Institut for Matematik og Datalogi
Syddansk Universitet

September 5, 2023
JBJ

DM551/MM851 – Fall 2023 – Weekly Note 2

Stuff covered in week 36
Rosen 6.1-6.5.

Lecture September 12

- Last part of Rosen 6.5 (if any remains)
- Rosen 7.1-7.2

Exercises in Week 37

Note that you have two exercise sections in Week 37

Remember to try to solve as many exercises as possible. It is important that you practice on the different ways of counting so that you can use these skills later in the course.

- Left over exercises from Week 36.
- Section 6.4: 9,16,24,26,28,32,34
- Section 6.5: 3,6,12,14,20,32,52
- Discuss the proofs of Theorem 4 on page 443.
- Read Section 6.6 in the book and be ready to discuss how one can produce all k-permutations and all k-combinations of an n-set efficiently.

1 Notes on Combinatorial proofs

The purpose of this part of Weekly note 2 is to demonstrate some proofs. A subset X of a set S is **even** (resp. **odd**) if $|X|$, the number of elements of X, is even (odd).

Theorem If S is a finite set with at least one element, then the number of even subsets of S is the same as the number of odd subsets of S.

Proof: We will give two different proofs. First note that the theorem does not hold for the empty set, so we must require that S is not empty. Let E_S be the number of even subsets of S and let O_S be the number of odd subsets of S. Finally let $n = |S|$.

1. There are exactly $\binom{n}{k}$ ways of choosing a set with k elements from S. Recall the binomial formula $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$. If we insert $x = 1$ and $y = -1$ in this formula we get $0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$. Now the theorem follows just by observing that, in the sum above, every even subset contributes with 1 and every odd subset contributes with -1.

2. The theorem clearly holds when $n = 1$, so consider a set S with $n > 1$ elements. Fix an element $s \in S$ and let $S' = S \setminus \{s\}$. Let e_s, o_s denote the number of even, respectively odd subsets of S that contain s. Clearly every even (odd) subset of S that contains s consists of s plus an odd (even) subset of S'. Hence we have $e_s = O_{S'}$ and $o_s = E_{S'}$. Finally observe that, by the sum rule, the number of even (odd) subsets of S equals the number of even (odd) subsets that contain s plus the number of even (odd) subsets of S that do not contain s. The later are subsets of S'. Thus we have
$$E_S = e_s + E_{S'} = O_{S'} + E_{S'} = O_{S'} + o_s = O_S$$

We now give another example of the usefulness of combinatorial arguments. For given natural numbers k, n we let $\mathcal{S}_{n,k} = \{(n_1, n_2, \ldots, n_k) | n_i \geq 0 \text{ and } n_1 + n_2 + \ldots + n_k = n\}$. Note that $\mathcal{S}_{n,k}$ is the set of all ordered k-tuples of non negative numbers for which the sum of the elements in the tuple is n. From Rosen Section 6.5.3 we know that there are $\binom{n+(k-1)}{n}$ of these.

Theorem

$$\sum_{(n_1,n_2,\ldots,n_k)\in\mathcal{S}_{n,k}} \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!} = k^n$$

Proof: We claim that both sides of the equality sign count the number of ways to distribute n distinct balls in k distinct boxes. This is easy to see for the right side: we have k choices for each of the n balls, so k^n in total. Now let us show that the left side also counts the number of ways to distribute n distinct balls in k distinct boxes. By Rosen Theorem 4 page 452 we have that for fixed $n_1, n_2, \ldots n_k$ such that $\sum_{i=1}^{k} n_i = n$ there are $\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$ ways to distribute the n distinct balls into boxes $1, 2, \ldots, k$ so that n_i balls are placed in box i. Now we see that the left hand side counts the number of ways to distribute n distinct balls in k distinct boxes by counting, for each of the $\binom{n+(k-1)}{n}$ possible choices $(n_1, n_2, \ldots, n_k)$ of numbers of elements to put in each of the k boxes, the number of ways to distribute the balls when we must put exactly n_i balls in box i for $i \in [n]$.
You can verify for yourself that when $k = 2$ the lefthand side is the same as $\sum_{r=0}^{n} \binom{n}{r}$ so the formula is a generalization of Corollary 1 page 439 in Rosen.