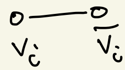
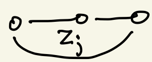
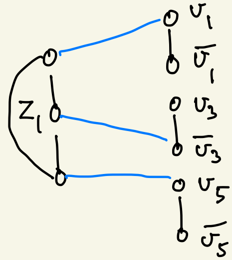
More NP-complete problems from Sipser 7.5 and Cormen 34.5

3-SAT $\leq_p$ Vertex-cover

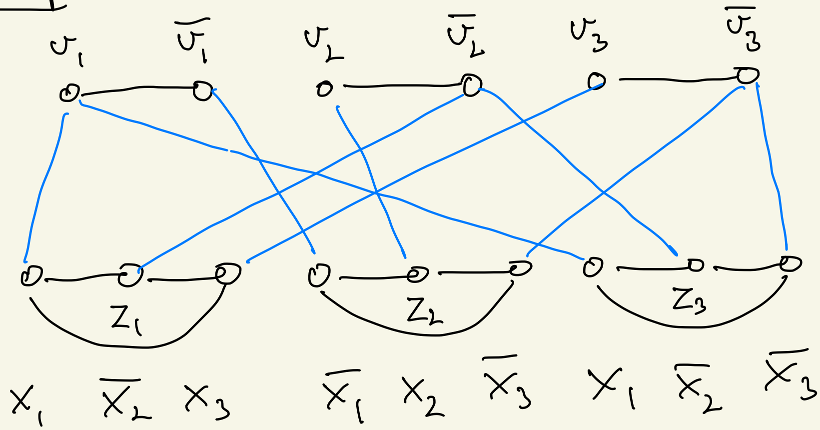
We have already seen that 3-SAT $\leq_p$ Clique $\leq_p$ Independent set $\leq$ Vertex-cover
Here we give a direct reduction from 3-SAT to Vertex-cover

- Given an instance $f = C_1 \wedge C_2 \wedge \dots \wedge C_m$ of 3-SAT with variables $x_1, x_2, \dots, x_n$
- Let $K = n + 2m$ and construct a graph $G = G(f)$ with $2n+3m$ vertices
  - Each variable x_i is represented by  in G
  - Each clause C_j is represented by  in G
  - connections between these in G:
    suppose $C_1 = (x_1 \vee \bar{x}_3 \vee x_5)$, then



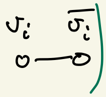
do this for each clause

Example $f = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3)$



$G(f)$

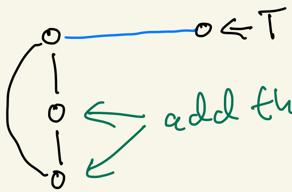
Claim $G(f)$ has a VC of size $n+2m$
$\Updownarrow$
f is satisfiable

Easy: Every vertex cover has size at least $n+2m$
(2 per clause triangle  and 1 per variable edge )

$\Uparrow$ suppose f is satisfied by the truth assignment $t:\{x_1,\dots,x_n\}\to\{T,F\}^n$
Construct $U\subseteq V(G)$ by adding v_i to U if $t(x_i)=T$
adding $\bar{v}_i$ to U if $t(x_i)=F$ ($t(\bar{x}_i)=T$)

Now U covers all edges of the kind $v_i - \bar{v}_i$ $i\in[n]$

For each clause triangle Z_j at least one of the blue edges e leaving it is now covered by U as C_j is satisfied by t



add these two to U

Now U is a vertex cover and $|U| = n+2m$

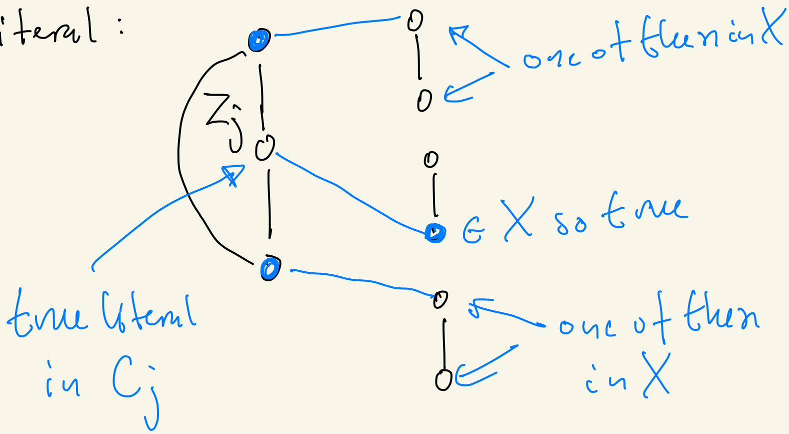
$\Downarrow$ Suppose G has a VC X s.t. $|X| = n+2m$

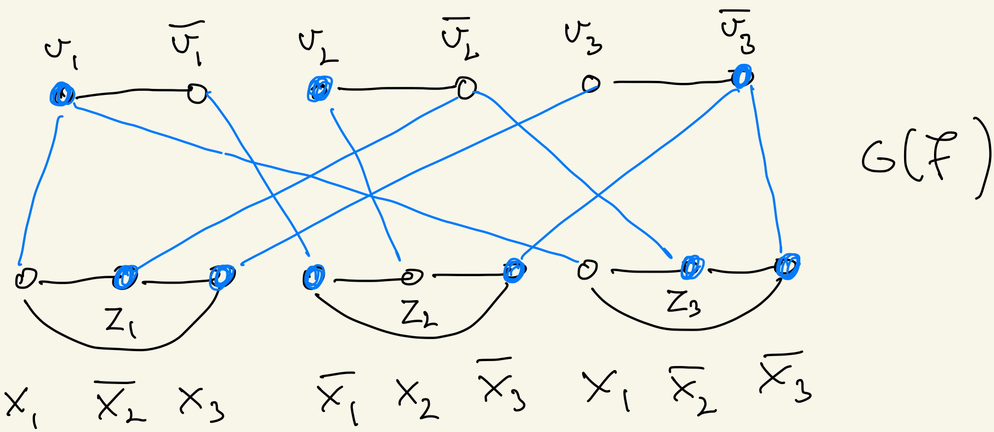
Then $|X \cap \{v_i, \bar{v}_i\}| = 1$ for $i \in [n]$

and X has exactly 2 vertices from Z_j for $j \in [m]$

Set $\varphi(x_i) = T$ iff $v_i \in X$ $\quad i \in [n]$

Then φ is a satisfying truth assignment, that is, each clause C_j has at least one true literal:





Subset-sum:

<u>Input</u>: $\langle S,t\rangle$ where S is a set of non-negative integers and $t\geq 0$ is an integer

<u>Question</u>: Does there exist $S'\subseteq S$ such that $\sum_{x\in S'} x = t$?

Theorem Subset-sum is NP-complete

P: 1. Subset-sum $\in$ **NP**: certificate is a set $S'\subseteq S$ s.t. $\sum_{x\in S'} x = t$

2. we show 3-SAT $\leq_p$ Subset-sum

Let $f = C_1\wedge C_2\wedge\ldots\wedge C_m$ be a 3-SAT instance with variables $x_1, x_2, \ldots, x_n$

We construct an instance $\langle S,t\rangle$ of Subset-sum such that f is satisfiable $\iff \langle S,t\rangle\in$ Subset-sum $\left(\exists S'\subseteq S: \sum_{x\in S'} x = t\right)$

Consider numbers base 10 and set $t = \underbrace{11\cdots1}_{n}\underbrace{44\cdots4}_{m}$

Each variable x_i of f is represented by v_i and v_i' where

$$\begin{array}{ll} & \quad\quad\quad x_i \quad\quad\quad\quad\quad C_j \\ v_i = & 0\cdots010\cdots0\cdots1\cdots0 \end{array}$$ 1 in pos j if x_i literal of C_j

$$\begin{array}{ll} & \quad\quad\quad\quad\quad\quad\quad\quad\quad C_{j'} \\ v_i' = & 0\cdots010\cdots0\cdots1\cdots0 \end{array}$$ 1 in pos $j' \iff \bar{x}_i$ literal of $C_{j'}$

Each clause C_j is represented by S_j, S_j' where

$$\begin{array}{ll} & \quad\quad\quad\quad\quad\quad\quad\quad C_j \\ S_j = & 00\cdots00\cdots010\cdots0 \\ S_j' = & 00\cdots00\cdots020\cdots0 \end{array}$$

idea: v_i can make a 4 in column j if and only if at least one of the rows corresponding to the literals in C_j is in S'

Example $f = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3)$

	x_1	x_2	x_3	C_1	C_2	C_3	C_4	
v_1	1			1		1		$\leftarrow$ $x_1 \leftarrow T$
v_1'	1				1		1	
v_2		1			1		1	$\leftarrow$ $x_2 \leftarrow T$
v_2'		1		1		1		
v_3			1	1				
v_3'			1		1	1	1	$\leftarrow$ $x_3 \leftarrow F$
s_1				$\boxed{1}$				
s_1'				$\boxed{2}$				
s_2					1			
s_2'					$\boxed{2}$			
s_3						1		
s_3'						$\boxed{2}$		
s_4							1	
s_4'							$\boxed{2}$	
t	1	1	1	4	4	4	4	

Claim f is satisfiable $\iff \langle S, t\rangle \in$ Subset-sum

$\Rightarrow$ For $i \in [n]$: if $x_i = T$ add v_i to S' else add v_i' to S'

For $j \in [m]$ if C_j has 1 true literal add s_j, s_j' to S'

if C_j has 2 true literals add s_j' to S'

if C_j has 3 true literals add s_j to S'

$\Leftarrow$ assume $S' \subseteq S$ satisfies $\sum_{x \in S'} x = t$

set $x_i \leftarrow T$ if $v_i \in S'$

$x_i \leftarrow F$ if $v_i' \in S'$

This is a satisfying truth assignment since we must have at least one 1 in upper part of column for C_j $\square$.

Hamilton cycle

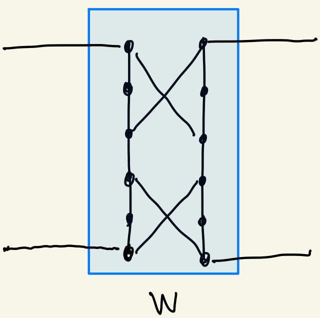
input: a graph $G=(V,E)$
Question: does G have a cycle C with $V(C)=V$?

Theorem Hamilton cycle is NPC

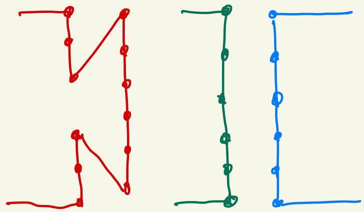
Proof 1. Hamilton cycle $\in$ **NP**: certificate is a permutation $v_1, v_2, \dots v_n$ s.t. $v_i v_{i+1} \in E$ for $i=1,2 \dots n-1$ and $v_1 v_n \in E$

2. Vertex-cover $\leq_p$ Hamilton cycle

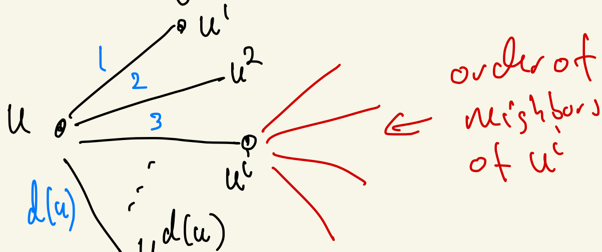
We will use the following so-called gadget W



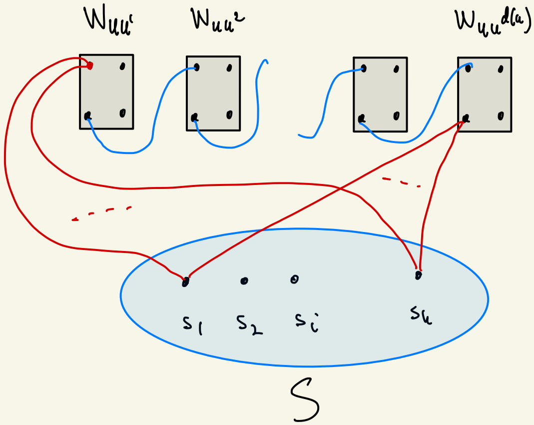
exactly 2 ways to traverse all 12 vertices of W



- Let $\langle G, k \rangle$ be an instance of Vertex-cover where $G=(V,E)$
- For each edge $uv \in E$ take a copy W_{uv} of W
- Fix an ordering of the neighbours around each vertex $u \in V$



For all $u \in V$ connect $W_{uu^1}, W_{uu^2}, \dots, W_{uu^{d(u)}}$ as follows.

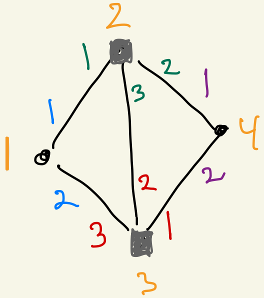


S is a fixed set of k vertices same for all u

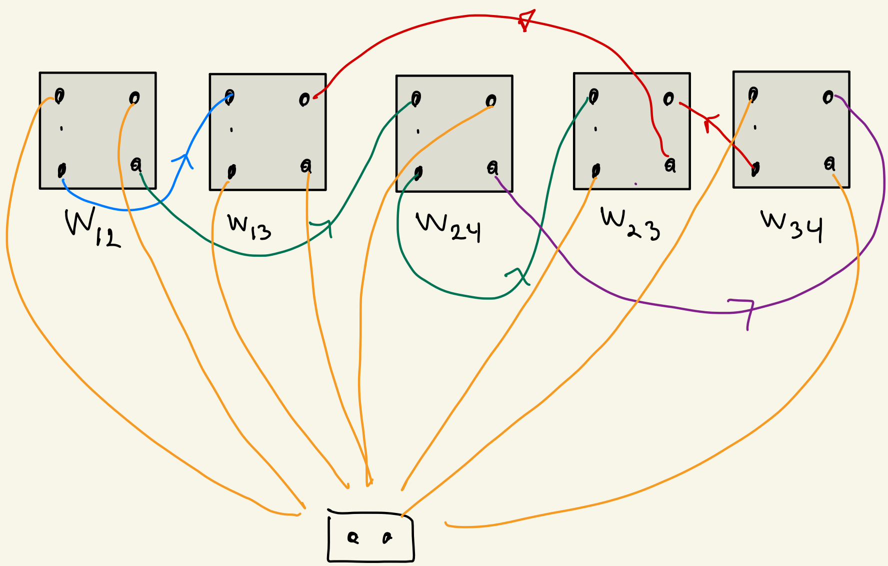
Note that $W_{uu^i} = W_{u^i u}$ so it appears in the chain for u and in the chain for u^i

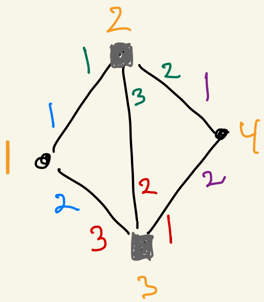
<u>Claim</u> resulting graph G' has a hamilton cycle $\iff$ G has a vertex cover of size k

<u>Example</u>

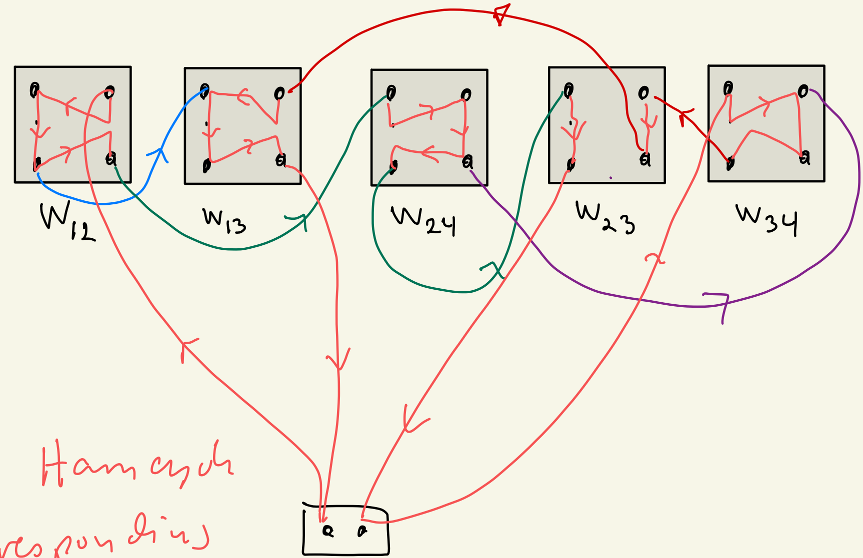


$k = 2$





$k = 2$



pink is a Ham cycle in G' corresponding to the VC $\{2,3\}$ in G.

suppose $U = \{u_1, u_2, \ldots u_k\}$ is a vertex cover in G

let $C = s_1 P_1 s_2 P_2 \cdots s_{k-1} P_{k-1} s_k P_k s_1$ be the cycle we obtain as indicated in the example above, that is

P_i traverses the gadgets $W_{u_i v_i^1}, W_{u_i v_i^2} \cdots W_{u_i v_i^{d(u_i)}}$ in that order and inside $W_{u_i v_i^j}$ P_i uses all vertices if $v_i^j \notin U$

and otherwise P_i uses only one side of $W_{u_i v_i^j}$

Suppose now that G' has a Hamilton cycle C

The set S has no edges so C is of the form

$s_{i_1} Q_{i_1} s_{i_2} Q_{i_2} \cdots s_{i_k} Q_{i_k} s_{i_1}$

- C picks up all vertices inside each W_{uv}
  - W_{uv} is only connected to other $W_{uv'}$ or $W_{u'v}$ via the edges of the chain corresponding to the neighbours of either u or v
  - So each $Q_{i_j} \iff$ chain for some vertex u_j
- By symmetry we can assume that Q_{i_j} starts at the first of the W's for u_i and ends at the last one
- $\{u_1, u_2 \cdots u_k\}$ is a vertex cover of G

TSP input: A complete graph K_n and $w: E(K_n) \to$ non negative integers and an integer K

Question: Does there exist a Hamilton cycle of weight $\leq K$?

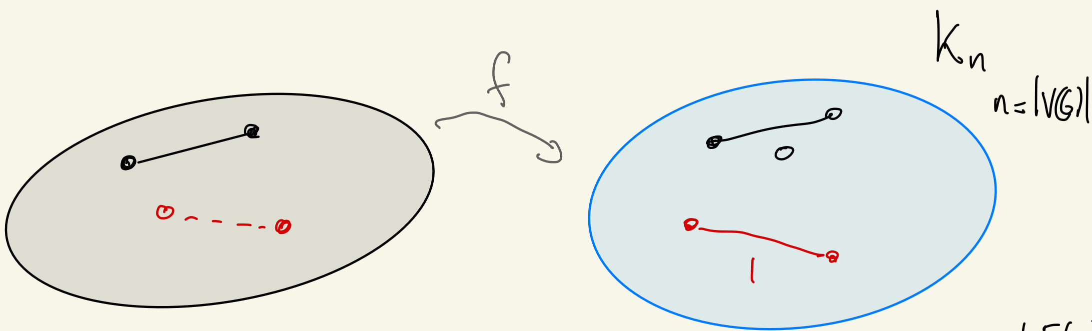
Theorem TSP is NP-complete

proof

1. TSP $\in$ **NP**

   Certificate is a permutation of the vertices corresponding to a Hamilton cycle of weight $\leq K$

2. Hamilton cycle $\leq_p$ TSP



G

$$w(ij) = \begin{cases} 1 & \text{if } ij \notin E(G) \\ 0 & \text{if } ij \in E(G) \end{cases}$$

$K = 0$

G has a Hamilton cycle

$\Updownarrow$

$\exists$ Hamilton cycle of weight 0 in (K_n, w)