A New Constructive Method for the One-Letter Context-Free Grammars

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***Abstract*— Constructive methods for obtaining the regular grammar counterparts for some sub-classes of the context free grammars (`cfg`) have been investigated by many researchers. An important class of grammars for which this is always possible is the one-letter `cfg`. We show in this paper a new constructive method for transforming arbitrary one-letter `cfg` to an equivalent regular expression of star-height 0 or 1. Our new result is considerably simpler than a previous construction by Leiss, and we also propose a new normal form for a regular expression with single-star occurrence. Through an alphabet factorization theorem, we show how to go beyond the one-letter `cfg` in a straight-forward way.**

***Index Terms*— reduction of a context-free grammar, one-letter context-free language, regular expression**

I. INTRODUCTION

The subclass of one-letter alphabet languages has been studied for many years. The result "*Each context-free one-letter language is regular*" was first proven in [13] and re-published in [14] using Parikh mappings. A second method based on the "pumping" lemma for the context-free languages (`cfl`'s) was presented in [10]. Salomaa ([15]) used the *systems of equations* (based on $\cup$, $\cdot$ and $*$ operators) to prove that the star-height of every one-letter alphabet language is equal to 0 or 1. Later, Leiss ([12]) gave the first constructive method by developing a theory of language equations over a one-letter alphabet. Several key theorems were proven and tied together to provide an algorithm which solves any equation of that type. In this paper, we shall present a new simpler method using only a single result, called the *Regularization Theorem*, with the help of a new normal form for one-letter equations.

Like [1], [7], [10], we will use systems of equations to denote `cfg`'s. It is known that for an arbitrary `cfg`, it is undecidable whether its least fixed point can be expressed as a regular expression, in general ([5]). We define a new normal form for the one-letter equations and a new theorem for solving them. Algorithm **A** (Section III) will use this normal form to determine precisely the least fixed point, as an equivalent regular expression. By considering the classes of *one-letter/one-variable factorizable*, we enlarge slightly the class of `cfg`'s for which the construction of a regular expression remains decidable.

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II. PRELIMINARIES

We suppose the reader is familiar with the basic notions of the formal language theory, but some important notions are briefly covered here.

A **context-free grammar** is denoted as $G = (V_N, V_T, S, P)$, where V_N/V_T are the alphabets of variables/terminals, ($V = V_N \cup V_T$ is the alphabet of all symbols of G), S is the start symbol and $P \subseteq V_N \times V^*$ is the set of productions. The productions $X \to \alpha_1$, $X \to \alpha_2$, ..., $X \to \alpha_k$ will be denoted by $X \to \alpha_1 \,|\, \alpha_2 \,|\, ... \,|\, \alpha_k$ and the right-hand side of X is denoted by $\mathtt{rhs}(X)$, that is $\{\alpha_1, \alpha_2, ..., \alpha_k\}$. A variable X is a **self-embedded variable** in G if there exists a derivation $X \overset{*}{\underset{G}{\Longrightarrow}} \alpha X \beta$, where α, $\beta \in V^+$ ([6]). G is a **self-embedded grammar** if there exists a self-embedded variable. G is a **reduced grammar** if $\forall\ X \in V$, $S \overset{*}{\underset{G}{\Longrightarrow}} \alpha\, X\, \beta$ and $\forall\ X \in V_N$, $X \overset{*}{\underset{G}{\Longrightarrow}} u$, with $u \in V_T^*$. The empty word is denoted by ϵ. A `cfg` is **proper** if it has no ϵ-productions (i.e. $X \to \epsilon$, $X \in V_N$) and no chain-productions (i.e. $X \to Y$, $X, Y \in V_N$). It is known that for every `cfg` (which doesn't generates ϵ) there exists an equivalent proper `cfg`.

The set of the terminal words attached to the variable X of the grammar G is $L_G(X) = \{w \in V_T^* \mid \exists\ X \overset{+}{\underset{G}{\Longrightarrow}} w\}$ ($\overset{m}{\underset{G}{\Longrightarrow}}$ ($\overset{+}{\underset{G}{\Longrightarrow}}$) means that m (at least one) productions have been applied in G). The set of **sentential forms** of X in G is $S_G(X) = \{\alpha \in V^* \mid \exists\ X \overset{*}{\underset{G}{\Longrightarrow}} \alpha\}$, the set of **sentential forms** of G is $S(G) = S_G(S)$. The **language** of G is $L(G) = S(G) \cap V_T^* = L_G(S)$. All the above sets can be easily extended to words, i.e. $L_G(\alpha) = \{\alpha \in V_T^* \mid \exists\ \alpha \overset{+}{\underset{G}{\Longrightarrow}} w\}$, a.s.o.

A **permutation** with n elements is an one-to-one correspondence from $\{1, ..., n\}$ to $\{1, ..., n\}$, the set of all permutations with n elements is denoted by Π_n. $\mathbf{N}$ denotes the set of natural numbers; $\overline{1,n}$ denotes the set $\{1, ..., n\}$, $i, j \in \overline{1,n}$ denotes $i \in \overline{1,n}$, $j \in \overline{1,n}$.

We continue by providing some results related to the *system of equations* ([1]). The systems of equations are extremely concise for modeling `cfl`'s ([7], [10]). The notions of substitution, solution, and equivalence can be found in [1], [11].

Definition 2.1: Let $G = (\{X_1, ..., X_n\}, V_T, X_1, P)$ be a `cfg`. A **system of (X_i-)equations** over G is a vector $\mathcal{P} = (\mathcal{P}_1, ..., \mathcal{P}_n)$ of subsets of V^*, usually written as: $X_i = \mathcal{P}_i,\ \forall\ i \in \overline{1,n}$, with $\mathcal{P}_i = \{\alpha \in V^* \mid X_i \to \alpha \in P\}$.

The next classical result gives one method for computing the minimal solution of a system of equations by derivations ([1]).

Theorem 2.1: Let $G = (\{X_1, ..., X_n\}, V_T, X_1, P)$ be a cfg. Then the vector $L_G = (L_G(X_1), ..., L_G(X_n))$ is the least solution of the associated system.

The next theorem refers to a well known transformation which "eliminates" X from a linear $X-$equation ([3], [15], [11]). From now on, unless specified otherwise, we will use notations $\alpha = \alpha_1 + ... + \alpha_m$, $\beta = \beta_1 + ... + \beta_n$, where m, and $n \in \mathbf{N}$. We shall use $X \notin \beta$ to mean $X \notin \beta_j, \forall\ j \in \overline{1,n}$.

Theorem 2.2: Let $X = \alpha\, X + \beta$ be an $X-$equation, where $X \notin \alpha$, and $X \notin \beta$. The least solution is $X = \alpha^* \beta$, and if $\epsilon \notin \alpha$, then this is unique.

III. One-letter CFG and its Regular Construction

In this section, we shall give a new constructive method for regularizing one-letter cfg's that is more concise and general than the method proposed by Leiss ([12]). The commutativity plays an important role for transforming the one-letter cfg's and this is covered in the following lemma.

Lemma 3.1: Let $G = (V_N, \{a\}, S, P)$ be a one-letter cfg. The set of all **commutative grammars** of G is $\mathcal{G}_{com}(G) = \{(V_N, \{a\}, S, P_{com}),$ where $P_{com} = \{X \rightarrow \alpha_{\pi(1)} ... \alpha_{\pi(k)} \mid X \rightarrow \alpha_1 ... \alpha_k \in P,\ \pi \in \Pi_k\}$. Then for every $G_{com} \in \mathcal{G}_{com}(G)$, it follows $L(G) = L(G_{com})$.

Proof It can be easily proved by induction on l, $l \geq 1$, that for any $X \in V_N$, we have: (1) $X \overset{l}{\underset{G}{\Longrightarrow}} a^n$ iff $X \overset{l}{\underset{G_{com}}{\Longrightarrow}} a^n$. Complete proof can be found in [2]. ∎

Lemma 3.2 allows the symbols of any sentential form to be re-ordered in an one-letter cfg. Its proof is similar to Lemma 3.1.

Lemma 3.2: Let $G = (V_N, \{a\}, S, P)$ be an one-letter cfg and let us consider the derivation $\alpha_1 ... \alpha_k \overset{*}{\underset{G}{\Longrightarrow}} a^n$. For any $\pi \in \Pi_k$, we have $\alpha_{\pi(1)} ... \alpha_{\pi(k)} \overset{*}{\underset{G}{\Longrightarrow}} a^n$.

The next lemma shows how star-operations are flattened in the one-letter cfg's.

Lemma 3.3: Let $G = (V_N, \{a\}, S, P)$ be an one-letter cfg and α_1, ..., α_n some words over $V_N \cup \{a\}$. Then the following properties hold: $L_G((\alpha_1 + ... + \alpha_n)^*) = L_G(\alpha_1^* ... \alpha_n^*) = L_G((\alpha_1^* ... \alpha_n^*)^*)$, $L_G((\alpha_1 \alpha_2^* ... \alpha_n^*)^*) = \epsilon + L_G(\alpha_1 \alpha_1^* \alpha_2^* ... \alpha_n^*)$.

Proof Focusing to the first equality, we have to prove that: $(\alpha_1 + ... + \alpha_n)^* \overset{*}{\underset{G}{\Longrightarrow}} a^m$ iff $\alpha_1^* ... \alpha_n^* \overset{*}{\underset{G}{\Longrightarrow}} a^m$. Based on Lemma 3.2, the words α_1, ..., α_n can be commuted in any order. We proceed by induction on n. First, let us suppose that $n = 2$. The inclusion $L_G((\alpha_1 + \alpha_2)^*) \supseteq L_G(\alpha_1^* \alpha_2^*)$ is obvious. For the other inclusion, let us take $\beta = (\alpha_1 + \alpha_2)^n$, $n \geq 0$. It can be rewritten $\beta = \alpha_1^{n_1}\, \alpha_2^{n_2} ... \alpha_1^{n_k - 1}\, \alpha_2^{n_k}$, where $n_i \in \overline{0,n}$, $\forall\ i \in \overline{1,k}$, and $\sum_{i=1}^{k} n_i = n$. Using $\alpha_1 \alpha_2 = \alpha_2 \alpha_1$ applied several times, we get $\beta = \alpha_1^{n_1 + ... + n_{k-1}}\, \alpha_2^{n_2 + ... + n_k}$. So $L(\beta) \subseteq L_G(\alpha_1^*\, \alpha_2^*)$, therefore $L_G((\alpha_1 + \alpha_2)^*) = L_G(\alpha_1^* \alpha_2^*)$.

Now, we suppose true for $n = m > 2$ and prove it for $n = m + 1$. We have $L_G((\alpha_1 + ... + \alpha_m + \alpha_{m+1})^*) = L_G(((\alpha_1 + ... + \alpha_m) + \alpha_{m+1})^*) = L_G((\alpha_1 + ... + \alpha_m)^* \alpha_{m+1}^*) = L_G((\alpha_1 + ... + \alpha_m)^*) \cdot L_G(\alpha_{m+1}^*) = L_G(\alpha_1^* ... \alpha_m^*) \cdot L_G(\alpha_{m+1}^*) = L_G(\alpha_1^* ... \alpha_m^* \alpha_{m+1}^*)$.

For the other identities, we shall use some equations for regular expressions from [15]: $(\alpha^*)^* = \alpha^*$ and $(\alpha\, \beta^*)^* = \epsilon + \alpha(\alpha + \beta)^*$. Therefore $L_G((\alpha_1^* ... \alpha_n^*)^*) = L_G(((\alpha_1 + ... + \alpha_n)^*)^*) = L_G((\alpha_1 + ... + \alpha_n)^*) = L_G(\alpha_1^* ... \alpha_n^*)$ and $L_G((\alpha_1 \alpha_2^* ... \alpha_n^*)^*) = L_G(((\alpha_1 (\alpha_2 + ... + \alpha_n)^*)^*) = L_G(\epsilon + \alpha_1 (\alpha_1 + ... + \alpha_n)^*) = \epsilon + L_G(\alpha_1 \alpha_1^* \alpha_2^* ... \alpha_n^*)$. ∎

Definition 3.1: We say that the equation $X = \mathcal{P}$ is in the **one-letter normal form** (abbreviated by **OLNF**) if $\mathcal{P} = \alpha\, X + \beta$, where $X \notin \beta$.

Theorem 3.1: Let $G = (\{X_1, ..., X_n\}, \{a\}, X_1, P)$ be an one-letter reduced cfg. Then every attached X_i-equation can be transformed into OLNF.

Proof Let $X_i = \alpha\, X_i + \beta$ be an arbitrary X_i-equation. Because G is reduced, it follows that $\beta \neq \emptyset$, otherwise there will be no terminal word in $L_G(X_i)$. Based on Lemma 3.2, it follows that the symbols of α can be commuted in $\mathcal{P}_i$ in such a way that X_i will be on the last position. Next, by distributivity ($\gamma_1 \cdot X_i + \gamma_2 \cdot X_i = (\gamma_1 + \gamma_2) \cdot X_i$), it is obvious that every X_i-equation can be transformed in this form. The only possible term of $\mathcal{P}_i$ for which X_i cannot be commuted until the last position is $\alpha' (\beta'\, X_i)^*$. In that case, $\alpha' (\beta'\, X_i)^*$ will be rewritten into $\alpha' (\epsilon + (\beta'\, X_i)^* (\beta'\, X_i)) = \alpha' + \alpha'\, \beta' (\beta'\, X_i)^* X_i$. Now, if $X_i \notin \alpha'$ then the X_i-equation is in OLNF, otherwise the transformation will continue and stop after a finite number of steps. ∎

By doing this transformation together with the (flattening) Lemma 3.3, Theorem 3.2 can be viewed as a generalization of Leiss's results (consisting of Theorems 3.1, 4.1, and 4.2 from [12]).

The next theorem is a tool for eliminating the occurrences of the variable X in a rhs of an $X-$equation. This is a generalization of Theorem 2.2, and a key ingredient of Algorithm **A**. Let us denote by $\alpha[\beta/X]$ the word obtained by replacing every $X-$occurrence in α with β. Of course, the substitution is valid iff X does not occur in β.

Theorem 3.2: (Regularization) Let $G = (V_N, \{a\}, S, P)$ be an one-letter reduced cfg, $X \in V_N$ and $X = \alpha\, X + \beta$ be an OLNF $X-$equation. Then, the least solution of the $X-$equation is $X = (\alpha[\beta/X])^* \beta$, and if G is proper, then this solution is unique.

Proof Before starting the proof, let us refer to the uniqueness of the solution. Because G is proper, it follows that G has no ϵ-productions and chain-productions, so $\epsilon \notin \alpha$, and $\epsilon \notin \beta$. Similarly to Theorem 2.2, it easily follows that the solution of the $X-$equation is unique. Without loss of generality, by applying finitely many times Lemmas 3.2 and 3.3, we suppose that α can be viewed as a regular expression over $V_N \cup \{a\}$ of star-height 0 or 1. So, its general form is $\alpha = \sum_{i=1}^{t} \alpha_{0,i} (\alpha_{1,i} X^{k_{1,i}})^* ... (\alpha_{m,i} X^{k_{m,i}})^*$. For simplicity, let us focus on $(\alpha_{1,i} X^{k_{1,i}})^*$. Based on commutativity, $(\alpha_{1,i} X^{k_{1,i}})^* = \{(\alpha_{1,i} X^{k_{1,i}})^{n_{1,i}} \mid n_{1,i} \geq 0\} = \{\alpha_{1,i}^{n_{1,i}} X^{k_{1,i} \cdot n_{1,i}} \mid n_{1,i} \geq 0\}$. Hence, $\alpha = \sum_{i=1}^{t} \alpha_{0,i} (\alpha_{1,i}^{n_{1,i}} X^{k_{1,i} \cdot n_{1,i}}) ... (\alpha_{m,i}^{n_{m,i}} X^{k_{m,i} \cdot n_{m,i}})$ $= \sum_{i=1}^{t} \alpha_{0,i} \alpha_{1,i}^{n_{1,i}} ... \alpha_{m,i}^{n_{m,i}} X^{k_{1,i} \cdot n_{1,i} + ... + k_{m,i} \cdot n_{m,i}}$. This will be denoted by $\alpha = \sum_{i=1}^{t} \alpha'_i X^{Q_i}$, where α'_i are words over $(V_N - \{X\}) \cup \{a\}$ and Q_i are (linear) polynomials in variables $n_{j,i} \in \mathbf{N}$, ($k_{j,i} \in \mathbf{N}$ are constants).

Therefore, the initial $X-$equation becomes $X = (\sum_{i=1}^{t} \alpha'_i X^{Q_i})X + \beta$, which corresponds to the following $X-$productions in G: $X \rightarrow \alpha'_1 X^{Q_1} X \mid ... \mid \alpha'_t X^{Q_t} X \mid \beta_1 \mid ... \mid \beta_n$. Because $X \notin \alpha'_i$, $\forall\ i \in \overline{1,t}$, and $X \notin \beta_j$, $\forall\ j \in \overline{1,n}$, it follows that $S_G(X)$ can be generated by applying several times (e.g. $s-$times) productions of the form $X \rightarrow \alpha'_i X^{Q_i} X$, $i \in \overline{1,t}$, followed by productions of the form $X \rightarrow \beta_j$, $j \in \overline{1,n}$ in order to remove all the occurrences of X. According to Lemma 3.2, we can re-order the symbols in any sentential form, and thus apply the current X-production to the last occurrence of the variable X, so we get the general $X-$derivations: $X \underset{G}{\overset{s}{\Longrightarrow}} \alpha'_{i_1} ... \alpha'_{i_s} X^{Q_{i_1}} ... X^{Q_{i_s}} X$, where i_1, ..., $i_s \in \overline{1,t}$. After applying $\underline{Q_{i_1}} + ... + Q_{i_s} + 1$ productions of type $X \rightarrow \beta_j$, $j \in \overline{1,n}$, we obtain the words $\alpha'_{i_1} ... \alpha'_{i_s} \beta_{j_{1,1}} ... \beta_{j_{1,Q_{i,1}}} ... \beta_{j_{s,1}} ... \beta_{j_{s,Q_{i,s}}} \beta_j$. According to Lemma 3.2, $L_G(\alpha'_{i_1} .. \alpha'_{i_s} \beta_{j_{1,1}} .. \beta_{j_{1,Q_{i,1}}} .. \beta_{j_{s,1}} .. \beta_{j_{s,Q_{i,s}}} \beta_j) = L_G(\alpha_{i_1} \beta_{j_{1,1}} .. \beta_{j_{1,Q_{i,1}}} .. \alpha_{i_s} \beta_{j_{s,1}} .. \beta_{j_{s,Q_{i,s}}} \beta_j)$. Because the words $\alpha_{i_1} \beta_{j_{1,1}} ... \beta_{j_{1,Q_{i,1}}} ... \alpha_{i_s} \beta_{j_{s,1}} ... \beta_{j_{s,Q_{i,s}}} \beta_j$ correspond to $(\alpha[\beta/X])^* \beta$, then it follows that the solution of the $X-$equation is $X = (\alpha[\beta/X])^* \beta$. ■

Algorithm **A** is based on the representation of the one-letter `cfg` as a system of equations. Then this system of equations is solved in order to obtain an equivalent regular expression. As we assume reduced `cfg`, each recursive $X-$equation must have at least one term without any occurrence of X.

Algorithm A

Input: $G = (\{X_1, ..., X_n\}, \{a\}, X_1, P)$ a reduced and proper one-letter `cfg`
Output: $L_G = (L_G(X_1), ..., L_G(X_n))$, and $L_G(X_i)$ is regular, $\forall\ i \in \overline{1,n}$
Method:
1. Construct $X_i = \mathcal{P}_i$, $\forall\ i \in \overline{1,n}$ as in Definition 2.1;
2. **for** $i := 1$ **to** n **do begin**
3. Transform X_i-equation into OLNF
4. $\mathcal{P}_i = (\alpha[\beta/X_i])^* \beta$;
5. Apply Lemma 3.3 to obtain the star-height 0 or 1 for $\mathcal{P}_i$
6. **for** $j := i+1$ **to** n **do** $\mathcal{P}_j = \mathcal{P}_j[\mathcal{P}_i/X_i]$;
endfor
7. **for** $i := n-1$ **downto** 1 **do**
8. **for** $j := n$ **downto** $i+1$ **do begin**
9. $\mathcal{P}_i = \mathcal{P}_i[\mathcal{P}_j/X_j]$;
10. Apply Lemma 3.3 to obtain the star-height 0 or 1 for $\mathcal{P}_i$
endfor
11. $L_G = (X_1, ..., X_n)$

Theorem 3.3: Algorithm **A** is correct and performs a finite number of steps.

Proof The lines 1, 11 are due to Definition 2.1 and Theorem 2.1, respectively. The instructions between lines 3-5 are based on Theorem 3.2 and Lemma 3.3 and imply that $\forall\ i \in \overline{1,n}$, $\mathcal{P}_i$ doesn't contain X_i. Line 6 ensures that $\forall\ i \in \overline{1,n}$, $\mathcal{P}_i$ doesn't contain any X_j with $j < i$. The occurrences of X_j from $\mathcal{P}_i$, where $j > i$ are replaced with terminal words at the lines 7-10. After the execution of Algorithm **A**, $\mathcal{P}_i$ is a regular expression over $\{a\}$ of star-height 0 or 1, thus $L_G(X_i)$ is regular, $\forall\ i \in \overline{1,n}$. By induction on i, it can be easily proved that according to Lemma 3.3, $\mathcal{P}_i$ has the star-height 0 or 1. ■

As a remark, due to the nested **for** instructions (2-6 and 7-10), if we suppose that the steps 3-6 and 9-10 require constant time in n, then the time-complexity of Algorithm **A** is $\mathcal{O}(n^2)$.

Example 3.1: Let us consider $G = (\{X_1, X_2\}, \{a\}, X_1, P)$ with P given by the following productions: $X_1 \rightarrow a\, X_1\, X_2 \mid a$, $X_2 \rightarrow X_1\, X_2 \mid a\, a$. Line 1 of Algorithm **A** will construct the system: $X_1 = a\, X_1\, X_2 + a$, $X_2 = X_1\, X_2 + a^2$. After executing line 4, we get $X_1 = (a\, X_2)^*\, a$, and after line 6, we obtain $X_2 = a\,(a\, X_2)^*\, X_2 + a^2$. At the next iteration, Algorithm **A** will provide $X_2 = (a\,(a^3)^*)^* a^2$, and after line 5, $X_2 = a^2 + a^3 \cdot a^*(a^3)^*$. At line 9, it follows $X_1 = a(a^3 + a^4 \cdot a^*(a^3)^*)^*$, and after line 10, $X_1 = (a^3)^* \cdot (a + a^5 \cdot a^* \cdot (a^3)^* \cdot (a^4)^*)$. ■

As a remark, in Algorithm **A** the order of eliminating X_i can be arbitrary. For instance, by eliminating X_2, followed by X_1, we get the equivalent simpler expressions: $X_1 = a + a^4 \cdot a^*$ and $X_2 = a^2 \cdot a^*$. We shall next show that every factor of the one-letter regular expression can be reduced to only one occurrence of $*$.

Definition 3.2: We say that $e = e_1 + ... + e_n$ (where each e_i contains only $\cdot$ and $*$ operators) is in **single-star normal form** iff $\forall\ i \in \overline{1,n}$, e_i has at most one occurrence of $*$.

This normalization is captured in the following theorem.

Theorem 3.4: Every regular expression over an one-letter alphabet can be transformed into an equivalent single-star normal form.

Proof If e is a regular expression of the star-height 1 (the case 0 is trivial) then it can be written as $e = e_1 + ... + e_n$, where $\forall\ i \in \overline{1,n}$, $e_i = a^{m_{0,i}} (a^{m_{1,i}})^* ... (a^{m_{k_i,i}})^*$, where $m_{1,i} < ... < m_{k_i,i}$. We suppose, without loss of generality, that the cases $m_{s,i} = m_{s+1,i}$ are excluded based on the property $\alpha^* \alpha^* = \alpha^*$. Let $G(a_1, ..., a_k)$ be the greatest number b such that the Diophantine equation $a_1\, x_1 + ... + a_k\, x_k = b$ has no solution in $\mathbf{N}$, where the greatest common divisor of $a_1, ..., a_k$ is 1 (notation $gcd(a_1, ..., a_k) = 1$). This means that for any $b > G(a_1, ..., a_k)$ the equation $a_1\, x_1 + ... + a_k\, x_k = b$ has always solution in $\mathbf{N}$. Let us denote by $F(a_1, ..., a_k)$ the set of all natural numbers less than $G(a_1, ..., a_k)$ such that the above equation has solution in $\mathbf{N}$. According to [8], if $a_1 < ... < a_k$ and $gcd(a_1, ..., a_k) = 1$, then $G(a_1, ..., a_k) \leq (a_k - 1)(a_1 - 1)$.

Denoting $d = gcd(m_{1,i}, ..., m_{k_i,i})$, due to the above Diophantine equation, it follows that e_i can be equivalently transformed into $a^{m_{0,i}} \cdot \left(\epsilon + a^{d \cdot n_1} + ... + a^{d \cdot n_s} + (a^d)^{(\frac{m_{k,i}}{d} - 1)(\frac{m_{1,i}}{d} - 1) + 1} \cdot (a^d)^*\right)$, where n_1, ..., $n_s \in F(\frac{m_{1,i}}{d}, ..., \frac{m_{k,i}}{d})$. In this way, each factor e_i of e has at most one star, so e is in single-star normal form. ■

A particular case of the above theorem is to reduce the expression $(a^m)^* \cdot (a^n)^*$ for which $m \equiv 0 (mod\, n)$. So, $gcd(m, n) = m$, hence by Theorem 3.4, it follows that $(a^m)^* \cdot (a^n)^* = \epsilon + (a^m) \cdot (a^m)^* = (a^m)^*$. Considering the `cfg` from Example 3.1, we can reduce $X_1 = a \cdot (a^3)^* + a^5 \cdot a^*$ and $X_2 = a^2 + a^3 \cdot a^*$.

Example 3.2: For instance, the following regular expressions of star-height 1 are reduced to the single-star normal form: $(a^2)^*(a^3)^* = \epsilon + a^2a^*$, $(a^4)^*(a^6)^* = \epsilon + a^4(a^2)^*$ and $(a^4)^*(a^6)^*(a^9)^* = \epsilon + a^4 + a^6 + a^8 + a^9 + a^{10} + a^{12} \cdot a^*$. ■

Our main result, based on Theorem 3.2, is considerably simpler and more general than the constructive method given by Leiss [12]. Firstly, we needed only a single (more general) theorem to facilitate the construction of an equivalent regular expression for an arbitrary one-letter `cfg`. Secondly, the substitution of all the $X-$occurrences by β is done in one step, as opposed to multiple steps used by Leiss's procedure. We now explore a straight-forward way to go beyond one-letter `cfg`'s through the use of alphabet factorisation.

IV. Beyond one-letter CFG's

As is well-known, the non self-embedded variables/`cfg`'s are easily converted to the regular sublanguages. Theorem 4.1 (proven in [2]) shows that any `cfg`, G, generates a regular language if all its self-embedded variables can be shown to generate regular languages.

Theorem 4.1: Let G be an arbitrary reduced and proper `cfg`. If for all self-embedded variables X the language $L_G(X)$ is regular, then $L(G)$ is regular.

In the following, we shall combine the property of an one-letter alphabet, together with self-embeddedness, in order to obtain a more powerful class of `cfg`'s which generates regular languages.

Definition 4.1: A `cfg` $G = (V_N, V_T, S, P)$ is called **one-letter factorizable** iff for every self-embedded variable X, $L_G(X) \subseteq \{a\}^*$, where $a \in V_T$.

In other words, if G is one-letter factorizable, then every self-embedded variable has the corresponding language defined over (only) one-letter alphabet.

Now, the notion of one-variable factorizable will be introduced. This notion is somehow *dual* to one-letter factorizable, by considering at most one occurrence of a variable A_i in $\mathtt{rhs}(X_i)$.

Definition 4.2: We say that $G = (V_N^1 \cup V_N^2, V_T, X_1, P)$ where $V_N^1 = \{X_1, ..., X_n\}$ and $V_N^2 = \{A_1, ..., A_n\}$ ($V_N^1 \cap V_N^2 = \emptyset$) is **one-variable factorizable** iff for every self-embedded variable X_i the $\mathtt{rhs}(X_i) \subseteq \{X_i, A_i\}^*$ and $\mathtt{rhs}(A_i) \subseteq V_T^*$.

Theorem 4.2: (Factorization) The following facts hold:

(a) An one-letter factorizable `cfg` generates a regular language.

(b) An one-variable factorizable `cfg` generates a regular language.

Proof (a) Let $G = (V_N, V_T, S, P)$ be a one-letter factorizable `cfg`. For every self-embedded variable $X \in V_N$, we know that $L_G(X) \subseteq \{a\}^*$. So due to Theorem 3.3, it follows that $L_G(X)$ is regular. Applying Theorem 4.1, it follows that $L(G)$ is regular.

(b) Let $G = (V_N^1 \cup V_N^2, V_T, X_1, P)$ be a one-variable factorizable `cfg`, where $V_N^1 = \{X_1, ..., X_n\}$, $V_N^2 = \{A_1, ..., A_n\}$ ($V_N^1 \cap V_N^2 = \emptyset$) and for every self-embedded variable X_i the $\mathtt{rhs}(X_i) \subseteq \{X_i, A_i\}^*$ and $\mathtt{rhs}(A_i) \subseteq V_T^*$.

Let us construct the `cfg` $G' = (V_N^1, V_N^2 \cup V_T, X_1, P')$, where $P' = P - \{A_i \rightarrow w \mid A_i \in V_N^2\}$. Because for every self-embedded variable X_i the $\mathtt{rhs}(X_i) \subseteq \{X_i, A_i\}^*$, it follows that $L_{G'}(X_i) \subseteq \{A_i\}^*$. Hence $L_{G'}(X_i)$ is an one-letter language, so based on Algorithm **A**, it results that $L_{G'}(X_i)$ is a regular language. By applying Theorem 4.1, it follows that $L(G')$ is regular.

Now, let us consider the substitution $\sigma : V_N^2 \cup V_T \rightarrow V_T^*$, such that $\sigma(A_i) = \{\mathtt{rhs}(A_i)\}$, $\forall\ i \in \overline{1,n}$ and $\sigma(a) = a$, $\forall\ a \in V_T$. Because $\{\mathtt{rhs}(A_i)\}$ is a finite set of words, it follows that σ is a regular substitution. Obviously, $L(G) = \sigma(L(G'))$ and according to closure of the regular languages under the regular substitutions, it results that $L(G)$ is regular. ■

Example 4.1: Let $G = (\{S, A, B\}, \{a, b, c\}, S, P)$ be a `cfg` with the following set of productions P: $S \rightarrow A\,B\,S \mid c$, $A \rightarrow a\,A\,a\,a\,A\,a \mid a$, $B \rightarrow b\,B\,B \mid b\,b\,b$. The set of the self-embedded variables is $\{A, B\}$, and $L_G(A) \subseteq \{a\}^*$, $L_G(B) \subseteq \{b\}^*$, so G is one-letter factorizable. Based on Algorithm **A**, we get $L_G(A) = \{(a^5)^{n_1}\,a \mid n_1 \geq 0\}$ and $L_G(B) = \{(b^4)^{n_2}\,b^3 \mid n_2 \geq 0\}$. Now, $L_G(S) = (L_G(A) \cdot L_G(B))^* \cdot c = \{((a^5)^{n_1}\,a\,(b^4)^{n_2}\,b^3)^{n_3}\,c \mid n_1, n_2, n_3 \geq 0\}$, so $L(G) = L_G(S)$ is regular. ■

Example 4.2: Let $G = (\{S, A\}, \{\mathtt{(}, \mathtt{)}\}, S, P)$ be a `cfg` with productions P given by $S \rightarrow S\,S \mid A\,S\,A \mid \epsilon$, and $A \rightarrow \mathtt{(} \mid \mathtt{)}$. Obviously, by Definition 4.2, G is one-variable factorizable. Similarly to the proof of Theorem 4.2, we get the equation $S = (S + A^2)S + \epsilon$. Now based on Algorithm **A**, it results that $S = (A^2)^*$, so according to the $A-$productions, we get the regular language $L(G) = \{\{\mathtt{(}, \mathtt{)}\}^2\}^*$. ■

V. Concluding Remarks

We summarize and compare some previous work on one-letter alphabet language. The class of one-letter alphabet languages were used by considering pushdown automata, whose memory consists of one-letter language. Boasson ([4]) called this kind of pushdown automata *counters* and the accepted language *one-counter* language. He proved that the family of one-counter languages is a proper subfamily of `cfl`'s.

The class of one-letter alphabet languages can be handled by considering finite-state automata. In [8], the problem of converting the (one-way) nondeterministic and two-way deterministic finite-state automata is hard to simulate by (one-way) deterministic finite-state automata, even for only one-letter alphabet languages. He proved that $\mathcal{O}(e^{\sqrt{n \log n}})$ states are sufficient to simulate an $n-$state (one-way) nondeterministic finite automaton recognizing a one-letter language by a (one-way) deterministic finite automaton.

The class of one-letter alphabet languages was covered in [9], where an efficient conversion from a finite-state automaton over one-letter alphabet to a context-free grammar in Chomsky normal form was proposed. The authors of [9] showed that any $n-$states one-letter deterministic finite automata can be simulated by a Chomsky normal form grammar with $\mathcal{O}(n^{2/3})$ variables, respectively the non-deterministic automata requires $\mathcal{O}(n^{1/3})$ variables. In our paper, Algorithm **A** takes in its input an one-letter reduced and proper `cfg` and provide the equivalent regular expression in single-star normal form.

The one-letter languages have been used recently in [16] for the decomposition of finite languages.

Our work has advanced the frontier of research in one-letter `cfg`'s by providing a much simpler constructive method for transforming into regular expressions using one-letter normal form. We also introduced a factorization result that enabled us to go beyond one-letter languages in a straight-forward way. This helps to enlarge the class of `cfg`'s that could be regularized.

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