_{\mathbb{C}} \mathrm{Mor}(v, w) = \dim_{\mathbb{C}}((V_w \otimes V_v')^{w \mathbin{\textcircled{T}} v^c})$$
$$\dim_{\mathbb{C}} \mathrm{Mor}(v^{cc}, w) = \dim_{\mathbb{C}}((V_v' \otimes V_w)^{v^c \mathbin{\textcircled{T}} w})$$

Here the right hand side denotes the linear dimension of the space of invariant vectors under the relevant coaction. These formulas are proved in [Wor87, 3.4] *for compact matrix quantum groups, but the same proof carries over to the case where the compact quantum group in question does not necessarily possess a fundamental corepresentation. Using the* *first formula, we get for* $\alpha, \beta, \gamma \in I$ *that*

$$\begin{aligned} N^{\gamma}_{\alpha,\beta} &= \dim_{\mathbb{C}} \mathrm{Mor}(u^{\gamma}, u^{\alpha} \textcircled{T} u^{\beta}) \\ &= \dim_{\mathbb{C}} (V_{\alpha} \otimes V_{\beta} \otimes V'_{\gamma})^{u^{\alpha} \textcircled{T} u^{\beta} \textcircled{T} (u^{\gamma})^c} \\ &= \dim_{\mathbb{C}} (V_{\gamma} \otimes V'_{\beta} \otimes V'_{\alpha})^{u^{\gamma} \textcircled{T} (u^{\beta})^c \textcircled{T} (u^{\alpha})^c} \\ &= \dim_{\mathbb{C}} \mathrm{Mor}(u^{\alpha}, u^{\gamma} \textcircled{T} (u^{\beta})^c) \\ &= N^{\alpha}_{\gamma,\bar{\beta}} \end{aligned}$$

The remaining identity in Frobenius reciprocity follows similarly using the second formula. The fusion algebra $\mathbb{Z}[\mathrm{Irred}(\mathbb{G})]$ *is called the corepresentation ring (or fusion ring) of* $\mathbb{G}$ *and is denoted* $R(\mathbb{G})$*.*

Recall that the character of a corepresentation $u \in \mathbb{M}_n(A)$ *is defined as* $\chi(u) = \sum_{i=1}^n u_{ii}$*. It follows from the Proposition 1.1 that the* $\mathbb{Z}$*-linear extension*

$$\chi \colon \mathbb{Z}[\mathrm{Irred}(\mathbb{G})] \longrightarrow A_0$$

is an injective homomorphism of $*$*-rings. I.e.* χ *is additive and multiplicative with* $\chi(u^{\bar{\alpha}}) = (\chi(u^{\alpha}))^*$*. This gives a link between the two* $*$*-algebras* $R(\mathbb{G})$ *and* A_0 *which will be of importance later.*

Other interesting examples of fusion algebras arise from inclusions of $\mathbf{II}_1$-factors. See [HI98] for details.

Convention 2.4. *In the following we shall only consider fusion algebras with an at most countable basis. This will therefore be assumed without further comments throughout the paper. Since we will primarily be interested in corepresentation rings of compact quantum groups, this is not very restrictive since the standing separability assumption (Remark 1.4) ensures that the corepresentation rings always have a countable basis.*

Consider again an abstract fusion algebra $R = \mathbb{Z}[I]$. For $\xi, \eta \in I$ we define the (weighted) convolution of the corresponding Dirac measures, δ_ξ and δ_η, as

$$\delta_\xi * \delta_\eta = \sum_{\alpha \in I} \frac{d(\alpha)}{d(\xi)d(\eta)} N^{\alpha}_{\xi,\eta} \delta_\alpha \in \ell^1(I).$$

This extends linearly and continuously to a submultiplicative product on $\ell^1(I)$. For $f \in \ell^\infty(I)$ and $\xi \in I$ we define $\lambda_\xi(f), \rho_\xi(f) \colon I \to \mathbb{C}$ by

$$\begin{aligned} \lambda_\xi(f)(\eta) &= \sum_{\alpha \in I} f(\alpha)(\delta_{\bar{\xi}} * \delta_\eta)(\alpha) \\ \rho_\xi(f)(\eta) &= \sum_{\alpha \in I} f(\alpha)(\delta_\eta * \delta_\xi)(\alpha). \end{aligned}$$

Denote by σ the counting measure on I scaled with d^2; that is $\sigma(\xi) = d(\xi)^2$. Combining Proposition 1.3, Remark 1.4 and Theorem 1.5 in [HI98] we get

Proposition 2.5 ([HI98]). *For each $f \in \ell^\infty(I)$ we have $\lambda_\xi(f) \in \ell^\infty(I)$ and for each $p \in \mathbb{N} \cup \{\infty\}$ the map $\lambda_\xi \colon \ell^\infty(I) \to \ell^\infty(I)$ restricts to a bounded operator on $\ell^p(I, \sigma)$ denoted $\lambda_{p,\xi}$. By linear extension, we therefore obtain a map $\lambda_{p,-} \colon \mathbb{Z}[I] \to B(\ell^p(I, \sigma))$. The map $\lambda_{p,-}$ respects the weighted convolution product. Moreover, for $p = 2$ the operator $U \colon \ell^2(I) \to \ell^2(I, \sigma)$ given by $U(\delta_\xi) = \frac{1}{d(\xi)}\delta_\xi$ is unitary and intertwines $\lambda_{2,\xi}$ with the operator*

$$l_\xi : \delta_\eta \longmapsto \frac{1}{d(\xi)} \sum_\alpha N^\alpha_{\xi,\eta} \delta_\alpha.$$

Remark 2.6. *Under the natural identification of $\ell^2(I)$ with the GNS space $L^2(\mathbb{C}[I], \tau)$, we see that $\pi_\tau(\xi) = d(\xi) l_\xi$. In particular the GNS representation consists of bounded operators. Here τ is the natural trace defined just after Definition 2.1.*

3. Amenability for Fusion Algebras

The notion of amenability for fusion algebras was introduced in [HI98], but only in the slightly restricted setting of finitely generated fusion algebras; a fusion algebra $R = \mathbb{Z}[I]$ is called *finitely generated* if there exists a finitely supported probability measure μ on I such that

$$I = \bigcup_{n \in \mathbb{N}} \operatorname{supp}(\mu^{*n}) \quad \text{and} \quad \forall \xi \in I : \mu(\bar{\xi}) = \mu(\xi).$$

That is, if the union of the supports of all powers of μ, with respect to convolution, is I and μ is invariant under the involution. The first condition is referred to as *nondegeneracy* of μ and the second condition is referred to as *symmetry* of μ.

In [HI98], amenability is defined, for a finitely generated fusion algebra, by requiring that $\|\lambda_{p,\mu}\| = 1$ for some $1 < p < \infty$ and some finitely supported, symmetric, non-degenerate probability measure μ. It is then proved that this is independent of the choice of μ and p, using the non-degeneracy property of the measure. If we consider a compact quantum group $\mathbb{G} = (A, \Delta)$ it is not difficult to prove that its corepresentation ring $R(\mathbb{G})$ is finitely generated exactly when $\mathbb{G}$ is a compact matrix quantum group. Since we are also interested in quantum groups without a fundamental corepresentation we will choose the following definition of amenability.

Definition 3.1. *A fusion algebra $R = \mathbb{Z}[I]$ is called amenable if $1 \in \sigma(\lambda_{2,\mu})$ for every finitely supported, symmetric probability measure μ on I.*

Here $\sigma(\lambda_{2,\mu})$ denotes the spectrum of the operator $\lambda_{2,\mu}$. From Proposition 1.3 and Corollary 4.4 in [HI98] it follows that our definition agrees with the one in [HI98] on the class of finitely generated fusion algebras. The relation between amenability for fusion algebras and the classical notion of amenability for groups will be explained later. See e.g. Remark 3.8 and Corollary 4.7.

Definition 3.2. *Let $R = \mathbb{Z}[I]$ be a fusion algebra. For two finite subsets $S, F \subseteq I$ we define the boundary of F relative to S as the set*

$$\begin{aligned}\partial_S(F) = &\{\alpha \in F \mid \exists\, \xi \in S : \mathrm{supp}(\alpha\xi) \nsubseteq F\}\\ &\cup \{\alpha \in F^c \mid \exists\, \xi \in S : \mathrm{supp}(\alpha\xi) \nsubseteq F^c\}\end{aligned}$$

Here, and in what follows, F^c denotes the set $I \setminus F$.

The modified definition of amenability allows the following extension of [HI98, 4.6] from where we also adopt some notation.

Theorem 3.3. *Let $R = \mathbb{Z}[I]$ be a fusion algebra with dimension function d. Then the following are equivalent:*

(A) *The fusion algebra is amenable.*

(FC1) *For every finitely supported, symmetric probability measure μ on I with $e \in \mathrm{supp}(\mu)$ and every $\varepsilon > 0$ there exists a finite subset $F \subseteq I$ such that*

$$\sum_{\xi \in \mathrm{supp}(\chi_F * \mu)} d(\xi)^2 < (1+\varepsilon) \sum_{\xi \in F} d(\xi)^2.$$

(FC2) *For every finite, non-empty subset $S \subseteq I$ and every $\varepsilon > 0$ there exists a finite subset $F \subseteq I$ such that*

$$\forall\, \xi \in S : \ \|\rho_{1,\xi}(\chi_F) - \chi_F\|_{1,\sigma} < \varepsilon \|\chi_F\|_{1,\sigma}.$$

(FC3) *For every finite, non-empty subset $S \subseteq I$ and every $\varepsilon > 0$ there exists a finite subset $F \subseteq I$ such that*

$$\sum_{\xi \in \partial_S(F)} d(\xi)^2 < \varepsilon \sum_{\xi \in F} d(\xi)^2.$$

The condition (FC3) was not present in [HI98]. It is to be considered as a fusion algebra analogue of the Følner condition for groups, as presented in [BP92, F.6]. The strategy for the proof of Theorem 3.3 is to prove the following implications.

$$(A) \Leftrightarrow (\mathrm{FC2}) \Rightarrow (\mathrm{FC3}) \Rightarrow (\mathrm{FC1}) \Rightarrow (\mathrm{FC2})$$

The proof of the implications (A) $\Leftrightarrow$ (FC2) and (FC1) $\Rightarrow$ (FC2) are small modifications of the corresponding proof in [HI98]. We first set out to prove the circle of implications

$$(\mathrm{FC2}) \Rightarrow (\mathrm{FC3}) \Rightarrow (\mathrm{FC1}) \Rightarrow (\mathrm{FC2})$$

For the proof we will need the following simple lemma.

Lemma 3.4. *If $N_{\xi,\eta}^{\alpha} > 0$ for some $\xi, \eta, \alpha \in I$ then $d(\alpha)d(\eta) \geq d(\xi)$.*

Proof. By Frobenius reciprocity, we have $N_{\xi,\eta}^{\alpha} = N_{\alpha,\bar{\eta}}^{\xi} > 0$ and hence

$$d(\alpha)d(\eta) = d(\alpha)d(\bar{\eta}) = \sum_{\gamma} N_{\alpha,\bar{\eta}}^{\gamma} d(\gamma) \geq N_{\alpha,\bar{\eta}}^{\xi} d(\xi) \geq d(\xi).$$

□

Proof of (FC2) ⇒ (FC3). We first note that (FC2), by the triangle inequality, implies the following condition:

For every finite, non-empty set $S \subseteq I$ and every $\varepsilon > 0$ there exists a finite set $F \subseteq I$ such that

$$\|\rho_{1,\chi_S}(\chi_F) - |S|\chi_F\|_{1,\sigma} < \varepsilon\|\chi_F\|_{1,\sigma}. \qquad (\dagger)$$

Here $|S|$ denotes the cardinality of S. Let S and $\varepsilon > 0$ be given and choose F such that (†) is satisfied. Define a map $\varphi\colon I \to \mathbb{R}$ by $\varphi(\xi) = \rho_{1,\chi_S}(\chi_F)(\xi) - |S|\chi_F(\xi)$. We note that

$$\begin{aligned}
\varphi(\xi) &= \Big(\sum_{\alpha \in I} \chi_F(\alpha)(\delta_\xi * \chi_S)(\alpha)\Big) - |S|\chi_F(\xi) \\
&= \Big(\sum_{\alpha \in F}\sum_{\eta \in S}(\delta_\xi * \delta_\eta)(\alpha)\Big) - |S|\chi_F(\xi) \\
&= \sum_{\alpha \in F}\sum_{\eta \in S} \frac{d(\alpha)}{d(\xi)d(\eta)} N_{\xi,\eta}^{\alpha} - |S|\chi_F(\xi).
\end{aligned}$$

We now divide into four cases.

(i) If $\xi \in F \cap \partial_S(F)^c$ then $\operatorname{supp}(\xi\eta) \subseteq F$ for all $\eta \in S$ and hence we get the relation $\sum_{\alpha \in F} \frac{d(\alpha)}{d(\xi)d(\eta)} N_{\xi,\eta}^{\alpha} = 1$. This implies $\varphi(\xi) = 0$.

(ii) If $\xi \in F^c \cap \partial_S(F)^c$ we see that $N_{\xi,\eta}^{\alpha} = 0$ for all $\alpha \in F$ and all $\eta \in S$ and hence $\varphi(\xi) = 0$.

(iii) If $\xi \in F^c \cap \partial_S(F)$ we have $\chi_F(\xi) = 0$ and there exist $\alpha_0 \in F$ and $\eta_0 \in S$ such that $N_{\xi,\eta_0}^{\alpha_0} \neq 0$. Using Lemma 3.4, we now get

$$\varphi(\xi) \geq \frac{d(\alpha_0)}{d(\xi)d(\eta_0)} N_{\xi,\eta_0}^{\alpha_0} \geq \frac{1}{d(\eta_0)^2} N_{\xi,\eta_0}^{\alpha_0} \geq \frac{1}{d(\eta_0)^2} \geq \frac{1}{M},$$

where $M = \max\{d(\eta)^2 \mid \eta \in S\}$.

(iv) If $\xi \in F \cap \partial_S(F)$ we have

$$\begin{aligned}\varphi(\xi) &= \sum_{\alpha\in F}\sum_{\eta\in S}\frac{d(\alpha)}{d(\xi)d(\eta)}N_{\xi,\eta}^{\alpha} - |S| \\ &= (-1)\sum_{\eta\in S}\Big(1-\sum_{\alpha\in F}\frac{d(\alpha)}{d(\xi)d(\eta)}N_{\xi,\eta}^{\alpha}\Big) \\ &= (-1)\sum_{\eta\in S}\sum_{\alpha\notin F}\frac{d(\alpha)}{d(\xi)d(\eta)}N_{\xi,\eta}^{\alpha},\end{aligned}$$

and because $\xi \in \partial_S(F)\cap F$ there exist $\eta_0 \in S$ and $\alpha_0 \notin F$ such that $N_{\xi,\eta_0}^{\alpha_0} \neq 0$. Using Lemma 3.4 again we conclude, as in (iii), that $|\varphi(\xi)| \geq \frac{1}{M}$.

We now get

$$\begin{aligned}\varepsilon\sum_{\xi\in F} d(\xi)^2 &= \varepsilon\|\chi_F\|_{1,\sigma} \\ &> \|\rho_{1,\chi_S}(\chi_F) - |S|\chi_F\|_{1,\sigma} && \text{(by (†))} \\ &= \sum_{\xi\in I}|\varphi(\xi)|d(\xi)^2 \\ &= \sum_{\xi\in\partial_S(F)}|\varphi(\xi)|d(\xi)^2 && \text{(by (i) and (ii))} \\ &\geq \frac{1}{M}\sum_{\xi\in\partial_S(F)} d(\xi)^2, && \text{(by (iii) and (iv))}\end{aligned}$$

and since ε was arbitrary the claim follows. □

Proof of (FC3) $\Longrightarrow$ (FC1). Given a finitely supported, symmetric probability measure μ, with $\mu(e) > 0$, and $\varepsilon > 0$ we put $S = \mathrm{supp}(\mu)$ and choose $F \subseteq I$ such that (FC3) is fulfilled with respect to ε. We have

$$(\chi_F * \mu)(\xi) = \sum_{\alpha\in F,\beta\in S}\mu(\beta)\frac{d(\xi)}{d(\alpha)d(\beta)}N_{\alpha,\beta}^{\xi},$$

so

$$\begin{aligned}(\chi_F*\mu)(\xi) = 0 &\Leftrightarrow \forall\alpha\in F\ \forall\beta\in S : N_{\alpha,\beta}^{\xi} = 0 \\ &\Leftrightarrow \forall\alpha\in F\ \forall\beta\in S : N_{\xi,\bar\beta}^{\alpha} = 0 && \text{(Frobenius)} \\ &\Leftrightarrow \forall\alpha\in F\ \forall\beta\in S : N_{\xi,\beta}^{\alpha} = 0 && \text{(S symmetric)} \\ &\Leftrightarrow \xi \in F^c \cap \partial_S(F)^c. && (e\in S)\end{aligned}$$

Hence $\mathrm{supp}(\chi_F * \mu) = (F^c \cap \partial_S(F)^c)^c = F \cup \partial_S(F)$ and we get

$$\begin{aligned}\sum_{\xi\in\mathrm{supp}(\chi_F*\mu)} d(\xi)^2 - \sum_{\xi\in F} d(\xi)^2 &= \sum_{\xi\in F\cup\partial_S(F)} d(\xi)^2 - \sum_{\xi\in F} d(\xi)^2 \\ &= \sum_{\xi\in\partial_S(F)\cap F^c} d(\xi)^2 \\ &\leq \sum_{\xi\in\partial_S(F)} d(\xi)^2 \\ &< \varepsilon \sum_{\xi\in F} d(\xi)^2. \qquad \text{(by (FC3))}\end{aligned}$$

□

Proof of (FC1) $\Rightarrow$ (FC2). Given $\varepsilon > 0$ and $S \subseteq I$ we define $\tilde{S} = S \cup \bar{S} \cup \{e\}$ and $\mu = \frac{1}{|\tilde{S}|}\chi_{\tilde{S}}$. Choose $F \subseteq I$ such that μ and F satisfy (FC1) with respect to $\frac{\varepsilon}{2}$. We aim to prove that (FC2) is satisfied for all $\xi \in \tilde{S}$. For arbitrary $\xi \in I$ we have

$$\begin{aligned}\|\rho_{1,\xi}(\chi_F) - \chi_F\|_{1,\sigma} &= \sum_{\alpha} |\rho_{1,\xi}(\chi_F)(\alpha) - \chi_F(\alpha)| d(\alpha)^2 \\ &= \sum_{\alpha} |(\sum_{\eta\in F} \frac{d(\eta)}{d(\alpha)d(\xi)} N^{\eta}_{\alpha,\xi}) - \chi_F(\alpha)| d(\alpha)^2 \\ &= \sum_{\alpha\in F} \Big(1 - \sum_{\eta\in F} \frac{d(\eta)}{d(\alpha)d(\xi)} N^{\eta}_{\alpha,\xi}\Big) d(\alpha)^2 \\ &+ \sum_{\alpha\notin F} \Big(\sum_{\eta\in F} \frac{d(\eta)}{d(\alpha)d(\xi)} N^{\eta}_{\alpha,\xi}\Big) d(\alpha)^2 \\ &= \sum_{\alpha\in F}\sum_{\eta\notin F} \frac{d(\eta)d(\alpha)}{d(\xi)} N^{\eta}_{\alpha,\xi} + \sum_{\alpha\notin F}\sum_{\eta\in F} \frac{d(\eta)d(\alpha)}{d(\xi)} N^{\eta}_{\alpha,\xi} \\ &= \sum_{\alpha\notin F}\sum_{\eta\in F} \frac{d(\eta)d(\alpha)}{d(\xi)} (N^{\eta}_{\alpha,\xi} + N^{\alpha}_{\eta,\xi}) \\ &= \sum_{\alpha\notin F}\sum_{\eta\in F} \frac{d(\eta)d(\alpha)}{d(\xi)} (N^{\alpha}_{\eta,\bar{\xi}} + N^{\alpha}_{\eta,\xi}). \qquad (\dagger)\end{aligned}$$

For $\xi \in \mathrm{supp}(\mu) = \tilde{S}$ and $\alpha \notin F$, it is easy to check that $(\chi_F * \mu)(\alpha) > 0$ if there exists an $\eta \in F$ such that $N^{\alpha}_{\eta,\bar{\xi}} + N^{\alpha}_{\eta,\xi} > 0$. Hence the calculation

(†) implies that

$$\begin{aligned}\|\rho_{1,\xi}(\chi_F)-\chi_F\|_{1,\sigma} &\leq \sum_{\alpha\in\mathrm{supp}(\chi_F*\mu)\setminus F}\ \sum_{\eta\in F}\frac{d(\eta)d(\alpha)}{d(\xi)}(N^{\alpha}_{\eta,\bar{\xi}}+N^{\alpha}_{\eta,\xi})\\ &\leq \sum_{\alpha\in\mathrm{supp}(\chi_F*\mu)\setminus F}\ \sum_{\eta\in I}\frac{d(\eta)d(\alpha)}{d(\xi)}(N^{\alpha}_{\eta,\bar{\xi}}+N^{\alpha}_{\eta,\xi})\\ &=2\sum_{\alpha\in\mathrm{supp}(\chi_F*\mu)\setminus F} d(\alpha)^2\\ &=2\Big(\sum_{\alpha\in\mathrm{supp}(\chi_F*\mu)} d(\alpha)^2-\sum_{\alpha\in F}d(\alpha)^2\Big)\\ &<\varepsilon\|\chi_F\|_{1,\sigma},\end{aligned}$$

where the last estimate follows from (FC1). Note that the condition $e\in\mathrm{supp}(\mu)$ was used to get the fourth step in the calculation above. □

We now set out to prove the last remaining equivalence in Theorem 3.3.

Proof of (A) ⇔ (FC2). In the end of this section, four formulas are gathered. These will be used during the proof and referred to as (F1) - (F4). For the actual proof we also need the following definitions. Consider a finitely supported, symmetric probability measure μ on I and define $p_\mu\colon I\times I\to\mathbb{R}$ by

$$p_\mu(\xi,\eta)=(\delta_\xi*\mu)(\eta)=\sum_{\omega}\mu(\omega)\frac{d(\eta)}{d(\xi)d(\omega)}N^{\eta}_{\xi,\omega}.$$

Note that the function p_μ satisfies the *reversibility condition*:

$$\sigma(\xi)p_\mu(\xi,\eta)=\sigma(\eta)p_\mu(\eta,\xi).$$

For a finitely supported function $f\in c_0(I)$ and $r\in\mathbb{N}$ we also define

$$\|f\|_{D_\mu(r)}=\Big(\frac{1}{2}\sum_{\xi,\eta}\sigma(\xi)p_\mu(\xi,\eta)|f(\xi)-f(\eta)|^r\Big)^{\frac{1}{r}}$$

Although this is referred to as the *generalized Dirichlet r-norm* of f, one should keep in mind that the function $\|\cdot\|_{D_\mu(r)}$ is only a semi norm. Denote by $c_0(I)$ the space of finitely supported functions on I and and by $\langle\cdot|\cdot\rangle_{r,\sigma}$ and $\|\cdot\|_{r,\sigma}$, respectively, the inner product and norm on $\ell^r(I,\sigma)$. We shall consider the following condition

For all finitely supported, symmetric, probability measures μ:

$$\inf\Big\{\frac{\|f\|_{D_\mu(r)}}{\|f\|_{r,\sigma}}\mid f\in c_0(I)\setminus\{0\}\Big\}=0. \qquad (\mathrm{NW}_r)$$

The reason for the name (NW_r), which appeared in [HI98], is that the condition is the negation of a so-called Wirtinger inequality. See [HI98] for more details. To prove (A) $\Leftrightarrow$ (FC2) we will actually prove the following equivalences

$$(\mathrm{FC2}) \Leftrightarrow (\mathrm{NW}_1) \quad \text{and} \quad \forall r : (\mathrm{NW}_1) \Leftrightarrow (\mathrm{NW}_r) \quad \text{and} \quad (\mathrm{A}) \Leftrightarrow (\mathrm{NW}_2).$$

For the latter of these equivalences the following lemma will be useful.

Lemma 3.5. *For all $f \in c_0(I)$ we have*

$$\|f\|^2_{D_\mu(2)} = \langle f \,|\, f\rangle_{2,\sigma} - \langle \rho_{2,\mu}(f) \,|\, f\rangle_{2,\sigma}.$$

Proof. This is proven by a direct calculation using the reversibility condition and the formula (F4) from the end of this section. □

Proof of (A)$\Leftrightarrow$ (NW_2). Let μ be a finitely supported, symmetric probability measure on I. By [HI98, 1.3,1.5], we have that $\rho_{2,\mu}$ is self-adjoint and $\|\rho_{2,\mu}\| \leq \|\mu\|_1 = 1$ so that $1 - \rho_{2,\mu} \geq 0$. We now get

$$\begin{aligned}
1 \in \sigma(\lambda_{2,\mu}) &\Leftrightarrow 1 \in \sigma(\rho_{2,\mu}) && ([\text{HI98, 1.5}])\\
&\Leftrightarrow 0 \in \sigma(1-\rho_{2,\mu})\\
&\Leftrightarrow 0 \in \sigma(\sqrt{1-\rho_{2,\mu}})\\
&\Leftrightarrow \exists x_n \in (\ell^2(I,\sigma))_1 : \|(\sqrt{1-\rho_{2,\mu}})x_n\|_{2,\sigma} \longrightarrow 0\\
&\Leftrightarrow \exists f_n \in (c_0(I))_1 : \|(\sqrt{1-\rho_{2,\mu}})f_n\|_{2,\sigma} \longrightarrow 0\\
&\Leftrightarrow \exists f_n \in (c_0(I))_1 : \langle (1-\rho_{2,\mu})f_n \,|\, f_n\rangle_{2,\sigma} \longrightarrow 0\\
&\Leftrightarrow \exists f_n \in (c_0(I))_1 : \|f_n\|_{D_\mu(2)} \longrightarrow 0 && (\text{Lem. 3.5})\\
&\Leftrightarrow \inf\{\frac{\|f\|_{D_\mu(2)}}{\|f\|_{2,\sigma}} \mid f \in c_0(I)\setminus\{0\}\} = 0.
\end{aligned}$$

Hence (A) $\Leftrightarrow$ (NW_2) as desired. □

Proof of (NW_1) $\Rightarrow$ (FC2). Given $\varepsilon > 0$ and $\xi_1,\dots,\xi_n \in I$, we choose a finitely supported, symmetric probability measure μ with $\xi_1,\dots,\xi_n \in \operatorname{supp}(\mu)$. Define

$$\varepsilon' = \frac{\varepsilon}{2}\min\{\mu(\xi) \mid \xi \in I\},$$

and choose, according to (NW_1), an $f \in c_0(I)$ such that

$$\|f\|_{D_\mu(1)} < \varepsilon'\|f\|_{1,\sigma}. \qquad (*)$$

Since $\||f|\|_{D_\mu(1)} \leq \|f\|_{D_\mu(1)}$ and $\||f|\|_{1,\sigma} = \|f\|_{1,\sigma}$ we may assume that f is positive. Since f can be approximated by a rational function we may actually assume that f has integer values. Put $N = \max\{f(\xi) \mid \xi \in I\}$ and define, for $k = 1,\dots,N$, $F_k = \{\xi \mid f(\xi) \geq k\}$. Then $f = \sum_1^N \chi_{F_k}$ and the following formulas hold.

$$\|f\|_{D_\mu(1)} = \sum_{k=1}^N \|\chi_{F_k}\|_{D_\mu(1)} \quad \text{and} \quad \|f\|_{1,\sigma} = \sum_{k=1}^N \|\chi_{F_k}\|_{1,\sigma}.$$

The first formula is proved by induction on the integer N and the second follows from a direct calculation using only the reversibility property of p_μ. Because of $(*)$, there must therefore exist some $j \in \{1,\dots,N\}$ such that

$$\|\chi_{F_j}\|_{D_\mu(1)} < \varepsilon'\|\chi_{F_j}\|_{1,\sigma} \tag{$**$}$$

For the sake of simplicity we denote this F_j by F in the following. We now get

$$\begin{aligned}
\|\chi_F\|_{D_\mu(1)} &= \frac{1}{2}\sum_{\xi,\eta}\sigma(\xi)p_\mu(\xi,\eta)|\chi_F(\xi)-\chi_F(\eta)| \\
&= \sum_{\xi\in F,\eta\notin F}\sigma(\xi)p_\mu(\xi,\eta) \qquad \text{(reversibility)}\\
&= \sum_{\xi\in F,\eta\notin F}\sigma(\xi)\Big(\sum_{\omega}\mu(\omega)\frac{d(\eta)}{d(\xi)d(\omega)}N^{\eta}_{\xi,\omega}\Big)\\
&= \sum_{\omega}\mu(\omega)\Big(\sum_{\xi\in F,\eta\notin F}\frac{d(\xi)d(\eta)}{d(\omega)}N^{\eta}_{\xi,\omega}\Big)\\
&= \frac{1}{2}\sum_{\omega}\mu(\omega)\Big(\sum_{\xi\in F,\eta\notin F}\frac{d(\xi)d(\eta)}{d(\omega)}(N^{\eta}_{\xi,\omega}+N^{\eta}_{\xi,\bar{\omega}})\Big)\\
&= \frac{1}{2}\sum_{\omega}\mu(\omega)\|\rho_{1,\omega}(\chi_F)-\chi_F\|_{1,\sigma} \qquad (\ddagger)
\end{aligned}$$

Here the last equality follows from the computation (†) in the proof of (FC1) ⇒ (FC2). The inequality $(**)$ therefore reads

$$\frac{1}{2}\sum_{\omega}\mu(\omega)\|\rho_{1,\omega}(\chi_F)-\chi_F\|_{1,\sigma} < \varepsilon'\|\chi_F\|_{1,\sigma}$$

For every $\omega\in I$ we therefore conclude, since $\varepsilon' = \frac{\varepsilon}{2}\min(\mu)$, that

$$\mu(\omega)\|\rho_{1,\omega}(\chi_F)-\chi_F\|_{1,\sigma} < \min(\mu)\varepsilon\|\chi_F\|_{1,\sigma}.$$

Since each of the given ξ_i's are in $\mathrm{supp}(\mu)$ we get

$$\forall i \ : \|\rho_{1,\xi_i}(\chi_F)-\chi_F\|_{1,\sigma} < \varepsilon\|\chi_F\|_{1,\sigma},$$

as desired. □

Proof of (FC2) ⇒ (NW_1). Assume now (FC2) and let μ and ε be given. Choose F such that

$$\|\rho_{1,\xi}(\chi_F)-\chi_F\|_{1,\sigma} < \varepsilon\|\chi_F\|_{1,\sigma},$$

for all $\xi \in \operatorname{supp}(\mu)$. Using the calculation (‡), from the proof of opposite implication, we get

$$\begin{aligned}
\|\chi_F\|_{D_\mu(1)} &= \frac{1}{2}\sum_{\omega} \mu(\omega)\|\rho_{1,\omega}(\chi_F) - \chi_F\|_{1,\sigma} \\
&< \frac{1}{2}\sum_{\omega} \mu(\omega)\varepsilon\|\chi_F\|_{1,\sigma} \\
&= \frac{\varepsilon}{2}\|\chi_F\|_{1,\sigma} \\
&< \varepsilon\|\chi_F\|_{1,\sigma}.
\end{aligned}$$

□

For the proof of the statement (NW_1) ⇔ (NW_r) we will need the following lemma.

Lemma 3.6 ([Ger88]). *For $r \geq 2$ and $f \in c_0(I)_+$ we have*

$$\|f^r\|_{D_\mu(1)} \leq 2r\|f\|_{r,\sigma}^{r-1}\|f\|_{D_\mu(r)}.$$

Proof. First note that

$$\begin{aligned}
\|f^r\|_{D_\mu(1)} &= \frac{1}{2}\sum_{\xi,\eta} \sigma(\xi)p_\mu(\xi,\eta)|f(\xi)^r - f(\eta)^r| \\
&\leq \frac{r}{2}\sum_{\xi,\eta} \sigma(\xi)p_\mu(\xi,\eta)(f(\xi)^{r-1} + f(\eta)^{r-1})|f(\xi) - f(\eta)|,
\end{aligned}$$

where the inequality follows from (F1). Define a measure ν on $I \times I$ by $\nu(\xi,\eta) = \frac{1}{2}\sigma(\xi)p_\mu(\xi,\eta)$ and consider the functions $\varphi, \psi \colon I \times I \to \mathbb{R}$ given by

$$\varphi(\xi,\eta) = f(\xi)^{r-1} + f(\eta)^{r-1} \quad \text{and} \quad \psi(\xi,\eta) = |f(\xi) - f(\eta)|.$$

Define $s > 1$ by the equation $\frac{1}{r} + \frac{1}{s} = 1$. Then the inequality above can be written as $\|f^r\|_{D_\mu(1)} \leq r\|\varphi\psi\|_{1,\nu}$ and, using Hölder's inequality, we therefore get

$$
\begin{aligned}
\|f^r\|_{D_\mu(1)} &\le r\|\varphi\psi\|_{1,\nu} \\
&\le r\|\varphi\|_{s,\nu}\|\psi\|_{r,\nu} \\
&= r\Big[\sum_{\xi,\eta}\frac{1}{2}\sigma(\xi)p_\mu(\xi,\eta)(f(\xi)^{r-1}+f(\eta)^{r-1})^s\Big]^{\frac{1}{s}} \\
&\quad \times \Big[\sum_{\xi,\eta}\frac{1}{2}\sigma(\xi)p_\mu(\xi,\eta)|f(\xi)-f(\eta)|^r\Big]^{\frac{1}{r}} \\
&\overset{1}{\le} r\Big[2^{s-1}\frac{1}{2}\sum_{\xi,\eta}\sigma(\xi)p_\mu(\xi,\eta)(f(\xi)^{(r-1)s}+f(\eta)^{(r-1)s})\Big]^{\frac{1}{s}}\|f\|_{D_\mu(r)} \\
&\overset{2}{=} r\Big[2^{s-1}\sum_{\xi,\eta}\sigma(\xi)p_\mu(\xi,\eta)f(\xi)^{(r-1)s}\Big]^{\frac{1}{s}}\|f\|_{D_\mu(r)} \\
&= r2^{\frac{s-1}{s}}\Big[\sum_{\xi}\sigma(\xi)\Big(\sum_{\eta}p_\mu(\xi,\eta)\Big)f(\xi)^{(r-1)s}\Big]^{\frac{1}{s}}\|f\|_{D_\mu(r)} \\
&= r2^{\frac{s-1}{s}}\Big[\sum_{\xi}\sigma(\xi)f(\xi)^{(r-1)s}\Big]^{\frac{1}{s}}\|f\|_{D_\mu(r)} \\
&\le 2r\Big[\sum_{\xi}\sigma(\xi)f(\xi)^r\Big]^{\frac{r-1}{r}}\|f\|_{D_\mu(r)} \qquad (\tfrac{1}{r}+\tfrac{1}{s}=1) \\
&= 2r\|f\|_{r,\sigma}^{r-1}\|f\|_{D_\mu(r)}.
\end{aligned}
$$

□

Also the following observation will be useful

Observation 3.7. *Under the assumptions of Lemma 3.6 we have*

$$
\begin{aligned}
\|f\|_{D_\mu(r)} &= \Big[\frac{1}{2}\sum_{\xi,\eta}\sigma(\xi)p_\mu(\xi,\eta)|f(\xi)-f(\eta)|^r\Big]^{\frac{1}{r}} \\
&\le \Big[\frac{1}{2}\sum_{\xi,\eta}\sigma(\xi)p_\mu(\xi,\eta)|f(\xi)^r-f(\eta)^r|\Big]^{\frac{1}{r}} \qquad \text{(by (F3))} \\
&= \|f^r\|_{D_\mu(1)}^{\frac{1}{r}}
\end{aligned}
$$

Having these results, we are now able to prove $(\mathrm{NW}_1) \Leftrightarrow (\mathrm{NW}_r)$.

Proof of $(\mathrm{NW}_1) \Rightarrow (\mathrm{NW}_r)$. Assume (NW_1) and let μ and $\varepsilon > 0$ be given. Put $\varepsilon' = \varepsilon^r$ and choose non-zero $f \in c_0(I)_+$ such that

$$\frac{\|f\|_{D_\mu(1)}}{\|f\|_{1,\sigma}} < \varepsilon'.$$

[1]by (FC2)

[2]by reversibility

Using Observation 3.7 we get

$$\frac{\|\sqrt[r]{f}\|_{D_\mu(r)}}{\|\sqrt[r]{f}\|_{r,\sigma}} \leq \frac{\|f\|_{D_\mu(1)}^{\frac{1}{r}}}{\|f\|_{1,\sigma}^{\frac{1}{r}}} < (\varepsilon')^{\frac{1}{r}} = \varepsilon.$$

□

Proof of (NW$_r$) $\Rightarrow$ (NW$_1$). Given μ and $\varepsilon > 0$ and put $\varepsilon' = \frac{1}{2r}\varepsilon$. Then choose non-zero $f \in c_0(I)_+$ with

$$\frac{\|f\|_{D_\mu(r)}}{\|f\|_{r,\sigma}} < \varepsilon'.$$

Using Lemma 3.6, we get

$$\frac{\|f^r\|_{D_\mu(1)}}{\|f^r\|_{1,\sigma}} \leq \frac{2r\|f\|_{r,\sigma}^{r-1}\|f\|_{D_\mu(r)}}{\|f\|_{r,\sigma}^r} < 2r\varepsilon' = \varepsilon.$$

□

Gathering all the results just proven we get (A) $\Leftrightarrow$ (FC2). □

This concludes the proof of Theorem 3.3.

Remark 3.8. *Consider a countable, discrete group Γ and the corresponding fusion algebra $\mathbb{Z}[\Gamma]$. It is not difficult to prove that $\mathbb{Z}[\Gamma]$ satisfies* (FC3) *from Theorem 3.3 if and only if Γ satisfies Følner's condition (for groups) as presented in* [BP92, F.6]. *Since a group is amenable if and only if it satisfies Følner's condition, we see from this that Γ is amenable if and only if the corresponding fusion algebra $\mathbb{Z}[\Gamma]$ is amenable.*

3.1. **Formulas used in the proof of Theorem 3.3.** We collect here four formulas used in the proof of Theorem 3.3. Let $r, s > 1$ and assume that $\frac{1}{r} + \frac{1}{s} = 1$. Then for all $z, w \in \mathbb{C}$, $a, b \geq 0$ and $n \in \mathbb{N}$ we have

$$|a^r - b^r| \leq r(a^{r-1} + b^{r-1})|a - b| \tag{F1}$$

$$(a + b)^r \leq 2^{r-1}(a^r + b^r) \tag{F2}$$

$$|a - b|^n \leq |a^n - b^n| \tag{F3}$$

$$|z - w|^2 + |w - z|^2 = 2(|z|^2 - z\bar{w}) + 2(|w|^2 - w\bar{z}). \tag{F4}$$

Proof. The inequality (F1) can be proved using the mean value theorem on the function $f(x) = x^r$ and the interval between a and b. To prove (F2), consider a two-point set endowed with counting measure. Using Hölder's inequality, we then get

$$a + b = 1 \cdot a + 1 \cdot b \leq (1^s + 1^s)^{\frac{1}{s}}(a^r + b^r)^{\frac{1}{r}}.$$

From this the desired inequality follows using the fact that $\frac{1}{s} = \frac{r-1}{r}$. The inequality (F3) follows using the binomial theorem. If, for instance, $a = b + k$ for some $k \geq 0$ we have

$$(a - b)^n = k^n \leq (b + k)^n - b^n = a^n - b^n.$$

The formula (F4) follows by splitting w and z into real and imaginary parts and calculating both sides of the equation. □

4. Coamenable Compact Quantum Groups

In this section we introduce the notion of coamenability for compact quantum groups and discuss the relationship between coamenability of a compact quantum group and amenability of its corepresentation ring. The notion of (co-)amenability has been treated in different quantum group settings by numerous people. A number of references for this subject are [BMT01], [Voi79], [Rua96], [Ban99a], [Ban99b], [ES92] and [BS93]. For our purposes, the approach of Bédos, Murphy and Tuset in [BMT01] is the most natural and we are therefore going to follow this reference throughout this section. We will assume that the reader is familiar with the basics on Woronowicz's theory of compact quantum groups. Definitions, notation and some basic properties can be found in Section 1 and a detailed treatment can be found in [Wor98], [MVD98] and [KT99].

Definition 4.1 ([BMT01])**.** *Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group and let A_{red} be the image of A under the GNS representation π_h arising from the Haar state h. Then $\mathbb{G}$ is said to be coamenable if the counit $\varepsilon\colon A_0 \to \mathbb{C}$ extends continuously to A_{red}.*

Remark 4.2. *It is well known that a discrete group Γ is amenable if and only if the trivial representation of $C^*_{\mathrm{full}}(\Gamma)$ factorizes through $C^*_{\mathrm{red}}(\Gamma)$. This amounts to saying that $(C^*_{\mathrm{red}}(\Gamma), \Delta_{\mathrm{red}})$ is coamenable if and only if Γ is amenable. Note also that the abelian compact quantum groups $(C(G), \Delta_c)$ are automatically coamenable since the counit is given by evaluation at the identity and therefore already globally defined and bounded.*

In the following theorem we collect some facts on coamenable compact quantum groups. For more coamenability criteria and a proof of the theorem below we refer to [BMT01].

Theorem 4.3 ([BMT01])**.** *For a compact quantum group $\mathbb{G} = (A, \Delta)$ the following are equivalent.*

(i) *$\mathbb{G}$ is coamenable.*

(ii) *The Haar state h is faithful and the counit is bounded with respect to the norm on A.*

(iii) *The natural map from the universal representation A_u to the reduced representation A_{red} is an isomorphism.*

If $\mathbb{G}$ is a compact matrix *quantum group with fundamental corepresentation $u \in \mathbb{M}_n(A)$ the above conditions are also equivalent to the following.*

(iv) *The number* n *is in* $\sigma(\pi_h(\mathrm{Re}(\chi(u)))$ *where* $\chi(u) = \sum_{i=1}^{n} u_{ii}$ *is the character map from Section 2.*

Recall that $\sigma(T)$ denotes the spectrum of a given operator T. Thus, when we are dealing with a coamenable quantum group the Haar state is automatically faithful and hence the corresponding GNS representation π_h is faithful. We therefore can, and will, identify A and A_{red}. The condition (iv) is Skandalis's quantum analogue of the so-called Kesten condition for groups (see [Kes59],[Ban99a]) which is proved in [Ban99b]. The next result is a generalization of the Kesten condition to the case where a fundamental corepresentation is not (necessarily) present. The proof draws inspiration from the corresponding proof in [BMT01].

Theorem 4.4. *Let* $\mathbb{G} = (A, \Delta)$ *be a compact quantum group. Then the following are equivalent:*

(i) $\mathbb{G}$ *is coamenable.*

(ii) *For any finite dimensional, unitary corepresentation* $u \in \mathbb{M}_{n_u}(A)$ *we have* $n_u \in \sigma(\pi_h(\mathrm{Re}(\chi(u))))$.

Proof. Assume $\mathbb{G}$ to be coamenable and let a finite dimensional, unitary corepresentation $u \in \mathbb{M}_{n_u}(A)$ be given. Since the counit extends to a character $\varepsilon \colon A_{\mathrm{red}} \to \mathbb{C}$ and since

$$\varepsilon(\mathrm{Re}(\chi(u))) = \varepsilon(\sum_{i=1}^{n_u} \frac{u_{ii} + u_{ii}^*}{2}) = n_u,$$

we must have $n_u \in \sigma(\pi_h(\mathrm{Re}(\chi(u))))$. Assume conversely that the property (ii) is satisfied and define, for a finite dimensional, unitary corepresentation u, the set

$$C(u) = \{\varphi \in \mathscr{S}(A_{\mathrm{red}}) \mid \varphi(\pi_h(\mathrm{Re}(\chi(u)))) = n_u\}.$$

Here $\mathscr{S}(A_{\mathrm{red}})$ denotes the state space of A_{red}. It is clear that each $C(u)$ is closed in the weak*-topology and we now prove that the family

$$\mathcal{F} = \{C(u) \mid u \text{ finite dimensional, unitary corepresentation}\}$$

has the finite intersection property. We first prove that each $C(u)$ is non-empty. For given u, we put $x_{ij} = u_{ij} - \delta_{ij}$ and $x = \sum_{ij} x_{ij}^* x_{ij}$. Then x is clearly positive and a direct calculation reveals that

$$x = 2(n_u - \mathrm{Re}(\chi(u))). \tag{†}$$

Hence, $n_u \in \sigma(\pi_h(\mathrm{Re}(\chi(u))))$ if and only if there exists ([KR83, 4.4.4]) a $\varphi \in \mathscr{S}(A_{\mathrm{red}})$ with

$$\varphi(\pi_h(\mathrm{Re}(\chi(u)))) = n_u.$$

Thus, $C(u) \neq \emptyset$. Let now $u^{(1)}, \dots, u^{(k)}$ be given and put $u = \oplus_{i=1}^k u^{(i)}$. We aim at proving that

$$C(u) \subseteq \bigcap_{i=1}^{k} C(u^{(i)}).$$

Let $\varphi \in C(u)$ be given and note that

$$\sum_{i=1}^{k} n_{u^{(k)}} = \varphi(\pi_h(\mathrm{Re}(\chi(u)))) = \sum_{i=1}^{k} \sum_{j=1}^{n_{u^{(j)}}} \frac{1}{2}\varphi(\pi_h(u_{jj}^{(i)}) + \pi_h(u_{jj}^{(i)*})).$$

Since the matrix $(\pi_h(u_{st}))_{s,t=1}^{n_u}$ is unitary we have $\|\pi_h(u_{st})\| \leq 1$ and hence

$$\frac{1}{2}\varphi(\pi_h(u_{jj}^{(i)}) + \pi_h(u_{jj}^{(i)*})) \in [-1, 1].$$

This forces $\frac{1}{2}\varphi(\pi_h(u_{jj}^{(i)}) + \pi_h(u_{jj}^{(i)*})) = 1$ and hence $\varphi(\pi_h(\mathrm{Re}(\chi(u^{(i)})))) = n_{u^{(i)}}$. Thus φ is in each of the sets $C(u^{(1)}), \dots, C(u^{(k)})$ and we conclude that $\mathcal{F}$ has the finite intersection property. By compactness of $\mathscr{S}(A_{\mathrm{red}})$, we may therefore find a state φ such that $\varphi(\pi_h(\mathrm{Re}(\chi(u)))) = n_u$ for every unitary corepresentation u. Denote by H the GNS space associated with this φ, by ξ_0 the natural cyclic vector and by π the corresponding GNS representation of A_{red}. Consider an arbitrary unitary corepresentation u and form as before the elements x_{ij} and x. Then the equation (†) shows that $\varphi(x_{ij}^* x_{ij}) = 0$ and hence $\pi(x_{ij})\xi_0 = 0$ and

$$\pi(u_{ij})\xi_0 = \delta_{ij}\xi_0.$$

From the Cauchy-Schwarz inequality we get

$$|\varphi(x_{ij})|^2 \leq \varphi(x_{ij}^* x_{ij})\varphi(1) = 0,$$

and hence $\varphi(u_{ij}) = \delta_{ij}$. We therefore have that $\pi(u_{ij})\xi_0 = \varphi(u_{ij})\xi_0$. Since the matrix coefficients span A_0 linearly we get $\pi(a)\xi_0 = \varphi(a)\xi_0$ for all $a \in A_0$. By density of A_0 in A_{red} it follows that $\pi(a)\xi_0 = \varphi(a)\xi_0$ for all $a \in A_{\mathrm{red}}$. From this we see that

$$H = \overline{\pi(A_{\mathrm{red}})\xi_0}^{\|\cdot\|_2} = \mathbb{C}\xi_0,$$

and it follows that $\varphi\colon A_{\mathrm{red}} \to \mathbb{C}$ is a bounded $*$-homomorphism coinciding with ε on A_0. Thus, $\mathbb{G}$ is coamenable.

□

The following result was mentioned, without proof, in [HI98, p.692] in the restricted setting of compact matrix quantum groups whose Haar state is a trace.

Theorem 4.5. *A compact quantum group $\mathbb{G} = (A, \Delta)$ is a coamenable if and only if the corepresentation ring $R(\mathbb{G})$ is amenable.*

For the proof we will need the following lemma. For this, recall from Section 2 that the $*$-algebra $\mathbb{C}[\mathrm{Irred}(\mathbb{G})]$ comes with a trace τ given by

$$\tau\Big(\sum_{u\in\mathrm{Irred}(\mathbb{G})} z_u u\Big) = z_e,$$

where $e \in \mathrm{Irred}(\mathbb{G})$ denotes the identity in $R(\mathbb{G})$. In what follows, we denote by $C^*_{\mathrm{red}}(R(\mathbb{G}))$ the enveloping C^*-algebra of $\mathbb{C}[\mathrm{Irred}(\mathbb{G})]$ on the GNS space $K = L^2(\mathbb{C}[I], \tau)$ arising from τ.

Lemma 4.6. *The character map $\chi\colon R(\mathbb{G}) \to A_0$ extends to an isometric $*$-homomorphism $\chi\colon C^*_{\mathrm{red}}(R(\mathbb{G})) \to A_{\mathrm{red}}$.*

Proof. Put $I = \mathrm{Irred}(\mathbb{G})$. For an irreducible, finite dimensional, unitary corepresentation u we have $h(u_{ij}) = 0$ unless u is the trivial corepresentation and therefore the following diagram commutes

$$\begin{array}{ccc} \mathbb{C}[I] & \xhookrightarrow{\ \chi\ } & A_0 \\ {\scriptstyle\tau}\downarrow & \swarrow{\scriptstyle h} & \\ \mathbb{C} & & \end{array}$$

Hence χ extends to an isometric embedding

$$K = L^2(\mathbb{C}[I], \tau) \hookrightarrow L^2(A_0, h) = H.$$

Denote by S the algebra $\chi(R(\mathbb{G}))$ and by $\bar{S}$ the closure of $\pi_h(S)$ inside A_{red}. Since S is a $*$-algebra that maps K into itself it also maps $K^\perp$ into itself and hence $\pi_h(\chi(a))$ takes the form

$$\begin{pmatrix} \pi_h(\chi(a))\big|_K & 0 \\ 0 & \pi_h(\chi(a))\big|_{K^\perp} \end{pmatrix}.$$

Thus

$$\begin{aligned} \|\pi_h(\chi(a))\| &= \max\{\|\pi_h(\chi(a))\big|_K\|, \|\pi_h(\chi(a))\big|_{K^\perp}\|\} \\ &\geq \|\pi_h(\chi(a))\big|_K\| \\ &= \|\pi_\tau(a)\|. \end{aligned}$$

This proves that the map $\kappa\colon \pi_h(S) \to \pi_\tau(\mathbb{C}[I])$ given by $\kappa(\pi_h(\chi(a))) = \pi_\tau(a)$ is bounded and it therefore extends to a contraction $\bar{\kappa}\colon \bar{S} \to C^*_{\mathrm{red}}(R(\mathbb{G}))$. We now prove that $\bar{\kappa}$ is injective. Since h is faithful on A_{red} and τ is faithful on $C^*_{\mathrm{red}}(R(\mathbb{G}))$ we get the following commutative diagram

$$\begin{array}{ccc} \pi_h(S) & \xrightarrow[\sim]{\ \kappa\ } & \pi_\tau(\mathbb{C}[I]) \\ \big\downarrow & & \big\downarrow \\ \bar{S} & \xrightarrow{\ \bar{\kappa}\ } & C^*_{\mathrm{red}}(R(\mathbb{G})) \\ \big\downarrow & & \big\downarrow \\ L^2(\bar{S}, h) & \longrightarrow & L^2(C^*_{\mathrm{red}}(R(\mathbb{G})), \tau) \end{array}$$

One easily checks that κ induces an isometry $L^2(\bar{S}, h) \to L^2(C^*_{\mathrm{red}}(\mathbb{G}), \tau)$ and it therefore follows that $\bar{\kappa}$ is injective and hence an isometry. Thus, for $\chi(a) \in S$ we have

$$\|\pi_h(\chi(a))\| = \|\bar{\kappa}(\pi_h(\chi(a)))\| = \|\pi_\tau(a)\|,$$

as desired.

□

Proof of Theorem 4.5. Assume first that $\mathbb{G}$ is coamenable and put $I = \mathrm{Irred}(\mathbb{G})$. Consider a finitely supported, symmetric probability measure μ on I. We aim to show that $1 \in \sigma(\lambda_{2,\mu})$, where $\lambda_{2,\mu}$ is the operator on $\ell^2(I, \sigma)$ defined in Section 2. Write μ as $\sum_{\xi \in I} t_\xi \delta_\xi$ and recall (Lemma 4.6) that the character map $\chi \colon \mathbb{C}[I] \to A_0$ extends to an injective $*$-homomorphism $\chi \colon C^*_{\mathrm{red}}(R(\mathbb{G})) \to A_{\mathrm{red}}$. Using this, and Proposition 2.5, we get that

$$\begin{aligned}
\sigma(\lambda_{2,\mu}) &= \sigma(l_\mu) \\
&= \sigma(\sum_{\xi \in I} t_\xi l_\xi) \\
&= \sigma(\sum_{\xi \in I} t_\xi \frac{1}{n_\xi} \pi_\tau(\xi)) \\
&= \sigma(\chi(\sum_{\xi \in I} \frac{t_\xi}{n_\xi} \pi_\tau(\xi))) \\
&= \sigma(\sum_{\xi \in I} \sum_{i=1}^{n_\xi} \frac{t_\xi}{n_\xi} \pi_h(\xi_{ii})).
\end{aligned}$$

Since $\mathbb{G}$ is coamenable, the counit extends to a character $\varepsilon \colon A_{\mathrm{red}} \to \mathbb{C}$ and we have

$$\varepsilon(\sum_{\xi \in I} \frac{t_\xi}{n_\xi} (\sum_{i=1}^{n_\xi} \xi_{ii})) = \sum_{\xi \in I} \frac{t_\xi}{n_\xi} n_\xi = 1.$$

Hence $1 \in \sigma\Big(\sum_{\xi \in I} \frac{t_\xi}{n_\xi} (\sum_{i=1}^{n_\xi} \pi_h(\xi_{ii}))\Big) = \sigma(\lambda_{2,\mu})$ and we conclude that $R(\mathbb{G})$ is amenable.

Assume, conversely, that $R(\mathbb{G})$ is amenable. We aim at proving that $\mathbb{G}$ fulfills the Kesten condition from Theorem 4.4. Let therefore $u \in \mathbb{M}_n(A)$ be an arbitrary, finite dimensional, unitary corepresentation. Denote by $(u_\alpha)_{\alpha \in S} \subseteq \mathrm{Irred}(\mathbb{G})$ the irreducible corepresentations occurring in the decomposition of u and by k_α the multiplicity of u_α in u. Now define

$$\mu_u(u_\alpha) = \begin{cases} \frac{k_\alpha n_\alpha}{n} & \text{if } \alpha \in S; \\ 0 & \text{if } \alpha \notin S. \end{cases}$$

Putting $\mu = \frac{1}{2}\mu_u + \frac{1}{2}\mu_{\bar{u}}$, we obtain a finitely supported, symmetric probability measure and by assumption we have that $1 \in \sigma(\lambda_{2,\mu})$. Using again that the character map extends to an injective $*$-homomorphism $\chi\colon C^*_{\mathrm{red}}(R(\mathbb{G})) \to A_{\mathrm{red}}$, we obtain

$$\begin{aligned}
\sigma(\lambda_{2,\mu}) &= \sigma\Big(\sum_{\alpha\in S} \frac{k_\alpha n_\alpha}{2n}\lambda_{2,u_\alpha} + \sum_{\alpha\in S}\frac{k_\alpha n_\alpha}{2n}\lambda_{2,u_{\bar\alpha}}\Big)\\
&= \sigma\Big(\sum_{\alpha\in S} \frac{k_\alpha n_\alpha}{2n} l_{u_\alpha} + \sum_{\alpha\in S}\frac{k_\alpha n_\alpha}{2n} l_{u_{\bar\alpha}}\Big) && \text{(Prop. 2.5)}\\
&= \sigma\Big(\sum_{\alpha\in S} \frac{k_\alpha n_\alpha}{2n}\frac{1}{n_\alpha}\pi_\tau(u_\alpha) + \sum_{\alpha\in S}\frac{k_\alpha n_\alpha}{2n}\frac{1}{n_\alpha}\pi_\tau(u_{\bar\alpha})\Big) && \text{(Rem. 2.6)}\\
&= \sigma\Big(\sum_{\alpha\in S} \frac{k_\alpha}{2n}\pi_h(\chi(u_\alpha)) + \sum_{\alpha\in S}\frac{k_\alpha}{2n}\pi_h(\chi(u_{\bar\alpha}))\Big)\\
&= \sigma\Big(\frac{1}{2n}\pi_h(\chi(u)) + \frac{1}{2n}\pi_h(\chi(\bar u))\Big)\\
&= \sigma\Big(\frac{1}{n}\pi_h(\mathrm{Re}(\chi(u)))\Big).
\end{aligned}$$

Thus

$$1 \in \sigma(\lambda_{2,\mu}) \iff n \in \sigma(\mathrm{Re}(\pi_h(\chi(u)))),$$

and the result now follows from Theorem 4.4. □

In particular we (re-)obtain the following.

Corollary 4.7. *A discrete group is amenable if and only if the group ring, considered as a fusion algebra, is amenable.*

Corollary 4.8 ([Ban99b])**.** *The quantum groups $SU_q(2)$ are coamenable.*

Proof. By Theorem 4.5, $SU_q(2)$ is coamenable if and only if $R(SU_q(2))$ is amenable. But, $R(SU_q(2)) = R(SU(2))$ (see e.g. [Wor88]) and since $(C(SU(2)), \Delta_c)$ is a coamenable quantum group $R(SU(2))$ is amenable. □

As seen from Theorem 4.5, the answer to the question of whether a compact quantum group is coamenable or not can be determined using only information about its corepresentations (a fact noted by Banica in the setting of compact matrix quantum groups in [Ban99a] and [Ban99b]). With this in mind, we now propose the following Følner condition for quantum groups.

Definition 4.9. *A compact quantum group $\mathbb{G} = (A, \Delta)$ is said to satisfy Følner's condition if for any finite, non-empty subset $S \subseteq \mathrm{Irred}(\mathbb{G})$ and any $\varepsilon > 0$ there exists a finite subset $F \subseteq \mathrm{Irred}(\mathbb{G})$ such that*

$$\sum_{u\in\partial_S(F)} n_u^2 < \varepsilon \sum_{u\in F} n_u^2.$$

Here n_u denotes the dimension of the irreducible corepresentation u and $\partial_S(F)$ is the boundary of F relative to S as defined in Definition 3.2.

We immediately obtain the following.

Corollary 4.10. *A compact quantum group is coamenable if and only if it satisfies Følner's condition.*

Proof. By Theorem 4.5, the compact quantum group $\mathbb{G}$ is coamenable if and only if $R(\mathbb{G})$ is amenable. By Theorem 3.3, $R(\mathbb{G})$ is amenable if and only if it satisfies (FC3) which is exactly the same as saying that $\mathbb{G}$ satisfies Følner's condition. □

In Section 6 we will use this Følner condition to deduce a vanishing result concerning L^2-Betti numbers of compact, coamenable quantum groups.

5. An Interlude

In this section we gather various notation and minor results which will be used in the following section to proof our main result, Theorem 6.1. Some generalities on von Neumann algebraic quantum groups are stated without proofs; we refer to [KV03] for the details.

Consider again a compact quantum group $\mathbb{G} = (A, \Delta)$ with tracial Haar state h. Denote by $\{u^\alpha \mid \alpha \in I\}$ a complete set of representatives for the equivalence classes of irreducible, unitary corepresentations of $\mathbb{G}$. Consider the dense Hopf $*$-algebra

$$A_0 = \operatorname{span}_{\mathbb{C}}\{u^\alpha_{ij} \mid \alpha \in I\},$$

and its discrete dual Hopf $*$-algebra $\hat{A}_0$. Since h is tracial, the discrete quantum group $\hat{A}_0$ is unimodular; i.e. the left- and right-invariant functionals are the same. Denote by $\hat{\varphi}$ the left- and right-invariant functional on $\hat{A}_0$ normalized such $\hat{\varphi}(h) = 1$. For $a \in A_0$ we denote by $\hat{a} \in A_0'$ the map

$$x \longmapsto h(ax).$$

Then, by definition, we have $\hat{A}_0 = \{\hat{a} \mid a \in A_0\}$. The algebra $\hat{A}_0$ is $*$-isomorphic to

$$\bigoplus_{\alpha \in I}^{\text{alg}} \mathbb{M}_{n_\alpha}(\mathbb{C}),$$

and because h is tracial the isomorphism has a simple description; if we denote by E^α_{ij} the standard matrix units in $\mathbb{M}_{n_\alpha}(\mathbb{C})$ then

$$\Phi(\widehat{(u^\alpha_{ij})^*}) = \tfrac{1}{n_\alpha} E^\alpha_{ij},$$

extends to a $*$-isomorphism [MVD98].

Denote by λ the GNS representation of A on $H = L^2(A_0, h)$, by η the canonical inclusion $A_0 \subseteq H$ and by M (or $\lambda(M)$) the enveloping von Neumann algebra $\lambda(A_0)''$. The map $\hat{\eta}\colon \hat{A}_0 \to H$ given by $\hat{a} \mapsto \eta(a)$ makes $(H, \hat{\eta})$ into a GNS pair for $(\hat{A}_0, \hat{\varphi})$ and the corresponding GNS representation L is given by

$$L(\hat{a})\eta(x) = \hat{\eta}(\hat{a}\hat{x}).$$

We denote by $\hat{M}$ (or $L(\hat{M})$) the enveloping von Neumann algebra $L(\hat{A}_0)''$. This is a discrete von Neumann algebraic quantum group and $\hat{\varphi}$ gives rise to a left- and right-invariant, normal, semifinite, faithful (n.s.f.) weight on $\hat{M}$. If W denotes the multiplicative unitary for $\lambda(M)$ then

$$\lambda(M) = \overline{\{(\mathrm{id}\otimes\omega)W \mid \omega \in B(H)_*\}}$$
$$L(\hat{M}) = \overline{\{(\omega \otimes \mathrm{id})W \mid \omega \in B(H)_*\}},$$

where both closures are in the σ-strong* topology. In particular we see that $W \in \lambda(M)\bar{\otimes}L(\hat{M})$.

Denote by κ the unitary antipode on M and by J the anti-unitary on H given by $J(\eta(x)) = \eta(x^*)$. Then the formula $\rho(a) = J\lambda(\kappa(a^*))J$ defines another representation of M on H. Similarly, the unitary antipode $\hat{\kappa}$ on $\hat{M}$ and the modular conjugation $\hat{J}$ for $\hat{\varphi}$ give rise to another representation $R(x) = \hat{J}L(\hat{\kappa}(x^*))\hat{J}$ of $\hat{M}$ on H. Note that by Tomita-Takesaki theory we have $\rho(M) = \lambda(M)'$ and $R(\hat{M}) = L(\hat{M})'$. Because h is tracial we have that $\hat{J}J = J\hat{J}$ ([ES92, 4.1.7]) and the self-adjoint unitary $U = J\hat{J}$ has the property that

$$\mathrm{Ad}_U\,\lambda(a) = U\lambda(a)U = \rho(a) \qquad \text{for all } a \in M$$
$$\mathrm{Ad}_U\,L(x) = UL(x)U = R(x) \qquad \text{for all } x \in \hat{M}$$

This can be seen using, for instance, [KV03, 2.1]. In the following we denote by Σ the flip-unitary on $H\bar{\otimes}H$ and by $\sigma = \mathrm{Ad}_\Sigma$ the flip-automorphism of $M\bar{\otimes}M$. We shall also consider the opposite quantum group $(M, \Delta^{\mathrm{op}})$ whose underlying von Neumann algebra is again M, but the comultiplication is $\Delta^{\mathrm{op}} = \sigma\Delta$. Define a $*$-homomorphism $\alpha\colon \rho(M) \longrightarrow \lambda(M)\bar{\otimes}\rho(M)$ by $\alpha(\rho(a)) = (\lambda \otimes \rho)(\Delta^{\mathrm{op}}a)$. It is easy to check that α is a left coaction of $(M, \Delta^{\mathrm{op}})$ on the von Neumann algebra $\rho(M)$ and we may therefore ([Vae01]) form the cross product

$$\begin{aligned} M \ltimes_\alpha \rho(M) &= \mathrm{vNa}\{\alpha(\rho(M)), L(\hat{M})' \otimes 1\} \\ &= \mathrm{vNa}\{(\lambda \otimes \rho)(\Delta^{\mathrm{op}}(M)), R(\hat{M}) \otimes 1\}. \end{aligned}$$

Lemma 5.1. *There exists a unitary V on $H\bar{\otimes}H$ such that Ad_V implements an isomorphism $M \ltimes_\alpha \rho(M) \simeq 1 \otimes B(H)$. More precisely we* *have*

$$\mathrm{Ad}_V(\alpha(\rho(a))) = 1 \otimes \rho(a) \quad \textit{and} \quad \mathrm{Ad}_V(R(x) \otimes 1) = 1 \otimes L(x),$$

for all $a \in M$ *and all* $x \in \hat{M}$.

Proof. Consider again the self-adjoint unitary U and the multiplicative unitary $W \in \lambda(M)\bar{\otimes}L(\hat{M})$ for $\lambda(M)$. Define $\overline{W} = (U \otimes U)W(U \otimes U) \in \rho(M)\bar{\otimes}R(\hat{M})$; it is easy to see that $\overline{W}$ becomes a multiplicative unitary for ρ in the sense that

$$\overline{W}^*(1 \otimes \rho(a))\overline{W} = \rho \otimes \rho(\Delta(a)).$$

Put $V = \overline{W}\Sigma(U \otimes 1)$. For $a \in M$ we have

$$\begin{aligned}
\mathrm{Ad}_V(\alpha(\rho(a))) &= \mathrm{Ad}_{\overline{W}} \mathrm{Ad}_\Sigma \mathrm{Ad}_{U\otimes 1}[(\lambda \otimes \rho)\Delta^{\mathrm{op}}(a)] \\
&= \mathrm{Ad}_{\overline{W}} \mathrm{Ad}_\Sigma[(\rho \otimes \rho)\Delta^{\mathrm{op}}(a)] \\
&= \mathrm{Ad}_{\overline{W}} \mathrm{Ad}_\Sigma[\Sigma(\rho \otimes \rho)\Delta(a)\Sigma] \\
&= \mathrm{Ad}_{\overline{W}}[(\rho \otimes \rho)\Delta(a)] \\
&= 1 \otimes \rho(a).
\end{aligned}$$

For $x \in \hat{M}$ we get

$$\begin{aligned}
\mathrm{Ad}_V(R(x) \otimes 1) &= \mathrm{Ad}_{\overline{W}} \mathrm{Ad}_\Sigma \mathrm{Ad}_{U\otimes 1}[R(x) \otimes 1] \\
&= \mathrm{Ad}_{\overline{W}} \mathrm{Ad}_\Sigma[L(x) \otimes 1] \\
&= \mathrm{Ad}_{\overline{W}}[1 \otimes L(x)] \\
&= 1 \otimes L(x). \qquad (\overline{W} \in (1 \otimes L(\hat{M}))')
\end{aligned}$$

We now just have to see that Ad_V surjects onto $1 \otimes B(H)$. The pair $(M, \hat{M})$ is a dual pair of locally compact von Neumann algebraic quantum groups and by [MvD02, 3.2,3.4,3.16] we therefore have

$$B(H) = \overline{\mathrm{span}_{\mathbb{C}}\{\lambda(a)L(x) \mid a \in M, x \in \hat{M}\}}^{\sigma\text{-weak}}$$

But,

$$J\lambda(M)L(\hat{M})J = (J\lambda(M)J)(JL(\hat{M})J) = \rho(M)L(\hat{M}),$$

where the last equality follows from [KV03, 2.1], and since J is anti-unitary we have

$$B(H) = \overline{\mathrm{span}_{\mathbb{C}}\{\rho(a)L(x) \mid a \in M, x \in \hat{M}\}}^{\sigma\text{-weak}}$$

This proves that Ad_V maps $M \ltimes_\alpha \rho(M)$ onto $1 \otimes B(H)$. □

Consider again the Haar state h on M. This state induces ([Vae01, 2.5, 3.1]) a dual n.s.f. weight θ on $M \ltimes_\alpha \rho(M)$ with the property ([Vae01, 3.2]) that

$$\theta(\alpha(a^*)(R(x^*x) \otimes 1)\alpha(a)) = h(a^*a)\hat{\varphi}(x^*x),$$

for all $a \in M$ and $x \in \mathfrak{N}_{\hat{\varphi}} = \{x \in \hat{M} \mid \hat{\varphi}(x^*x) < \infty\}$. We therefore have the following.

Lemma 5.2. *There exists an n.s.f. weight ν on $B(H)$ such that*

$$\nu(\rho(a^*)L(x^*x)\rho(a)) = h(a^*a)\hat{\varphi}(x^*x),$$

for all $a \in M$ and all $x \in \mathfrak{N}_{\hat{\varphi}}$.

Proof. Put $\nu(T) = \theta \circ \mathrm{Ad}_{V^*}(1 \otimes T)$ for $T \in B(H)^+$. It now follows from Lemma 5.1 that ν has the desired properties. □

6. A Vanishing Result

In this section we investigate the L^2-Betti numbers of coamenable quantum groups. The notion of L^2-Betti numbers for compact quantum groups was introduced in [Kye06] and we refer to that paper (and Section 0) for the definitions and basic results. We will also freely use Lück's extended Murray-von Neumann dimension, but whenever explicit properties are used there will be a reference. These references will be to the original work [Lüc98a] and [Lüc97], but for the reader who wants to learn the subject, the book [Lüc02] is probably a better general reference.

Consider again a compact quantum group $\mathbb{G} = (A, \Delta)$ with Haar state h and denote by M the enveloping von Neumann algebra in the GNS representation arising from h. As promised in the introduction, we will now prove the following theorem which should be considered as a quantum group analogue of [Lüc98a, 5.1].

Theorem 6.1. *If $\mathbb{G}$ is coamenable and h is tracial then for any left A_0-module Z and any $k \geq 1$ we have*

$$\dim_M \mathrm{Tor}_k^{A_0}(M, Z) = 0,$$

where $\dim_M(\cdot)$ is Lück's extended dimension function arising from the extension of the trace-state h.

If M were flat as a module over A_0 we would have $\mathrm{Tor}_k^{A_0}(M, Z) = 0$ for any Z and any $k \geq 1$, and the property in Theorem 6.1 is therefore referred to as *dimension flatness of the von Neumann algebra over the algebra of matrix coefficients.*

The proof of Theorem 6.1 is divided into three parts and draws inspiration from the proof of [Lüc98a, 5.1]. Part I consists of reductions while part II contains the central argument carried out in detail in a special case. Part III shows how to boost the argument from part II to the general case. Throughout the proof, we will use freely the quantum group notation developed in the previous sections without further reference.

Proof of Theorem 6.1.

Part I

We begin with some reductions. Let an arbitrary A_0-module Z be given and choose a free module F that surjects onto Z. Then we have a short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow Z \longrightarrow 0,$$

and since F is free (in particular flat) the corresponding long exact Tor-sequence gives an isomorphism

$$\mathrm{Tor}_{k+1}^{A_0}(M,Z) \simeq \mathrm{Tor}_k^{A_0}(M,K) \text{ for } k \geq 1.$$

It is therefore sufficient to prove the theorem for arbitrary Z and $k = 1$. Moreover, we may assume that Z is finitely generated since Tor commutes with direct limits, every module is the directed union of its finitely generated submodules and $\dim_M(\cdot)$ is well behaved with respect to direct limits ([Lüc98a, 2.9]). Actually, we can assume that Z is finitely presented since any finitely generated module Z is a direct limit of finitely presented modules. To see this, choose a short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow Z \longrightarrow 0,$$

with F finitely generated and free. Denote by $(K_j)_{j\in J}$ the directed system of finitely generated submodules in K. Then F/K_j is finitely presented for each $j \in J$ and

$$Z = \varinjlim_{j} F/K_j.$$

Because of this and the direct limit formula for the dimension function ([Lüc98a, 2.9]) we may, and will, therefore assume that Z is finitely presented. Choose a finite presentation

$$A_0^n \xrightarrow{f} A_0^m \longrightarrow Z \longrightarrow 0.$$

Put $H = L^2(A,h)$, $K = \ker(f) \subseteq A_0^n \subseteq H^n$ and denote by $f^{(2)}\colon H^n \to H^m$ the continuous extension of f. Then we have

$$\mathrm{Tor}_1^{A_0}(M,Z) = \frac{\ker(\mathrm{id}_M \otimes f)}{M \underset{A_0}{\otimes} K},$$

and hence

$$\begin{aligned}\dim_M \mathrm{Tor}_1^{A_0}(M,Z) &= \dim_M \ker(\mathrm{id}_M \otimes f) - \dim_M M \underset{A_0}{\otimes} K \\ &= \dim_M \ker(f^{(2)}) - \dim_M \overline{K}^{\|\cdot\|_2},\end{aligned}$$

where the second equality follows from [CS05, 2.11]. See also [Lüc98a, p.158-159]. So we need to prove that $\overline{K}^{\|\cdot\|_2} = \ker(f^{(2)})$.

Part II

We first treat the case $m = n = 1$. Then the map f has the form R_a (right-multiplication by a) for some $a \in A_0$. If $a = 0$ we have $\overline{K}^{\|\cdot\|_2} = H = \ker(f^{(2)})$ so we may assume $a \neq 0$. Since $\{u_{ij}^\alpha \mid \alpha \in I\}$ is a linear basis for A_0 ([MVD98, 7.3]) this a has a unique, finite, linear expansion as $a = \sum_{\alpha,i,j} z_{ij}^\alpha u_{ij}^\alpha$. Consider now the finite, non-empty set

$$S = \{\alpha \in I \mid \exists\, 1 \leq i,j \leq n_\alpha \colon z_{ij}^\alpha \neq 0\}.$$

Since $\mathbb{G}$ is assumed coamenable it satisfies Følner's condition and we may therefore choose a finite set $F \subseteq I$ such that

$$\sum_{u \in \partial_S(F)} n_u^2 < \frac{1}{2} \sum_{u \in F} n_u^2. \qquad (\dagger)$$

In the following we will write ∂ in stead of $\partial_S(F)$ for simplicity. Denote by H_0 the space $\ker(f^{(2)})$, by $q_0 \in M'$ the projection onto H_0 and by $q \in M'$ the projection onto $H_0 \cap K^\perp$. We need to show that $q = 0$.

Recall the isomorphism $\Phi \colon \hat{A}_0 \to \bigoplus_\alpha^{\mathrm{alg}} \mathbb{M}_{n_\alpha}(\mathbb{C})$ from Section 5. For a finite subset $E \subseteq I$ we denote by $p_E \in \hat{A}_0$ the central projection $\Phi^{-1}(\sum_{\alpha \in I} \chi_E(\alpha) 1_{n_\alpha})$. A direct calculation shows that $L(p_E) \in B(H)$ projects onto the finite dimensional linear subspace

$$\mathrm{span}_{\mathbb{C}}\{u_{ij}^\alpha \mid 1 \leq i,j \leq n_\alpha, \alpha \in \bar{E}\}. \qquad \text{(note the "bar" on } E)$$

Since h is tracial, Woronowicz's quantum Peter-Weyl Theorem ([KT99, 3.2.3]) takes a particular simple form and states that the set

$$\{\sqrt{n_\alpha} u_{ij}^\alpha \mid 1 \leq i,j \leq n_\alpha, \alpha \in I\}$$

constitutes an orthonormal basis for H. Hence every $x \in H$ has an ℓ^2-expansion

$$x = \sum_{\alpha \in I} \sum_{i,j=1}^{n_\alpha} x_{ij}^\alpha \sqrt{n_\alpha} u_{ij}^\alpha. \qquad (x_{ij}^\alpha \in \mathbb{C})$$

Consider a vector $x = \sum_{i \in I} x_{ij}^\alpha \sqrt{n_\alpha} u_{ij}^\alpha \in H$ and assume that $L(p_{\bar{\partial}})x = 0$ such that $x = \sum_{\alpha \notin \partial} \sum_{i,j=1}^{n_\alpha} x_{ij}^\alpha \sqrt{n_\alpha} u_{ij}^\alpha$. For $\gamma \in S$ and $1 \leq p,q \leq n_\gamma$ we then have

$$(R^{(2)}_{u^\gamma_{pq}} \circ L(p_{\bar{F}}))x = \sum_{\alpha \notin \partial, \alpha \in F} \sum_{i,j=1}^{n_\alpha} x_{ij}^\alpha \sqrt{n_\alpha} u_{ij}^\alpha u_{pq}^\gamma$$

$$(L(p_{\bar{F}}) \circ R^{(2)}_{u^\gamma_{pq}})x = L(p_{\bar{F}})(\sum_{\alpha \notin \partial} \sum_{i,j=1}^{n_\alpha} x_{ij}^\alpha \sqrt{n_\alpha} u_{ij}^\alpha u_{pq}^\gamma).$$

Since $u_{ij}^\alpha u_{pq}^\gamma$ is contained in the linear span of the matrix coefficients of $u^\alpha \circledT u^\gamma$ and since $\alpha \notin \partial = \partial_S(F)$ and $\gamma \in S$ we see that the two expressions above are equal. By linearity and continuity we obtain

$$(f^{(2)} \circ L(p_{\bar{F}}))x = (L(p_{\bar{F}}) \circ f^{(2)})x.$$

This holds for all $x \in \ker(L(p_{\bar{\partial}}))$. Thus, if $x \in H_0 \cap \ker(L(p_{\bar{\partial}}))$ we have

$$0 = (f^{(2)} \circ L(p_{\bar{F}}))x = f(L(p_{\bar{F}})x),$$

where the last equality is due to the fact that $\mathrm{rg}(L(p_{\bar{F}})) \subseteq A_0 \subseteq H$. This proves that $L(p_{\bar{F}})x \in K = \ker(f)$ and since q was defined as the projection onto $H_0 \cap K^{\perp}$ we get $qL(p_{\bar{F}})x = 0$. Since this holds whenever $x \in H_0 = q_0(H)$ and $L(p_{\bar{\partial}})x = 0$ we get $qL(p_{\bar{F}})(q_0 \wedge (1 - L(p_{\bar{\partial}}))) = 0$; that is

$$qL(p_{\bar{F}})q_0 = qL(p_{\bar{F}})(q_0 \wedge L(p_{\bar{\partial}})) \tag{$\ddagger$}$$

Since $q, q_0 \in \lambda(M)' = \rho(M)$ there exist $\tilde{q}, \tilde{q}_0 \in M$ such that $q = \rho(\tilde{q})$ and $q_0 = \rho(\tilde{q}_0)$, where ρ is the representation of M constructed in Section 5. The equation ($\ddagger$) then reads

$$\rho(\tilde{q})L(p_{\bar{F}})\rho(\tilde{q}_0) = \rho(\tilde{q})L(p_{\bar{F}})(\rho(q_0) \wedge L(p_{\bar{\partial}})) \tag{$\ddagger'$}$$

Denote by ν the n.s.f. weight on $B(H)$ given by Lemma 5.2. We then get

$$\begin{aligned}
h(\tilde{q})\hat{\varphi}(p_{\bar{F}}) &= \nu(\rho(\tilde{q})L(p_{\bar{F}})\rho(\tilde{q})) && \text{(by 5.2)}\\
&= \nu(\rho(\tilde{q})L(p_{\bar{F}})\rho(\tilde{q}_0)\rho(\tilde{q})) && (q \leq q_0)\\
&= \nu(\rho(\tilde{q})L(p_{\bar{F}})(\rho(\tilde{q}_0) \wedge L(p_{\bar{\partial}}))\rho(\tilde{q})) && \text{(by ($\ddagger'$))}\\
&= \nu([L(p_{\bar{F}})\rho(\tilde{q})]^*[(\rho(\tilde{q}_0) \wedge L(p_{\bar{\partial}}))\rho(\tilde{q})])\\
&\leq \nu(\rho(\tilde{q})L(p_{\bar{F}})\rho(\tilde{q}))^{\frac{1}{2}}\nu(\rho(\tilde{q})(\rho(\tilde{q}_0) \wedge L(p_{\bar{\partial}}))\rho(\tilde{q}))^{\frac{1}{2}}\\
&= h(\tilde{q})^{\frac{1}{2}}\hat{\varphi}(p_{\bar{F}})^{\frac{1}{2}}\nu(\rho(\tilde{q})(\rho(\tilde{q}_0) \wedge L(p_{\bar{\partial}}))\rho(\tilde{q}))^{\frac{1}{2}} && \text{(by 5.2)}\\
&\leq h(\tilde{q})^{\frac{1}{2}}\hat{\varphi}(p_{\bar{F}})^{\frac{1}{2}}\nu(\rho(\tilde{q})L(p_{\bar{\partial}})\rho(\tilde{q}))^{\frac{1}{2}} && (\nu \text{ positive})\\
&= h(\tilde{q})\hat{\varphi}(p_{\bar{F}})^{\frac{1}{2}}\hat{\varphi}(p_{\bar{\partial}})^{\frac{1}{2}} && \text{(by 5.2)}
\end{aligned}$$

Here we used the Cauchy-Schwarz inequality to get the fifth step. Since $\hat{\varphi}$ is faithful and $p_{\bar{F}} > 0$ this gives

$$h(\tilde{q})^2 \leq \frac{\hat{\varphi}(p_{\bar{\partial}})}{\hat{\varphi}(p_{\bar{F}})}h(\tilde{q})^2. \tag{$*$}$$

Because h is tracial, the left invariant weight $\hat{\varphi}$ on $\hat{A}_0 \simeq \overset{\text{alg}}{\bigoplus}_{\alpha \in I}\mathbb{M}_{n_\alpha}(\mathbb{C})$ has the particular simple form ([VKV$^+$, p.47])

$$\hat{\varphi} = \sum_{\alpha \in I} n_\alpha \mathrm{Tr}_{n_\alpha},$$

where Tr_{n_α} is the unnormalized trace on $\mathbb{M}_{n_\alpha}(\mathbb{C})$. In particular

$$\hat{\varphi}(p_E) = \sum_{\alpha \in E} n_\alpha^2 = \hat{\varphi}(p_{\bar{E}}),$$

for any finite subset $E \subseteq I$. The inequalities (†) and (∗) therefore implies

$$h(\tilde{q})^2 \leq \frac{1}{2} h(\tilde{q})^2,$$

and since h is faithful this forces $\tilde{q} = 0$. Hence $0 = \rho(\tilde{q}) = q$ as desired.

Part III

We now treat the general case of a finitely presented A_0-module Z with finite presentation

$$A_0^n \xrightarrow{f} A_0^m \longrightarrow Z \longrightarrow 0.$$

In this case f is given by right multiplication by an $n \times m$ matrix $T = (t_{ij})$ with entries in A_0. Each t_{ij} has a unique linear expansion as $t_{ij} = \sum_{\alpha,k,l} t_{\alpha,k,l}^{(i,j)} u_{kl}^{\alpha}$ and we put

$$S = \{\alpha \in I \mid \exists\, i, j, k, l, \alpha \ : t_{\alpha,k,l}^{(i,j)} \neq 0\}.$$

As in Part II, we may assume that $T \neq 0$ so that $S \neq \emptyset$. According to the Følner condition, there exists an $F \subseteq I$ such that

$$\sum_{u \in \partial_S(F)} n_u^2 < \frac{1}{2} \sum_{u \in F} n_u^2.$$

Put $\partial = \partial_S(F)$, denote by H_0 the space $\ker(f^{(2)}) \subseteq H^n$, by $q_0 \in \mathbb{M}_n(M')$ the projection onto H_0 and by $q \in \mathbb{M}_n(M')$ the projection onto $H_0 \cap K^{\perp}$. We need to show that $q = 0$. By repeating the argument from the beginning of Part II we arrive at the equation

$$qL(p_{\bar{F}})^n q_0 = qL(p_{\bar{F}})^n (q_0 \wedge L(p_{\bar{\partial}})^n),$$

where $L(x)^n$ denotes the diagonal $n \times n$-matrix

$$\begin{pmatrix} L(x) & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & L(x) \end{pmatrix} \qquad (x \in \hat{M})$$

Since $\lambda(M)' = \rho(M)$ there exist $\tilde{q}, \tilde{q}_0 \in \mathbb{M}_n(M)$ such that $q = \rho(\tilde{q})$ and $q_0 = \rho(\tilde{q}_0)$. Here ρ is the representation constructed in Section 5 and the matrix-equations are to be interpreted entrywise. Consider again the n.s.f. weight ν on $B(H)$ from Lemma 5.2. It induces an n.s.f. weight $\nu_n = \nu \otimes \mathrm{Tr}_n$ on $\mathbb{M}_n(B(H)) = B(H) \odot \mathbb{M}_n(\mathbb{C})$ and a direct calculation shows that for $A \in \mathbb{M}_n(M)$ and $x \in \mathfrak{N}_{\hat{\varphi}}$ we have

$$\nu_n(\rho(A)^* L(x^*x)^n \rho(A)) = (h \otimes \mathrm{Tr}_n)(A^*A)\hat{\varphi}(x^*x).$$

Put $h_n = h \otimes \mathrm{Tr}_n$. By repeating the calculation from Part II, with ν_n and h_n in stead of ν and h, we arrive at the equation

$$h_n(\tilde{q})\hat{\varphi}(p_{\bar{F}}) \leq h_n(\tilde{q})\hat{\varphi}(p_{\bar{F}})^{\frac{1}{2}} \hat{\varphi}(p_{\bar{\partial}})^{\frac{1}{2}}$$

Thus

$$h_n(\tilde{q})^2 \leq \frac{\hat{\varphi}(p_{\bar{\partial}})}{\hat{\varphi}(p_{\bar{F}})} h_n(\tilde{q})^2 \leq \frac{1}{2} h_n(\tilde{q})^2,$$

and since $h_n = h \otimes \mathrm{Tr}_n$ is faithful we get $\tilde{q} = 0$. Hence $0 = \rho(\tilde{q}) = q$ as desired.

□

By putting $Z = \mathbb{C}$ in Theorem 6.1, we immediately obtain the following corollary.

Corollary 6.2. *Let $\mathbb{G} = (A, \Delta)$ be a compact, coamenable quantum group with tracial Haar state. Then $\beta_n^{(2)}(\mathbb{G}) = 0$ for all $n \geq 1$. Here $\beta_n^{(2)}(\mathbb{G})$ is the n-th L^2-Betti number of $\mathbb{G}$ as defined in* [Kye06].

In particular we obtain the following extension of [Kye06, 6.3].

Corollary 6.3. *If $\mathbb{G}$ is an abelian, compact quantum group then $\beta_n^{(2)}(\mathbb{G}) = 0$ for $n \geq 1$.*

Proof. Since $\mathbb{G}$ is abelian it is of the form $(C(G), \Delta_c)$ for some compact (second countable) group G. Since the counit, given by evaluation at the identity, is already globally defined and bounded it is clear that $\mathbb{G}$ is coamenable and the result now follows from Corollary 6.2. □

We also obtain the classical result.

Corollary 6.4. [Lüc98a, 5.1] *If Γ is an amenable, countable, discrete group then for all $\mathbb{C}\Gamma$-modules Z and all $n \geq 1$ we have*

$$\dim_{\mathscr{L}(\Gamma)} \mathrm{Tor}_n^{\mathbb{C}\Gamma}(\mathscr{L}(\Gamma), Z) = 0.$$

In particular, $\beta_n^{(2)}(\Gamma) = 0$ for $n \geq 1$.

Proof. Put $\mathbb{G} = (C^*_{\mathrm{red}}(\Gamma), \Delta_{\mathrm{red}})$. Then $\mathbb{G}$ is coamenable if and only if Γ is amenable and the result now follows from Theorem 6.1 and Corollary 6.2 □

In [CS05], Connes and Shlyakhtenko introduced a notion of L^2-Betti numbers for tracial $*$-algebras. From the above results we also obtain vanishing of these Connes-Shlyakhtenko L^2-Betti numbers for certain Hopf $*$-algebras. More precisely we get the following.

Corollary 6.5. *Let $\mathbb{G} = (A, \Delta)$ be a compact, coamenable quantum group with tracial Haar state h. Then $\beta_n^{(2)}(A_0, h) = 0$ for all $n \geq 1$, where $\beta_n^{(2)}(A_0, h)$ is the n-th Connes-Shlyakhtenko L^2-Betti number of the $*$-algebra A_0 with respect to the trace h.*

Proof. By [Kye06, 4.1] we have $\beta_n^{(2)}(\mathbb{G}) = \beta_n^{(2)}(A_0, h)$ and the claim therefore follows from Corollary 6.2. □

7. Examples

A concrete example of a non-commutative, non-cocommutative, coamenable (matrix) quantum group with tracial Haar state is the orthogonal quantum group $A_o(2) \simeq SU_{-1}(2)$. It follows from [Ban99a, 5.1] that $A_o(2)$ is coamenable. To see that the Haar state is tracial, one observes that the orthogonality property of the canonical fundamental corepresentation implies that the antipode has period two.

7.1. **Examples arising from tensor products.** If $\mathbb{G}_1 = (A_1, \Delta_1)$ and $\mathbb{G}_2 = (A_2, \Delta_2)$ are compact quantum groups then the (minimal) tensor product $A = A_1 \otimes A_2$ may be turned into a quantum group $\mathbb{G}$ by defining the comultiplication $\Delta\colon A \longrightarrow A \otimes A$ to be

$$\Delta(a) = (\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\Delta_1 \otimes \Delta_2)(a),$$

where σ denotes the flip-isomorphism from $A_1 \otimes A_2$ to $A_2 \otimes A_1$. The Haar state is the tensor product of the two Haar states and the counit is the tensor product of the counits. Using these facts, it is not difficult to see ([BMT01]) that if both $\mathbb{G}_1$ and $\mathbb{G}_2$ are coamenable and have tracial Haar states, then the same is true for $\mathbb{G}$. See e.g. [KR86, 11.3.2].

7.2. **Examples arising from bicrossed products.** Another way to obtain examples of compact, coamenable quantum groups is via *bicrossed products*. We therefore briefly sketch the bicrossed product construction following [VV03] closely. In [VV03], Vaes and Vainerman consider the more general notion of *cocycle* bicrossed products, but since we will mainly be interested in the case where the cocycles are trivial we will restrict our attention to this case in the following. The more general situation will be discussed briefly in Remark 7.5. The bicrossed product construction is defined using the language of von Neumann algebraic quantum groups. We will use this language freely in the following and refer to [KV03] for the background material.

Let (M_1, Δ_1) and (M_2, Δ_2) be locally compact (l.c.) von Neumann algebraic quantum groups. Let $\tau\colon M_1 \bar{\otimes} M_2 \to M_1 \bar{\otimes} M_2$ be a faithful $*$-homomorphism and denote by $\sigma\colon M_1 \bar{\otimes} M_2 \to M_2 \bar{\otimes} M_1$ the flip-isomorphism. Then τ is called a *matching* from M_1 to M_2 if the following holds.

- The map $\alpha\colon M_2 \longrightarrow M_1 \bar{\otimes} M_2$ given by $\alpha(y) = \tau(1 \otimes y)$ is a (left) coaction of (M_1, Δ_1) on the von Neumann algebra M_2.
- Defining $\beta\colon M_1 \longrightarrow M_1 \bar{\otimes} M_2$ as $\beta(x) = \tau(x \otimes 1)$ the map $\sigma\beta$ is a (left) coaction of (M_2, Δ_2) on the von Neumann algebra M_1.
- The coactions satisfy the following two *matching conditions*

$$\tau_{(13)}(\alpha \otimes 1)\Delta_2 = (1 \otimes \Delta_2)\alpha \qquad \text{(M1)}$$

$$\tau_{(23)}\sigma_{(23)}(\beta \otimes 1)\Delta_1 = (\Delta_1 \otimes 1)\beta \qquad \text{(M2)}$$

Here we use the standard leg numbering convention (see e.g. [MVD98]). If $\tau\colon M_1\bar{\otimes}M_2 \to M_1\bar{\otimes}M_2$ is a matching from M_1 to M_2 then it is easy to see that $\sigma\tau\sigma^{-1}$ is a matching from M_2 to M_1. We will therefore just refer to the pair (M_1, M_2) as a matched pair and to τ as a matching of the pair. Let (M_1, Δ_1) and (M_2, Δ_2) be such a matched pair of l.c. quantum groups and denote by τ the matching. We denote by H_i the GNS space of M_i with respect to the left invariant weight φ_i and by W_i and $\hat{W}_i$ the natural multiplicative unitaries on H_i for M_i and $\hat{M}_i$ respectively. By H we denote $H_1\bar{\otimes}H_2$ and by Σ the flip-unitary on $H\bar{\otimes}H$. We may now form two crossed products:

$$M = M_1 \ltimes_\alpha M_2 = \mathrm{vNa}\{\alpha(M_2), \hat{M}_1 \otimes 1\} \subseteq B(H_1\bar{\otimes}H_2)$$

$$\tilde{M} = M_2 \ltimes_{\sigma\beta} M_1 = \mathrm{vNa}\{\sigma\beta(M_1), \hat{M}_2 \otimes 1\} \subseteq B(H_2\bar{\otimes}H_1)$$

Some of the main results in [VV03] is summarized in the following:

Theorem 7.1 ([VV03]). *Define operators*

$$\hat{W} = (\beta \otimes 1 \otimes 1)(W_1 \otimes 1)(1 \otimes 1 \otimes \alpha)(1 \otimes \hat{W}_2)$$

and $W = \Sigma\hat{W}^*\Sigma$ *on* $H\bar{\otimes}H$. *Then* W *and* $\hat{W}$ *are multiplicative unitaries and the map* $\Delta\colon M \to B(H\bar{\otimes}H)$ *given by* $\Delta(a) = W^*(1\otimes 1\otimes a)W$ *defines a comultiplication on* M *turning it into a l.c. quantum group. Denoting by* Σ_{12} *the flip-unitary from* $H_1\bar{\otimes}H_2$ *to* $H_2\bar{\otimes}H_1$*, the dual quantum group* $\hat{M}$ *becomes* $\Sigma_{12}^*\tilde{M}\Sigma_{12}$ *with comultiplication implemented by* $\hat{W}$.

Thus, up to a flip the two crossed products above are in duality. In [DQV02], Desmedt, Quaegebeur and Vaes studied (co)amenability of bicrossed products. Combining their Theorem 15 with [VV03, 2.17] we obtain the following: If (M_1, M_2) is a matched pair with M_1 discrete and M_2 compact then the bicrossed product M is compact, and M is coamenable if and only if both M_2 and $\hat{M}_1$ are. Here a von Neumann algebraic compact quantum group is said to be coamenable if the corresponding C^*-algebraic quantum group is. Collecting the results discussed above we obtain the following.

Proposition 7.2. *If* (M_1, M_2) *is a matched pair of l.c. quantum groups in which* $\hat{M}_1$ *and* M_2 *are compact and coamenable, then the bicrossed product* $M = M_1 \ltimes_\alpha M_2$ *is coamenable and compact. So if the Haar state on* M *is tracial the quantum group* (M, Δ) *has vanishing* L^2*-Betti numbers in all positive degrees.*

In order to produce more concrete examples, we will now discuss a special case of the bicrossed product construction in which one of the coactions comes from an actual group action. This part of the theory is due to De Cannière ([DC79]) and is formulated using the language of Kac algebras. We remind the reader, that a *compact* Kac algebra is nothing but a von Neumann algebraic, compact quantum group with tracial Haar state. A discrete, countable group Γ acts on a compact Kac algebra (M, Δ, S, h) if the group acts on the von Neumann algebra M and the action commutes with both the coproduct and the antipode. Denoting the action by ρ, this means that

$$\Delta(\rho_\gamma(x)) = \rho_\gamma \otimes \rho_\gamma(\Delta(x)),$$
$$S(\rho_\gamma(x)) = \rho_\gamma(S(x)),$$

for all $\gamma \in \Gamma$ and all $x \in M$. In this situation, the action of Γ on M induces a coaction $\alpha\colon M \longrightarrow \ell^\infty(\Gamma)\bar{\otimes}M$. Denoting by H the Hilbert space on which M acts and identifying $\ell^2(\Gamma)\bar{\otimes}H$ with $\ell^2(\Gamma, H)$, this coaction is given by the formula

$$\alpha(x)(\xi)(\gamma) = \rho_{\gamma^{-1}}(x)(\xi(\gamma)),$$

for $\xi \in \ell^2(\Gamma, H)$. The crossed product, which is defined as

$$\Gamma \ltimes_\rho M = \{\alpha(M), \mathscr{L}(\Gamma) \otimes 1\}'',$$

becomes again a Kac algebra ([DC79, Thm.1]). One should note at this point that De Cannière works with the *right* crossed product acting on $H\bar{\otimes}\ell^2(\Gamma)$ where we work with the *left* crossed product acting on $\ell^2(\Gamma)\bar{\otimes}H$. But, one can come from one to the other by conjugation with the flip-unitary and we may therefore freely transport all results from [DC79] to the setting of *left* crossed products. We now prove that De Cannière's crossed product can also be considered as a bicrossed product. This is probably well known to experts in the field, but we were unable to find an explicit reference.

Proposition 7.3. *Defining* $\tau\colon \ell^\infty(\Gamma)\bar{\otimes}M \longrightarrow \ell^\infty(\Gamma)\bar{\otimes}M$ *by*

$$\tau(\delta_\gamma \otimes x) = \delta_\gamma \otimes \rho_{\gamma^{-1}}(x)$$

we obtain a matching with the above defined α *as the corresponding coaction of* $\ell^\infty(\Gamma)$ *on* M *and trivial coaction of* (M, Δ) *on* $\ell^\infty(\Gamma)$.

Proof. A direct calculation shows that $\alpha(x) = \tau(1 \otimes x)$ and $\beta(f) = \tau(f \otimes 1) = f \otimes 1$. Therefore the two maps $x \mapsto \tau(1 \otimes x)$ and $f \mapsto \sigma\tau(f \otimes 1)$ are coactions as required. We therefore just have to check that the matching conditions are fulfilled. Denote the coproduct on $\ell^\infty(\Gamma)$ by Δ_1 and choose $f \in \ell^\infty(\Gamma)$ such that $\Delta_1(f) \in \ell^\infty(\Gamma) \odot \ell^\infty(\Gamma)$. Then

$$\begin{aligned}\tau_{(23)}\sigma_{(23)}(\beta \otimes 1)\Delta_1 f &= \tau_{(23)}\sigma_{(23)}(\beta \otimes 1)(f_{(1)} \otimes f_{(2)}) \\ &= \tau_{(23)}\sigma_{(23)}(f_{(1)} \otimes 1 \otimes f_{(2)}) \\ &= \tau_{(23)}(f_{(1)} \otimes f_{(2)} \otimes 1) \\ &= f_{(1)} \otimes f_{(2)} \otimes 1 \\ &= (\Delta_1 \otimes 1)\beta(f),\end{aligned}$$

and hence (M2) is satisfied. An analogous, but slightly more cumbersome, calculation proves that (M1) is also satisfied. □

Thus, as von Neumann algebras, we have $\ell^\infty(\Gamma) \ltimes_\alpha M = \Gamma \ltimes_\rho M$. Using the fact that β is trivial, one can prove that the elements $\lambda_\gamma \otimes 1$ are group-like and it therefore follows from [DC79, 3.3] that also the comultiplications agree. Hence the two crossed product constructions are identical as l.c. quantum groups. In particular, the the bicrossed product $\ell^\infty(\Gamma) \ltimes_\alpha M$ is a Kac algebra so if (M, Δ) is compact then $\ell^\infty(\Gamma) \ltimes_\alpha M$ is also compact ([VV03, 2.7]) and the Haar state is tracial. We therefore have the following.

Proposition 7.4. *If $\mathbb{G} = (M, \Delta, S, h)$ is a compact, coamenable Kac algebra and Γ is a countable, discrete, amenable group acting on $\mathbb{G}$ then the crossed product $\Gamma \ltimes M$ is again a compact, coamenable Kac algebra.*

Proof. That $\Gamma \ltimes M$ is a Kac algebra follows from the discussion above and the coamenability of the crossed product follows from [DQV02, 15] since $\widehat{\ell^\infty(\Gamma)} = \mathscr{L}(\Gamma)$ is coamenable if (and only if) Γ is amenable. □

Remark 7.5. *It is also possibly to construct examples using the more general notion of cocycle crossed products introduced in* [VV03, 2.1]*. It is shown in* [DQV02, 13] *that weak amenability (i.e. the existence of an invariant mean) is preserved under cocycle bicrossed products. In general it is not known whether or not weak amenability is equivalent to strong amenability, the latter being defined as the dual quantum group being coamenable in the sense of Definition 4.1. But for discrete quantum groups this equivalence has been proven by Tomatsu in* [Tom06] *(and also by Blanchard and Vaes in unpublished work). Therefore, if (M_1, M_2) is a cocycle matched pair of l.c. quantum groups with both $\hat{M}_1$ and M_2 compact and coamenable, then the cocycle crossed product is also compact and coamenable.*

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