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The minimum spanning strong subdigraph problem is fixed parameter tractable[☆]

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Abstract

A digraph D is strong if it contains a directed path from x to y for every choice of vertices x, y in D. We consider the problem (MSSS) of finding the minimum number of arcs in a spanning strong subdigraph of a strong digraph. It is easy to see that every strong digraph D on n vertices contains a spanning strong subdigraph on at most $2n-2$ arcs. By reformulating the MSSS problem into the equivalent problem of finding the largest positive integer $k \leq n-2$ so that D contains a spanning strong subdigraph with at most $2n-2-k$ arcs, we obtain a problem which we prove is fixed parameter tractable. Namely, we prove that there exists an $\mathcal{O}(f(k)n^c)$ algorithm for deciding whether a given strong digraph D on n vertices contains a spanning strong subdigraph with at most $2n-2-k$ arcs.

We furthermore prove that if $k \geq 1$ and D has no cut vertex then it has a kernel of order at most $(2k-1)^2$. We finally discuss related problems and conjectures.
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1. Introduction

The *Minimum Spanning Strong Subgraph (MSSS) problem* is the following problem. Given a strong digraph D find a strong spanning subdigraph D' of D such that D' has as few arcs as possible. This problem, which generalizes the hamiltonian cycle problem and hence is *NP*-hard, is of practical interest and has been considered several times in the literature, see e.g. [1,4–6,10,11,15,16,18,19]. The MSSS problem is an essential subproblem of the so-called *minimum equivalent digraph problem*. Here one is seeking a spanning subdigraph with as few arcs as possible in which the reachability relation is the same as in the original digraph (i.e. there is a path from x to y if and only if the original digraph has such a path).

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Khuller, Raghavachari and Young [15,16] gave a 1.62-approximation algorithm for the size of a minimum spanning strong subdigraph of any strong digraph. This was improved to 3/2 by Vetta [19] and this is currently the best approximation rate for general digraphs.

In view of the *NP*-hardness of the MSSS problem, it makes sense to study the problem in the framework of parameterized complexity. We recall some basic notions of parameterized complexity here, for a more in-depth treatment of the topic we refer the reader to [8,9,12,13,17].

A parameterized problem Π can be considered as a set of pairs (I, k) where I is the *problem instance* and k (usually an integer) is the *parameter*. Π is called *fixed parameter tractable (FPT)* if membership of (I, k) in Π can be decided in time $O(f(k)|I|^c)$, where $|I|$ is the size of I, $f(k)$ is a computable function, and c is a constant independent from k and I. Let Π and Π' be parameterized problems with parameters k and k', respectively. An *fpt-reduction R from Π to Π'* is a many-to-one transformation from Π to Π', such that (i) $(I, k) \in \Pi$ if and only if $(I', k') \in \Pi'$ with $|k'| \leq g(k)$ for a fixed computable function g and (ii) R is of complexity $O(f(k)|I|^c)$. A *reduction to a problem kernel* (or *kernelization*) is a polynomial time fpt-reduction R from a parameterized problem Π to itself such that $|I'| \leq h(k)$ for a fixed computable function h. In kernelization, an instance (I, k) is reduced to another instance (I', k'), which is called the *problem kernel*; $|I'|$ is the *size* of the kernel.

Generally one tries to develop kernelizations that yield problem kernels of small size. The survey of Guo and Niedermeier [14] on kernelization lists some problem for which linear size kernels (the size here is the number of vertices), polynomial size kernels and exponential size kernels were obtained. For many parameterized problems, optimal size kernels have probably not been obtained yet; for example, Guo and Niedermeier [14] ask whether the feedback vertex set problem admits a linear size kernel.

Since the hamiltonian cycle problem is *NP*-complete we cannot hope to find an algorithm of complexity $\mathcal{O}(f(k)n^c)$ for deciding whether a given digraph has a strong spanning subdigraph with at most $n + k$ arcs, unless $P = NP$. The purpose of this paper is to give a natural reformulation of the MSSS problem which allows us to consider fixed parameter tractability questions regarding the MSSS problem. This formulation was first mentioned by the first author at the Dagstuhl seminar "Structure Theory and FPT Algorithmics for Graphs, Digraphs and Hypergraphs" in July 2007, who asked if this problem was fixed parameter tractable (see [7]).

Problem 1.1. Given a strong digraph D on n vertices and a natural number $k \leq n-2$. Does D have a spanning strong subdigraph on at most $2n - 2 - k$ arcs?

We will prove that this problem is indeed fixed parameter tractable and furthermore we will show how to obtain a kernel of size $(2k - 1)^2$ in polynomial time when D is 2-connected (i.e. the underlying graph of D has no cut vertex).

2. Terminology and preliminaries

Notation on digraphs which is not given here is consistent with [3]. Generally a digraph is denoted by $D = (V, A)$ where V is the set of vertices in D and A is the set of arcs. We also use $V(D)$ $(A(D))$ to denote these two sets. An arc from x to y is denoted xy or $x \to y$. We say that x **dominates** y if $xy \in A(D)$. If $X \subseteq V$ then we denote by $D\langle X\rangle$ the subdigraph of D induced by X. The **underlying graph** $UG(D)$ of a digraph $D = (V, A)$ is the graph with vertex set V and edge set $E = \{xy | x \to y \in A \text{ or } y \to x \in A\}$. A **cut vertex** in a digraph $D = (V, A)$ is a vertex $v \in V$ with the property that $UG(D) - v$ is not connected. Let x, y be vertices in a digraph D. An (x, y)-path is a directed path from x to y. Similarly, if $X, Y \subset V(D)$ are disjoint sets of vertices in D, then an (X, Y)-path is a directed (u, v)-path P such that $u \in X$, $v \in Y$ and P has no other vertex in $X \cup Y$.

Definition 2.1. An **ear decomposition** of a strong digraph $D = (V, A)$ is a sequence $\mathcal{E} = \{P_0, P_1, P_2, \ldots, P_t\}$, where P_0 is a cycle and each P_i is a path, or a cycle with the following properties:

(a) P_i and P_j are arc-disjoint when $i \neq j$.
(b) For each $i = 1, \ldots, t$: Let D_i denote the digraph with vertices $\bigcup_{j=0}^{i} V(P_j)$ and arcs $\bigcup_{j=0}^{i} A(P_j)$. If P_i is a cycle, then it has precisely one vertex in common with $V(D_{i-1})$. Otherwise the end vertices of P_i are distinct vertices of $V(D_{i-1})$ and no other vertex of P_i belongs to $V(D_{i-1})$.
(c) $\bigcup_{j=0}^{t} A(P_j) = A$.

Each P_i, $0 \le i \le t$ is called an **ear** of $\mathcal{E}$. The **size** of an ear P_i is the number $|A(P_i)|$ of arcs in the ear. The **number of ears** in $\mathcal{E}$ is the number $t+1$. An ear P_i is **trivial** if $|A(P_i)| = 1$. All other ears are **non-trivial**.

The following lemma follows from the definition above.

Lemma 2.2. *Let D be a strong digraph and let $\mathcal{E} = \{P_0, P_1, P_2, \dots, P_k, a_1, a_2, \dots, a_p\}$ be an ear decomposition of D where P_0 is a cycle, each P_i is a path or a cycle and has length at least 2 and $a_1, a_2, \dots a_p$ are arcs (the trivial ears). Then the digraph induced by $P_0 \cup P_1 \cup P_2 \cup \dots \cup P_k$ is a strong spanning subdigraph of D with $\sum_{i=0}^{k} |A(P_i)|$ arcs.*

Lemma 2.3. *Let D be a strong digraph on n vertices, let $P_0, P_1, P_2, \dots, P_r$ be the non-trivial ears of an ear decomposition $\mathcal{E}$ of D and assume that $|A(P_i)| \ge 3$ for $i = 0, \dots, s$ $(\le r)$. Then D has a strong spanning subdigraph D' on at most $2n - (|V(P_0)| + s)$ arcs.*

Proof. By Lemma 2.2 we can let D' be the strong spanning subdigraph of D formed by the union of $P_0, P_1, \dots, P_r$, which implies that D' has n vertices and $\sum_{i=0}^{k} |A(P_i)|$ arcs. Note that P_0 contributes $|V(P_0)|$ vertices and $|V(P_0)|$ arcs to D' and each P_i, $1 \le i \le r$, contributes $|A(P_i)| - 1$ vertices and $|A(P_i)|$ arcs to D'. This implies that $|A(D')| - |V(D')| = r$.

As $P_1, P_2, \dots, P_s$ all contribute at least two vertices to D' and $P_{s+1}, P_{s+2}, \dots, P_r$ all contribute one vertex to D' we get that $n = |V(D')| \ge |V(P_0)| + 2s + (r - s)$. This implies the following.

$$|A(D')| = n + r \le n + (n - |V(P_0)| - s) \le 2n - (|V(P_0)| + s) \quad \diamond$$

Corollary 2.4. *Every strong digraph D on n vertices has a strong spanning subdigraph with at most $2n-2$ arcs and equality holds only if the longest cycle in D has length 2 in which case $UG(D)$ is a tree.*

Proof. Let $P_0, P_1, P_2, \dots, P_r$ be the non-trivial ears of an ear decomposition $\mathcal{E}$ of D, such that P_0 is a longest cycle in D. Lemma 2.3 implies that D has a strong spanning subdigraph, D', with $|A(D')| \le 2n - |V(P_0)|$. This implies the $2n-2$ bound and we note that we only have equality if the longest cycle in D has length two. As D is strong this only happens if $UG(D)$ is a tree. $\diamond$

3. The main results

We will first consider the case where there is no cut vertex in D. We then consider the general case in Theorem 3.3

Theorem 3.1. *Let D be a strong digraph on $n \ge 3$ vertices with no cut vertex and let $k \le n-2$ be a non-negative integer. In polynomial time in n we can either decide that (D, k) is a 'yes' instance of Problem 1.1 or find a strong digraph D_{ker} such that $|V(D_{\text{ker}})| \le f(k) = (2k-1)^2$ and (D, k) is a 'yes' instance of Problem 1.1 if and only if (D_{ker}, k) is a 'yes' instance of Problem 1.1.*

Proof. Given D and k we proceed as follows. Since D has no cut vertex and $n \ge 3$ it contains a cycle of length at least 3. Let P_0 be such a cycle and add to it a maximal sequence $P_1, P_2, \dots, P_s$ of ears of size at least three (as defined in Definition 2.1, but without ears of size two or one added). Let $X = V(P_0) \cup \dots \cup V(P_s)$.

We claim that $Y = V(D) - X$ is an independent set. Assume this is not true and let uv be an arc of Y so that v dominates some vertex in X. Since we cannot add any new ear of size at least three to X, it follows that every (X, u)-path must pass through v (in fact, v is the first vertex after X on any such path) and thus uv is contained in a strong component W of $D\langle Y \rangle$. Now it follows from the fact that v is not a cut vertex that there is an edge e in $UG(D)$ with precisely one vertex p in W and $p \ne v$. Let q be the other vertex of e. If $q \in X$ then it is easy to see that we can add another ear of size at least three to X, contradicting the maximality of the sequence $P_1, P_2, \dots, P_s$. So $q \in Y$. But now it follows from the fact that D is strong and $q \notin W$ that if $p \to q \in A(D)$ then there is a directed path from q to X which avoids W and if $q \to p \in A(D)$ then there is a directed path from X to q which avoids W and again we find a new ear of size at least three. Thus in both cases we obtain a contradiction to the maximality of $P_1, P_2, \dots, P_s$. Therefore Y is independent.

By Lemma 2.3 D contains a strong spanning subdigraph D' on at most $2n - (|V(P_0)| + s) = 2n - 2 - (|V(P_0)| + s - 2)$ arcs. Therefore we may assume that $|V(P_0)| + s - 2 < k$ (as otherwise (D, k) is a 'yes' instance), which implies that $s < k - 1$, as $|V(P_0)| \geq 3$. Furthermore as Y is independent, we can add the vertices of Y one by one as an ear of size two to X, implying that we can obtain a strong spanning subdigraph D on $2|Y| + |X| + s = 2n - 2 - (|X| - s - 2)$ arcs. As before we may assume that $|X| - s - 2 < k$, which implies that $|X| \leq k + s + 1 \leq 2k - 1$.

We now build a new undirected bipartite graph G as follows. Let $Z = \{z_{uv} : u, v \in X, u \neq v$ and there exists $y \in Y$ so that $u \to y, y \to v \in A(D)\}$ and let $V(G) = Y \cup Z$ and let $E(G)$ consist of all edges yz_{uv} where $y \in Y$, $z_{uv} \in Z$ and $u \to y, y \to v \in A(D)$. Consider the following procedure.

Let $i = 0$, $Z_0 = Z$, $Y_0 = Y$ and denote for all values of i below the graph G_i by $G_i = G\langle Y_i \cup Z_i\rangle$. If G_i has a matching M meeting all vertices in Z_i, then let $M_Y \subseteq Y_i$ denote the set of end vertices of M in Y_i, let M_Z denote the end vertices of M in Z and stop. Otherwise, by Hall's theorem (see e.g. [3, Theorem 3.13]), there is some $Z' \subseteq Z_i$ so that $|N_{G_i}(Z')| < |Z'|$, where $N_{G_i}(Z')$ is the set of neighbours of Z' in G_i (they are all in Y_i). Now let $X_{i+1} = Z'$, $W_{i+1} = N_{G_i}(X_i)$, $Y_{i+1} = Y_i - W_{i+1}$, $Z_{i+1} = Z_i - X_{i+1}$, set $i = i + 1$ and repeat the process above for the new graph G_i.

When the procedure ends (with $i = r \geq 0$) we have generated (possibly empty sequences of) disjoint subsets $X_1, X_2, \ldots, X_r$ of Z, disjoint subsets $W_1, W_2, \ldots, W_r$ of Y and (possibly empty) subsets $M_Y \subseteq Y - (W_1 \cup \cdots \cup W_r)$ and $M_Z = Z - (X_1 \cup \cdots \cup X_t)$. Let $Y' = Y - M_Y - (W_1 \cup \cdots \cup W_r)$.

Note that if $M_Y = \emptyset$ then it follows from the procedure above and the fact that in G every vertex $y \in Y$ is adjacent to at least one vertex of Z (since D is strong and has no cut vertex and Y is independent in D, every vertex in $y \in Y$ has at least one arc to and from distinct vertices in X) that $Y' = \emptyset$ and therefore $|Y| < |Z| \leq (2k - 1)(2k - 2)$. Hence D itself has less than $f(k)$ vertices and we are done. Hence we may assume that $M_Y \neq \emptyset$.

Claim. *There exists a minimum spanning strong subdigraph D^* of D so that $D^* - Y'$ is strong and each vertex in Y' is incident to precisely two arcs in D^*.*

Proof of Claim. Let D' be a minimum strong spanning subdigraph of D. If $D' - Y'$ is strong we are done (clearly D' must add each vertex of Y' using only two arcs each). Otherwise we proceed as follows.

1. Remove all arcs incident to M_Y from D'.
2. Let $M_Z = \{z_{x_1,x_1'}, \ldots, z_{x_{|M|},x_{|M|}'}\}$ and $M_Y = \{y_1, \ldots, y_{|M|}\}$ such that $z_{x_1,x_1'}y_1, \ldots, z_{x_{|M|},x_{|M|}'}y_{|M|}$ form a matching in G. Add the following arcs $x_i \to y_i \to x_i'$, $i = 1, \ldots, |M|$ and let $\tilde{D}$ be the resulting digraph.

Observe that $\tilde{D}$ is strong and has the same number of arcs as D' because of the following. By adding the arcs $x_i \to y_i \to x_i'$, $i = 1, \ldots, |M|$ we provide all possible two step connections between vertices in X which can be made via a vertex in M_Y (recall that, by the definition of X_i and W_i only vertices from M_Z are adjacent to vertices in M_Y in G).

It follows from the fact that no vertex in $Z - M_Z$ is adjacent to Y' that $\tilde{D} - Y'$ is strong (again vertices in Y' can only be used to make connection between those pairs in X which correspond to M_Z in Z and by adding the arcs $x_i \to y_i \to x_i'$, $i = 1, \ldots, |M|$ we have already provided all these in $\tilde{D}$). Thus we have proved the claim. $\diamond$

Clearly $D^* - Y'$ can play the role of D_{ker}, so to complete the proof we only have to observe that the size of $V(D^* - Y')$ is at most $|Z| + |X| \leq (2k - 1)(2k - 2) + (2k - 1) = f(k)$. $\diamond$

The following lemma is trivially true.

Lemma 3.2. *Let D be a strong digraph on n vertices and suppose v is a cut vertex of $UG(D)$ and let $X_1, X_2 \ldots, X_r$ be the connected components of $UG(D) - v$. Every strong spanning subdigraph of D induces a strong spanning digraph of $D\langle X_i \cup \{v\}\rangle$ for each $i = 1, 2, \ldots, r$.*

Theorem 3.3. *Problem 1.1 is fixed parameter tractable with respect to the parameter k given in the formulation of Problem 1.1.*

Proof. Let (D, k) be an instance of Problem 1.1. If D has no cut vertex then the result follows from Theorem 3.1, so let v be a cut vertex of D. Let $X_1, X_2 \ldots, X_r$ be the connected components of $UG(D) - v$ and let $D_i = D\langle X_i \cup \{v\}\rangle$ for all $i = 1, 2, 3, \ldots, r$. Let D_i' be a strong spanning subgraph of D_i, such that $|A(D_i')|$ is minimum and let k_i be defined such that $|A(D'_i)| = 2|V(D'_i)| - 2 - k_i$. Let D' be defined such that $V(D') = V(D)$ and $A(D') = A(D'_1) \cup \cdots \cup A(D'_r)$. Note that D' is strong and that the following holds, where $k' = k_1 + k_2 + \cdots + k_r$.

$$|A(D')| = \sum_{i=1}^{r}(2|V(D'_i)| - 2 - k_i) = \left(2\sum_{i=1}^{r}(|V(D'_i)| - 1)\right) - k' = 2(|V(D)| - 1) - k'.$$

If $k' \geq k$ then (D, k) is a 'yes' instance of Problem 1.1, so we may assume that $k' < k$. Assume that the ordering $X_1, X_2, \ldots, X_r$ is such that $|X_i| \geq |X_j|$ whenever $i < j$ and let q be defined such that $|V(D_i)| \geq 3$ if and only if $i \leq q$. Note that for all j, with $q < j \leq r$ we have that D_j is a 2-cycle and $k_j = 0$. Let $\mathcal{D} = \{D_1, D_2, \ldots, D_q\}$.

Assume that some D_i $(1 \leq i \leq q)$ has a cut vertex u. Assume that $Y_1, Y_2, \ldots, Y_s$ are the connected components of $UG(D_i) - u$. Now remove D_i from $\mathcal{D}$ and add all the digraphs $D_i\langle Y_j \cup \{u\}\rangle$ with at least 3 vertices. Continue this process until there is no digraph in $\mathcal{D}$ with a cut vertex. As each digraph, D^*, in $\mathcal{D}$ has no cut vertex and order at least three we must have a strong spanning subgraph of D' with at most $2|V(D^*)| - 3$ arcs by Corollary 2.4. By the argument above we note that if there is more than k digraphs in $\mathcal{D}$ then (D, k) is a 'yes' instance. Therefore there is at most k digraphs in $\mathcal{D}$ and for each one we can find a kernel of size at most $(2k - 1)^2$. We can therefore solve the problems to optimality for each digraph in $\mathcal{D}$ (using any algorithm) in time $O(g(k))$ for some function g, which implies that to decide if (D, k) is a 'yes' instance or not is FPT. $\diamond$

Note that the algorithm implicitly given in Theorem 3.3 can be made more efficient, but our main purpose is to prove that Problem 1.1 is fixed parameter tractable, which we have done.

4. Remarks and open problems

An **out-branching** (**in-branching**) of a digraph D is a spanning subdigraph in which all but one vertex (called the **root**) have in-degree (out-degree) exactly one.

Note that a digraph D is strong if and only if it contains an out-branching and an in-branching rooted at some vertex v. Hence the following holds.

Lemma 4.1. *Let D be a strong digraph on n vertices, let $v \in V$ be arbitrary and let $k \leq n - 2$ be a natural number. There exists a strong spanning subdigraph of D with at most $2n - 2 - k$ arcs if and only if D contains a in-branching F_v^- with root v and an out-branching F_v^+ with the same root v so that $|A(F_v^-) \cap A(F_v^+)| \geq k$.* $\diamond$

It is also worth noting that given any out-branching F_v^+ we can find an in-branching F_v^- maximizing $|A(F_v^-) \cap A(F_v^+)|$ in polynomial time using any polynomial algorithm for finding a minimum cost in-branching in a digraph [3, Section 9.10].

Theorem 4.2 (*[2]*). *It is NP-complete to decide whether a digraph contains arc-disjoint branchings F_v^-, F_v^+ rooted at the same vertex v, where F_v^- (F_v^+) is an in-branching (out-branching).*

Similarly to the approach used in this paper for the MSSS problem, one can consider the following parametrized version of the arc-disjoint in- and out-branching problem above. Note that no pair F_v^-, F_v^+ in a digraph on n vertices can share more than $n - 2$ arcs.

Problem 4.3. Given a strong digraph D on n vertices and a natural number $k \leq n - 2$. Does D contain in- and out-branchings F_v^-, F_v^+, rooted at the same vertex v, so that $|A(F_v^-) \cap A(F_v^+)| \leq n - 2 - k$?

Problem 4.4. Does there exist an $\mathcal{O}(f(k)n^c)$ algorithm for Problem 4.3?

By **path-contracting** the arc xy in a digraph $D = (V, A)$ we mean the operation that deletes x, y and all their incident arcs and then adds a new vertex z together with the all arcs of the kind $u \to z$ where $u \to x \in A(D)$ and all arcs of the kind $z \to v$ where $y \to v \in A(D)$ [3, Section 5.1.1]. The resulting digraph is denoted $D//xy$.

Note that if a digraph D is not strong, then $D//a$ will be non-strong no matter which arc $a \in A$ we path-contract. However, a strong digraph D may have no arc a such that $D//a$ is strong. This holds e.g. for every strong digraph which is obtained from an undirected tree by replacing each edge by a directed 2-cycle.

By **path-contracting a sequence of arcs** $(a_1, a_2, \ldots, a_r)$ in a digraph D we mean the following process. Let $D_0 = D$ and for $i = 1, 2, \ldots, r$ let $D_i = D_{i-1}//a_i$, where a_i is an arc of D_{i-1}. In particular $a_1 \in A(D)$. Observe that a digraph D on n vertices has a hamiltonian cycle if and only if we can find a sequence $(a_1, a_2, \ldots, a_{n-2})$ of arcs so that path-contracting $(a_1, a_2, \ldots, a_{n-2})$ we are left with a directed cycle of length 2.

Problem 4.5. Given a strong digraph D on n vertices and a positive integer $k \leq n - 2$. Does there exists a sequence of arcs $(a_1, a_2, \ldots, a_k)$ of length k in D so that path-contracting this sequence leaves a strong digraph?

Problem 4.6. Does there exist an $\mathcal{O}(f(k)n^c)$ algorithm for Problem 4.5?

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