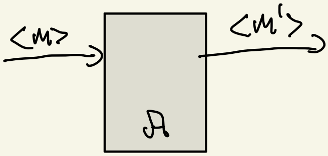
Sipser Section 5.1

Halting problem: $HALT = \{\langle M\rangle\langle w\rangle \mid M \text{ is a TM and } M \text{ stops on } w\}$

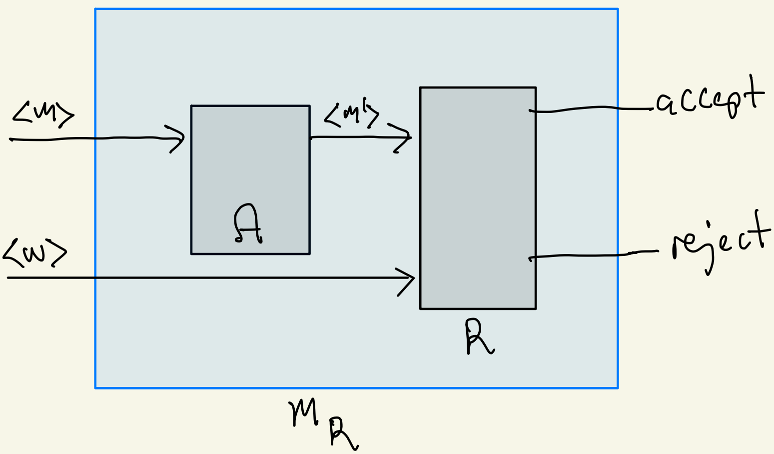
Theorem 5.1 HALT is undecidable

P: suppose the TM R decides HALT

We show that this would imply that A_{TM} is decidable



A: if $\langle M\rangle$ is not a TM
then $\langle M'\rangle = \langle M\rangle$
Else $\langle M'\rangle$ is TM that
has $L(M') = L(M)$ and
s.t M' loops on all strings
not in $L(M')$



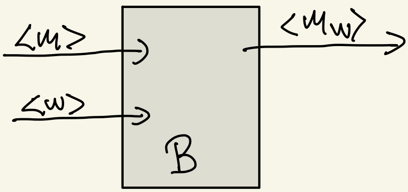
M_R

M_R is a decider for A_{TM} $\}$

$E_{TM} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

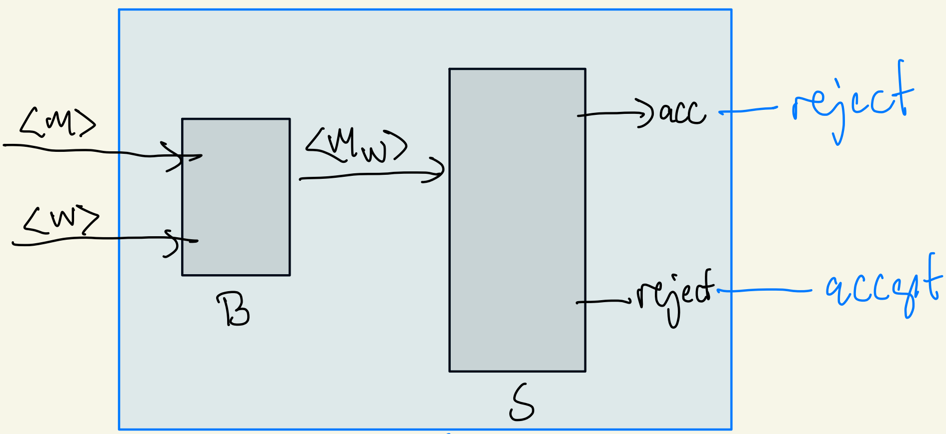
Theorem 5.2 E_{TM} is undecidable

P: suppose S is a TM which decides E_{TM}. We will use S to construct a decider for A_{TM} and reach a contradiction.



B: check whether $\langle M\rangle$ codes a TM. If not output M_w a TM with $L(M_w) = \emptyset$
If $\langle M\rangle$ is a TM output TM M_w with $L(M_w) = \begin{cases} \{w\} & \text{if } w \in L(M) \\ \emptyset & \text{if } w \notin L(M) \end{cases}$

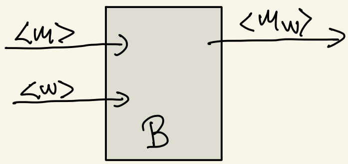
M_w: 1. check if input x equals w
2. If $x \neq w$ reject
3. Else simulate M on w and accept w iff M accepts w



S^*

S^* decides A_{TM} ↯

Consider again the TM B:



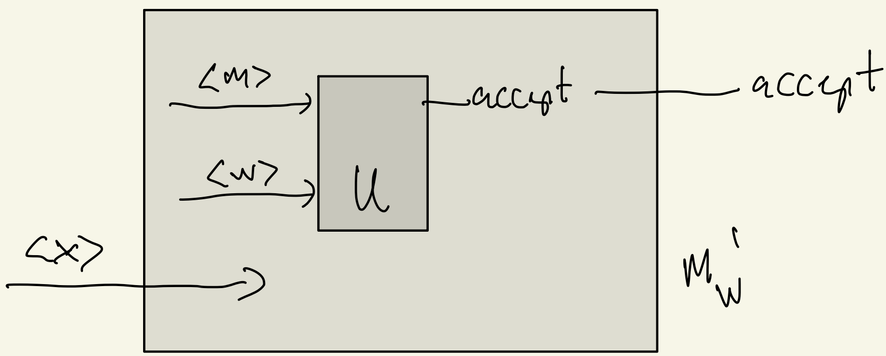
We made it such that the output $\langle M_w \rangle$ satisfies

$$L(M_w) = \begin{cases} \emptyset & \text{if } w \notin L(M) \\ \{w\} & \text{if } w \in L(M) \end{cases}$$

The construction works no matter what $L(M_w)$ is when $w \in L(M)$ as long as we have

$$L(M_w) \neq \emptyset \iff w \in L(M)$$

because this is what we need to be able to use S to decide A_{TM}



$$L(M'_w) = \begin{cases} \emptyset & \text{if } w \notin L(M) \\ \Sigma^* & \text{if } w \in L(M) \end{cases}$$

Remark $\overline{E}_{TM}$ is Turing recognizable

$$\overline{E}_{TM} = \{ \langle w \rangle \mid \text{Either } \langle w \rangle \text{ is not a TM or } \langle W \rangle = \langle M \rangle \text{ for some TM with } L(M) \neq \emptyset \}$$

Here is how to recognize $\overline{E}_{TM}$:

1. check whether $\langle w \rangle$ codes a TM
   if not accept $\langle w \rangle$
2. let M be TM coded by $\langle w \rangle$
3. simulate M on strings over Σ^* (input alphabet of M)
   in lex order in 'parallel':

   For $i = 1, 2, \dots$
      simulate M for i steps on strings $w_1, w_2, \dots w_i$
      according to the lex order $w_1, w_2, w_3 \dots$
      stop once a string is accepted

Note that if $w \in L(M)$ then the algorithm above will stop after at most p steps where $p = \max\{t, q\}$ and M accepts w in t steps and $w = w_q$

Corollary E_{TM} is not Turing recognizable

Two special Turing machines

$M_\emptyset$: goes directly to its reject state no matter what the input is (including empty input)

so $\delta(q_0, a) = q_{reject} \quad \forall a \in \Gamma$

$$L(M_\emptyset) = \emptyset$$

M_{Σ^*} : goes directly to its accept state no matter what the input is

$\delta(q_0, a) = q_{accept} \quad \forall a \in \Gamma$

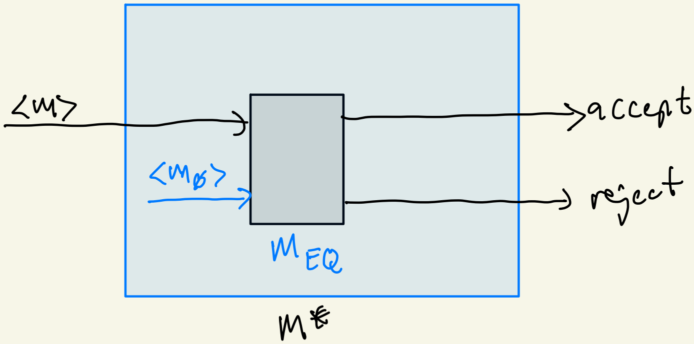
$$L(M_{\Sigma^*}) = \Sigma^*$$

$EQ_{TM} = \{\langle M_1\rangle\langle M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs s.t. } L(M_1) = L(M_2)\}$

<u>Theorem</u> EQ_{TM} is undecidable

P: We show how to 'reduce' E_{TM} to EQ_{TM}

Suppose M_{EQ} decides EQ_{TM}, then we can make the TM M^*:



M^* is a decider for E_{TM} ☹

Can we formalize the idea of using a hypothetical TM X to solve a known undecidable problem, hence proving that X cannot exist?

Mapping reducibility (Sipser 5.3)

Definition 5.17

$f : \Sigma^* \to \Sigma^*$ is computable if $\exists$ TM M_f which started with w ends with $f(w)$ on its tape : $q_0 w \overset{*}{\Rightarrow} q_{acc} f(w)$

Note that M_f always stops!

Example 1. $f : \mathbb{N} \to \mathbb{N}$ $\quad f(x) = x^2$

2. M_f takes as input a string $\langle w \rangle$ and

a. check whether $\langle w \rangle$ is a TM description

b. If yes M_f changes $\langle w \rangle = \langle M \rangle$ to $\langle M' \rangle$ when M' loops on all strings that M does not accept

c. If not M_f outputs the string $\langle w \rangle$

So M_f calculates the function

$$f : \langle w \rangle \to \langle w' \rangle$$

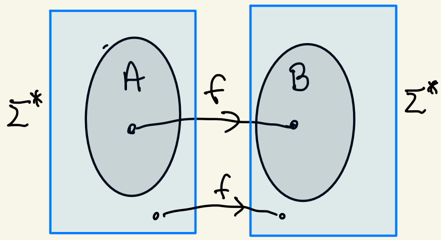
when $\langle w' \rangle$ is the output from M_f

Note M_f is the same as the TM that we used to prove HALT is undecidable.

Definition 5.20

Let A, B be languages. We say that A is mapping reducible to B if $\exists$ a computable function $f : \Sigma^* \to \Sigma^*$ s.t. $w \in A \iff f(w) \in B$

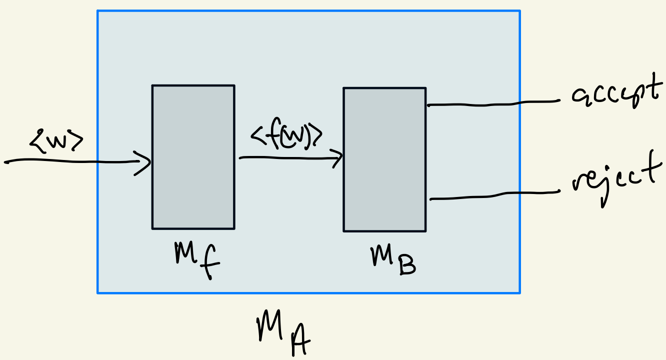
We write $A \leq_m B$ and call f a reduction of A to B



Example $\langle M \rangle \xrightarrow{f} \langle M \rangle \langle M_\emptyset \rangle$

is a mapping reduction from E_{TM} to EQ_{TM}

Theorem 5.22 If $A \leq_m B$ and B is decidable, then A is decidable



If M_B exists and $A \leq_m B$ then M_A decides A

Corollary 5.23 $A \le_m B$ and A undecidable

$\Downarrow$

B is undecidable

Theorem 5.28 $A \le_M B$ and B recognizable

$\Downarrow$

A is recognizable

Corollary 5.29 $A \le_M B$ and A not recognizable

$\Downarrow$

B is not recognizable

Recall that $\overline{E_{TM}} = \{ \langle w \rangle \mid$ either w is not a TM or $\langle w \rangle = \langle M \rangle$ for a TM with $L(M) \neq \emptyset \}$

We have seen that $\overline{E_{TM}}$ is recognizable

Claim $A_{TM} \le_m \overline{E_{TM}}$

Given $\langle M \rangle \langle w \rangle$ we construct $\langle M'_w \rangle$ s.t. M'_w is a TM and

$$L(M'_w) = \begin{cases} \emptyset & \text{if } M \text{ is not a TM or } M \text{ is a TM but } w \notin L(M) \\ \Sigma^* & \text{otherwise} \quad (w \in L(M)) \end{cases}$$

Now $\langle M \rangle \langle w \rangle \in A_{TM} \iff \langle M'_w \rangle \in \overline{E_{TM}}$

Clearly $\langle M \rangle \langle w \rangle \xrightarrow{f} \langle M'_w \rangle$ is computable

(we can store codes of TM's $M_\emptyset$ and M_{Σ^*} in the TM which computes f)

Remark $A \le_m B \iff \overline{A} \le_m \overline{B}$

Let f be such that $w \in A \iff f(w) \in B$

then $w \in \overline{A} \iff w \notin A \iff f(w) \notin B \iff f(w) \in \overline{B}$

so $w \in \overline{A} \iff f(w) \in \overline{B}$

Theorem 5.30 None of EQ_{TM} and $\overline{EQ_{TM}}$ are recognizable

Proof: We know that $\overline{A_{TM}}$ is not recognizable

so it suffices to prove that $\overline{A_{TM}} \le_m EQ_{TM}$ and $\overline{A_{TM}} \le_m \overline{EQ_{TM}}$

By the remark this the same as showing

$A_{TM} \le_m \overline{EQ_{TM}}$ and $A_{TM} \le_m EQ_{TM}$

$A_{TM} \le_m \overline{EQ_{TM}}$: Given $\langle M \rangle \langle w \rangle$ construct $\hat{M}_w$ s.t

$$L(\hat{M}_w) = \begin{cases} \emptyset & \text{if } w \notin L(M) \text{ or } \langle M \rangle \text{ not a TM} \\ \Sigma^* & \text{if } w \in L(M) \end{cases}$$

Then $\langle M \rangle \langle w \rangle \in A_{TM} \iff \langle \hat{M}_w \rangle \langle M_\emptyset \rangle \in \overline{EQ_{TM}}$

$A_{TM} \le EQ_{TM}$: Given $\langle M \rangle \langle w \rangle$ construct $\hat{M}_w$ as above

Then $\langle M \rangle \langle w \rangle \in A_{TM} \iff \langle \hat{M}_w \rangle \langle M_{\Sigma^*} \rangle \in EQ_{TM}$