Cormen Chapter 5

5.1 The hiring problem

Setup:

- there are n candidates for a job
- you interview them one by one
- after interviewing the i'th candidate you must decide immediately whether you will hire person i
- If you had previously hired a candidate j, $j < i$ and want to hire candidate i you must fire candidate j again

Goal: hire the best candidate while spending as little money on the process

- We can ignore the cost of interviews as we must interview all candidates.
- assume it costs c_h to hire a person (no matter whether we later fire the person again)
- Then we want to minimize $m \cdot c_h$ when m is the # of persons we hire

Worst case: We hire all so cost $n \cdot c_h$

This will happen if $q_1 < q_2 < \cdots < q_n$ when q_i is the quality of the i'th candidate

How do we avoid this?

Randomize the order in which we interview the candidates

Worst case still the same but what about the expected cost?

Let $a_1, a_2, \ldots, a_n$ be a random permutation of the candidates

$$X_i = \begin{cases} 1 & \text{if we hire candidate } i \\ 0 & \text{else} \end{cases}$$

$$X = \sum_{i=1}^{n} X_i$$ is the number of candidates we hire

cost of process is $X \cdot c_h$

When will we hire candidate i?

If this is the best we have seen so far so $P(X_i = 1) = \frac{1}{i}$

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right)$$

$$= \sum_{i=1}^{n} E(X_i)$$

$$= \sum_{i=1}^{n} p(X_i = 1)$$

$$= \sum_{i=1}^{n} \frac{1}{i}$$

$$= \ln n + O(1)$$

So $E(X) \in O(\ln n)$

New problem:

How to construct a random permutation?

Cormen 5.3

Permute by sorting(A)

1. $n \leftarrow |A|$
2. For $i \leftarrow 1$ to n do
3. $\quad P(i) \leftarrow \text{Random}(1, n^3)$
4. sort A using P as keys
5. return A

Lemma $p\left(P(i) \neq P(j) \ \forall\ 1 \le i < j \le n\right) \ge 1 - \frac{1}{n}$

P: A_{ij} is the event that $P(i) = P(j)$

$$p(A_{ij}) = \frac{1}{n^3}$$

$$A = \bigcup_{1 \le i < j \le n} A_{ij}$$

$$p(A) = p\left(\bigcup_{1 \le i < j \le n} A_{ij}\right) \le \sum_{1 \le i < j \le n} p(A_{ij})$$

$$= \sum_{1 \le i < j \le n} \frac{1}{n^3} = \frac{\binom{n}{2}}{n^3} < \frac{1}{n}$$

Lemma 5.4 Permute by sorting produces a uniform random permutation when all the priorities $P(1), P(2), \ldots, P(n)$ are distinct.

P: suppose first that $P(1) < P(2) < \cdots < P(n)$

We want to show that the permutation $1\,2\,3 \ldots n$ occurs with probability $\frac{1}{n!}$

For $i = 1, 2, \ldots, n$ let X_i be the event that $A[i]$ receives the i'th smallest P-value

Then we seek $p(X_1 \cap X_2 \cap \ldots \cap X_n)$

$$p(X_1 \cap X_2 \cap \cdots \cap X_n) = p(X_1)\, p(X_2 | X_1) \cdot p(X_3 | X_1 \cap X_2) \cdots \cdots p(X_n | X_1 \cap \cdots \cap X_{n-1})$$

$$p(X_1) = \frac{1}{n}, \quad p(X_2 | X_1) = \frac{1}{n-1}$$

$$p(X_{i+1} | X_1 \cap \cdots \cap X_i) = \frac{1}{n-i}$$

Hence

$$p(X_1 \cap \dots \cap X_n) = p(X_1) \cdot p(X_2 | X_1) \cdot \dots \cdot p(X_n | X_1 \cap \dots \cap X_{n-1})$$

$$= \frac{1}{n} \cdot \frac{1}{n-1} \cdot \quad \dots \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{n!}$$

Same argument works for an arbitrary permutation $\sigma(1)\sigma(2)\dots\sigma(n)$ of $1, 2 \dots n$:

- The element with index $\sigma(1)$ has probability $\frac{1}{n}$ of receiving the lowest P-value
- The element with index $\sigma(i)$ has probability $\frac{1}{n-i+1}$ of receiving the i'th lowest P-value

etc $\square$.

Permute by sorting runs in time $O(n \log n)$ but only works if all P-values are distinct otherwise we must repeat the algorithm

Expected # of repetitions is at most

$$\frac{1}{1 - \frac{1}{n}} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1} \leq 2$$

Can we do better?

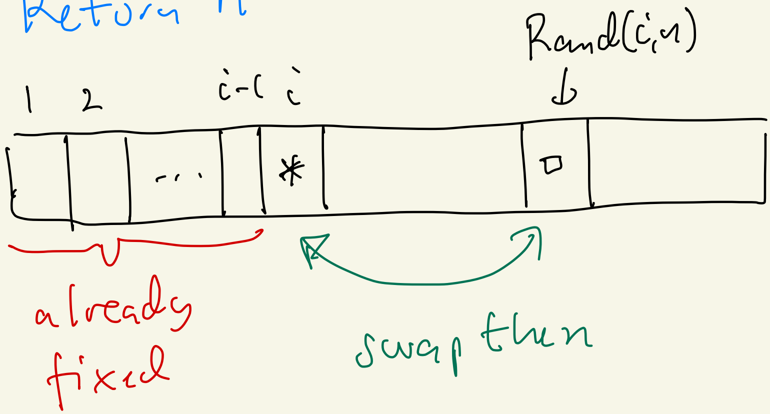
Randomize in place(A)

$n \leftarrow |A|$

For $i \leftarrow 1$ to n do

swap $A[i]$ with $A[\text{Random}(i,n)]$

Return A



Definition A k-permutation of an n-set A is any ordered k-subset of A

Then are $P(n,k) = \frac{n!}{(n-k)!}$ of the

Lemma 5.5 Randomize in place produces a random permutation

P: We prove the following loop invariant

(I) just before iteration i of the for loop every $(i-1)$-permutation of A has probability

$\frac{(n-i+1)!}{n!}$ of being the elements in

$A[1,\dots i-1]$ in that order.

We start with $i=1$ and the only possible 0-permutation is $\emptyset$ and it has probability 1 of forming the first 0 elements of A

Can also take $i=2$ as base case as every element has probability $\frac{1}{n}$ of landing in position 1 of A

Maintaining (I):

(*) assume that after iteration $i-1$ each $(i-1)$-permutation appears as the elements in $A[1,2,..,i-1]$ in that order with probability $\frac{(n-i+1)!}{n!}$

Consider a particular i-permutation $\langle x_1 x_2 \cdots x_i \rangle$ and define events E_1 and E_2.

E_1: $A[1,2,..,i-1] = \langle x_1 x_2 \cdots x_{i-1} \rangle$ after $i-1$ steps

E_2: in ith iteration $A[i]$ becomes equal to x_i

By (*) we get $p(E_1) = \frac{(n-i+1)!}{n!}$ and

$p(E_2 | E_1) = \frac{1}{n-i+1}$ (there are $n-i+1$ choices for $A[i]$)

So $p(E_1 \cap E_2) = p(E_2|E_1) \cdot p(E_1)$

$$= \frac{(n-i+1)!}{n!} \cdot \frac{1}{n-i+1} = \frac{(n-i)!}{n!}$$

so (I) holds after i steps if it holds after $i-1$ steps

Termination $i = n+1$

(I) $\Rightarrow$ $A[1, 2, \ldots n] = \langle x_1, x_2 \ldots x_n \rangle$ for fixed permutation $\langle x_1, x_2 \ldots x_n \rangle$ with probability

$$\frac{(n-n)!}{n!} = \frac{0!}{n!} = \frac{1}{n!} \qquad \square.$$