More undecidable problems and Rice's theorem

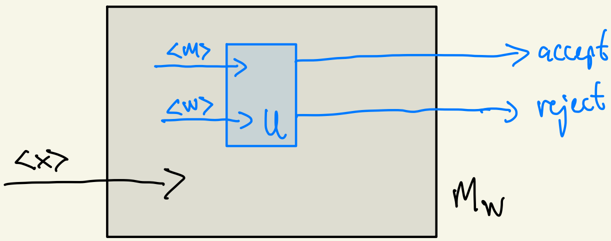
Based on Sipser 5.1 and JBJ notes on homepage

$H_\varepsilon = \{ \langle M \rangle \mid M \text{ is a TM and } \varepsilon \in L(M) \}$

__Theorem__ H_ε is undecidable

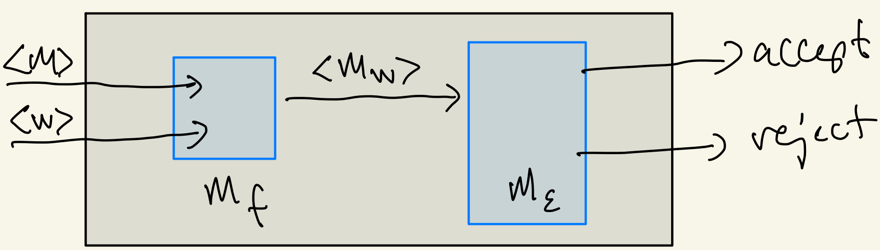
P: We give a mapping reduction from A_{TM} to H_ε

$$\langle M \rangle \langle w \rangle \xrightarrow{f} \langle M_w \rangle \qquad L(M_w) = \begin{cases} \emptyset & \text{if } w \notin L(M) \\ & \text{or } M \text{ is not a TM} \\ \Sigma^* & \text{if } w \in L(M) \end{cases}$$



So $A_{TM} \leq_m H_\varepsilon$ (via f above)

and A_{TM} is undecidable Hence H_ε is undecidable



M_ε cannot exist as otherwise $\hat{M}$ decides A_{TM}

$$\text{Regular}_{TM} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is regular}\}$$

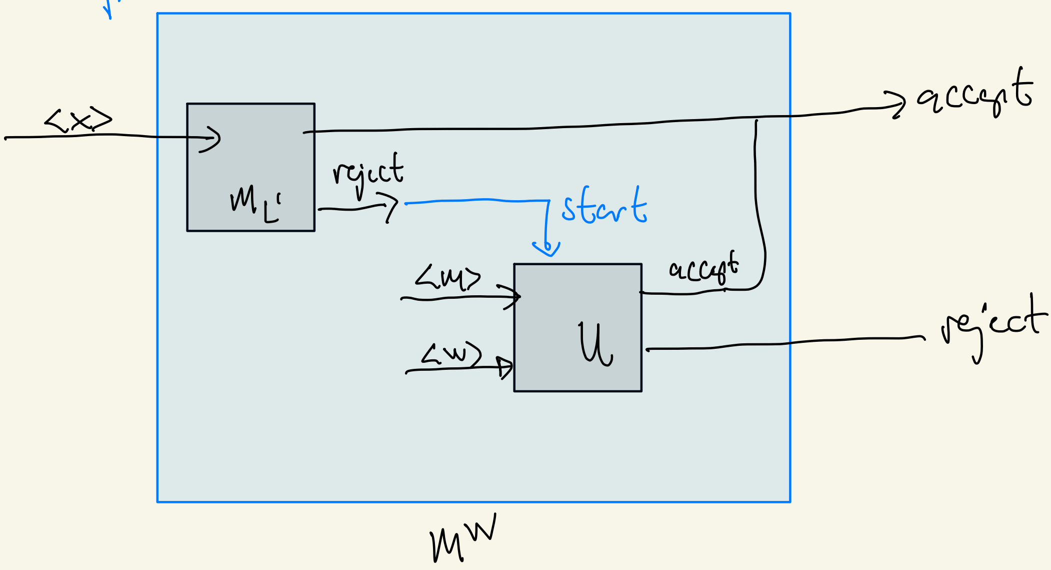
__Theorem 5.3__ Regular_{TM} is undecidable

P: We give a mapping reduction from A_{TM} to Regular_{TM}, by showing how to construct a TM M^w s.t.

$$L(M^w) = \begin{cases} L' = \{a^n b^n \mid n \geq 0\} & \text{if } w \notin L(M) \text{ or } \langle M\rangle \text{ not a TM} \\ \Sigma^* & \text{if } w \in L(M) \end{cases}$$

__Easy:__ $L(M^w)$ is regular if and only if $\langle M\rangle\langle w\rangle \in A_{TM}$

So we need to show that a TM can construct M^w from $\langle M\rangle\langle w\rangle$



Given $\langle M\rangle\langle w\rangle$ and a

TM a decider $M_{L'}$ with $L(M_{L'}) = \{a^n b^n \mid n \geq 0\}$

we can construct M^w with a TM

So $\langle M\rangle\langle w\rangle \longrightarrow \langle M^w\rangle$

is a mapping reduction from

A_{TM} to $Regular_{TM}$

A_{TM} is undecidable

$\Downarrow$

$Regular_{TM}$ is undecidable

Rice's Theorem (intro)

A property P concerns the language of a Turing machine M if P is about the language of M (that is, about L(M))

Examples of such properties

1. L(M) is regular
2. $L(M) = \emptyset$
3. L(M) contains strings x, y s.t $|x| = |y|$
4. L(M) contains a string x with $|x| = 22$
5. $\forall i = 1, 2, \ldots$ L(M) contains x_i with $|x_i| = i$

The following property is NOT about the language of a TM:

$\exists w \in \Sigma^*$ s.t when M is started on w it will visit all its states except one of q_{acc}, q_{reject}

Definition A property P about the language of TM is **non-trivial** if

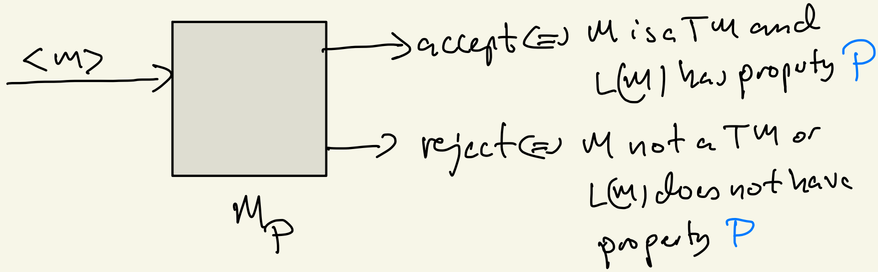
- $\exists$ TM M_1 s.t $L(M_1)$ has property P
- $\exists$ TM M_2 s.t $L(M_2)$ does not have property P

Theorem (Rice)

Every non-trivial property P about the language of Turing machines is undecidable

Proof:

- We may assume that the empty language ($L(M) = \emptyset$) does not have property P otherwise consider the complementary property $\overline{P}$
- Assume that the TM M_P can decide for a given coding $\langle M \rangle$ of a TM whether $L(M)$ has property P



Let M_1 be a TM s.t $L(M_1)$ has property P

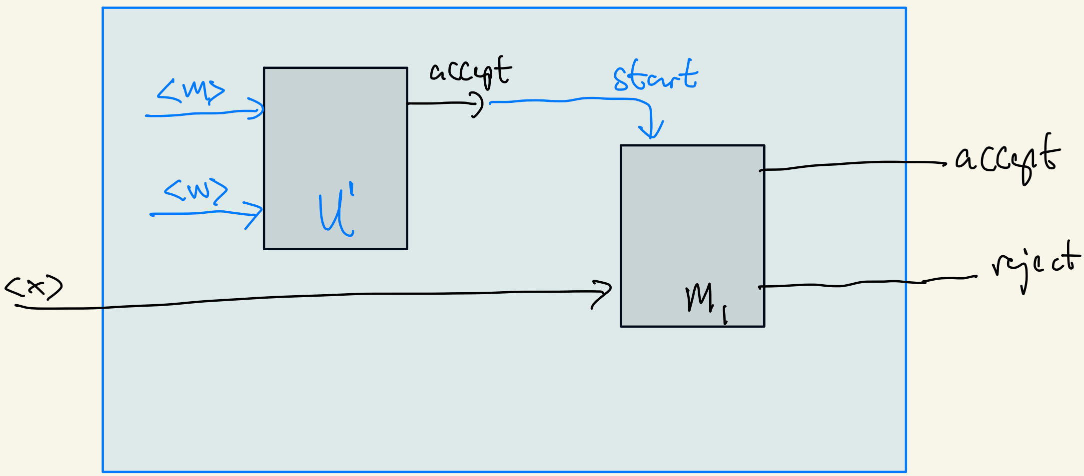
(by assumption on P $L(M_\emptyset)$ does not have " " " —

We show how to construct a TM which

given <M><w> constructs a TM $M(w)$

s.t

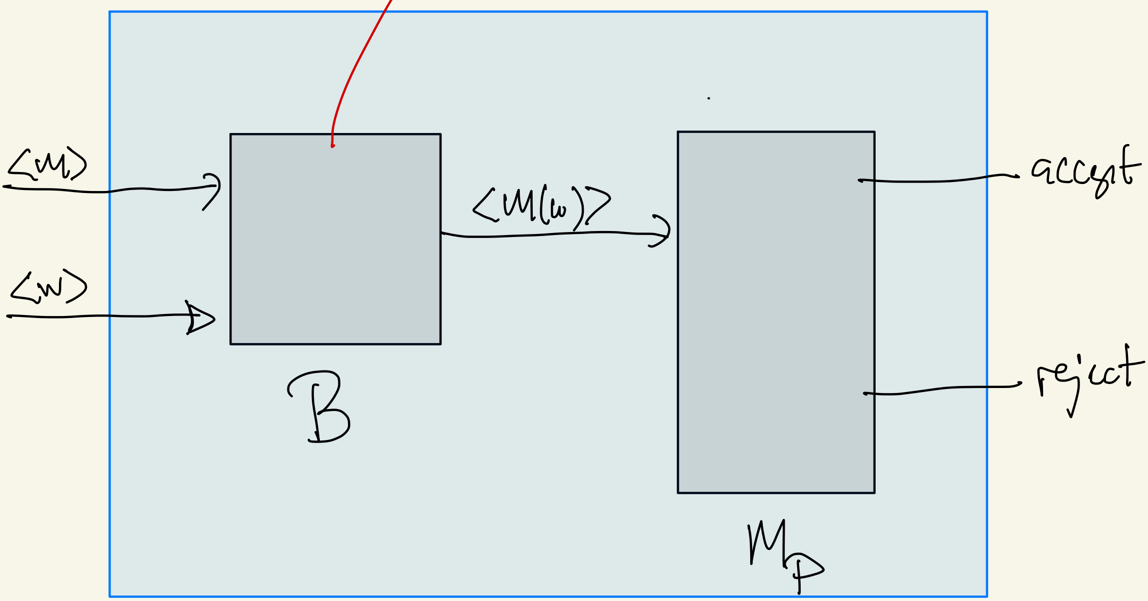
$$L(M(w)) = \begin{cases} \emptyset & \text{if } w \notin L(M) \text{ or } M \text{ is not a TM} \\ L(M_1) & \text{if } w \in L(M) \end{cases}$$



$M(w)$

$L(M(w))$ has property $P \iff \langle M\rangle\langle w\rangle \in A_{TM}$

So $\langle M\rangle\langle w\rangle \longrightarrow M(w)$ is a mapping reduction if <M> does not code a TM then B outputs $M(w) = M_\emptyset$



M^t

M^t decides A_{TM} $\Downarrow$ $\square$

Note proof is 'identical' to that for $Regular_{TM}$

$$L_{ALL} = \{ \langle M \rangle \mid \text{M is a TM an when started on the empty string, M will visit all its states except one} \}$$

Theorem L_{ALL} is undecidable

P: Idea make a mapping reduction from A_{TM} to L_{ALL}

Given a TM R with r states we can change its code $\langle R \rangle$ to the code $\langle R^* \rangle$ of TM R^* as follows

- add a new state q^* not in R's state set
- add a new symbol $\hat{a}$ not in R's alphabet
- Modify δ such that
  - Every transition $\delta(q', \beta) = q_{accept}$ is changed to $\delta(q', \beta) = (q^*, \tilde{a}, S)$
  - $\delta(q^*, \tilde{a}) = (q_1, \hat{a}, S)$
  - $\delta(q_i, \tilde{a}) = (q_{i+1}, \tilde{a}, S)$ if $i < r-2$
  - $\delta(q_{r-2}, \tilde{a}) = q_{accept}$

Then R^* visits all states except one if and only if it accepts its input

Given $\langle M\rangle\langle w\rangle$ we can construct a TM $U_{M,w}$ which has the codes of M and w in its own code.

When $U_{M,w}$ is started on the empty string

$U_{M,w}$: · First print $\langle w\rangle$ on tape 1 and $\langle M\rangle$ on tape 2

· Then simulate M on w as the universal TM would do it

· accept if and only if M accepts w

Now we have:

$$\langle U^*_{M,w}\rangle \in L_{ALL}$$
$$\Updownarrow$$
$$\langle M\rangle\langle w\rangle \in A_{TM}$$

and $\langle M\rangle\langle w\rangle \longrightarrow \langle U^*_{M,w}\rangle$ is a mapping reduction

Hence L_{ALL} is undecidable $\square$