Institut for Matematik og Datalogi
Syddansk Universitet, Odense

February 15, 2024
JBJ

DM553/MM850 – 1. Exam assignment

Hand in by Friday March 8, 10:00.

Rules

This is the first of three sets of problems which together with the oral exam in June constitute the exam in DM553/MM850. This first set of problems may be solved in groups of up to three. Any collaboration between different groups will be considered as exam fraud. Thus you are not allowed to show your solutions to fellow students, not from your group and you may not discuss the solutions with other groups. On the other hand, you can learn a lot from discussing the problems with each other so you may do this to some extend, such as which methods can be used or similar problems from the book or exercise classes.

It is important that you **argue so that the reader can follow your explanations**. It is not enough just to say that the solution follows directly from an example in the book or similar. In such a case you should repeat the argument in your own words.

Remember that this (and the two comming sets of assignments) counts as part of your exam, so do a good job and try to answer all questions carefully. Also note that at the exam each member of a group is responsible for all of the handed in solutions from the group. This means that I may ask questions in any of the assignments to check your understanding of that particular topic.

How to hand in your report

Your report **should be written in english or danish** and must be handed in via Itslearning by Friday March 8 at 10:00.

Hand in one report per group. On the first page you must write the **names** of all participants in the group as well as the first 6 digits of your **CPR-numbers**. Do not write the last 4 digits!

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Exam problems

Solve the following problems. **Remember to justify all answers.**

PROBLEM 1 (20%)

Question a:

Which of the following languages are regular? For those that are regular it suffices to describe in words how a DFA for these will work (you do not have to make a drawing, but may do so of course). For those that are not regular, you should give a short argument, for example via the pumping lemma or closure properties of regular languages.

- $L_1 = \{a^{4k} | k \geq 0\}$
- $L_2 = \{a^{k^2} | k \geq 0\}$
- $L_3 = \{a^i b^j | i, j \geq 0 \text{ and } |i - j| \text{ is even }\}$
- $L_4 = \{a^i b^j | i, j \geq 0 \text{ and } i \leq 2j\}$

Question b:

Which of the languages L_1, L_2, L_3, L_4 are context-free? Remember to justify your answers.

PROBLEM 2 (15%)

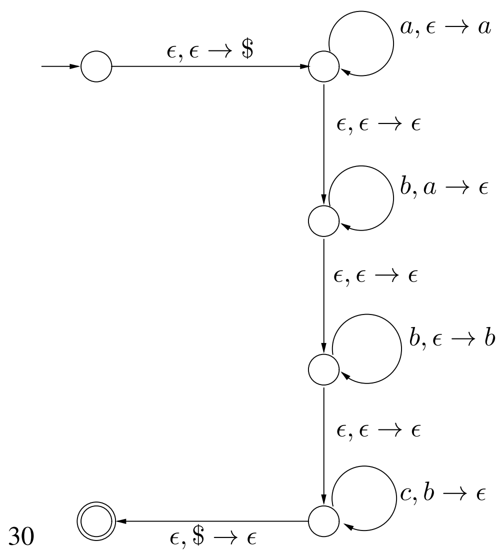

Figure 1: A PDA M_1.

Question a:

Argue that the language of the PDA shown in Figure 1 is given by
$L(M_1) = \{a^n b^{n+m} c^m | n, m \geq 0\}$.

In Figure 2 another PDA is given. The 7 different lines at the loop indicates 7 different transitions.

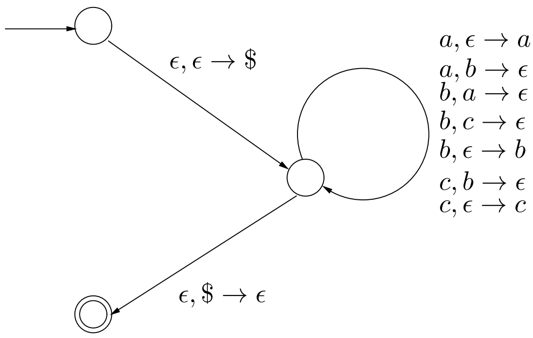

Figure 2: A PDA M_2.The 7 different lines at the loop indicates 7 different transitions.

Question b:

What is $L(M_2)$? You must justify your answer.

PROBLEM 3 (20 %)

Let L_1 be the set of strings over $\Sigma = \{a, b, c\}$ which have the same number of a's and b's. Similarly, L_2 is the set of strings over $\Sigma = \{a, b, c\}$ with the same number of b's and c's. That is,

$L_1 = \{w \in \{a, b, c\}^* | \#_a(w) = \#_b(w)\}$,
$L_2 = \{w \in \{a, b, c\}^* | \#_b(w) = \#_c(w)\}$.

Question a:

Prove that L_1 is context-free, by describing, on diagram form, a PDA M such that $L_1 = L(M)$. Hint: use the stack to keep track of which of the two symbols you have currently seen the most of.

Question b:

Which of the following languages are context-free?

1. $L_1 \cap a^* b^*$.
2. $L_1 \cup L_2$.
3. $L_1 \cap L_2$.
4. $\overline{L_1}$.

5. $L_1 - L_2$.

You must justify your answers! You may use, without proof, that the language $L' = \{a^n b^n c^m | n, m \geq 0, m \neq n\}$ is not context-free:

PROBLEM 4 (10%)

Let L_1, L_2, L_3 be regular languages over an alphabet Σ. Suppose we are given DFAs M_1, M_2, M_3 such that $L(M_i) = L_i$ for $i = 1, 2, 3$.

Question a:

Describe in your own words how you can use the DFA's above to construct algorithms for answering the following questions:

(1) Is $L_1 \setminus L_2 = \emptyset$?

(2) Is $L_3 = L_1 \cup L_2$?

(3) Is $L_1 \cup L_2 \cup L_3 = \Sigma^*$?

(4) Is $L_1 \cap L_2 \subseteq L_3$?

Question b:

Can we still make such algorithms if M_i is just an NFA for each $i = 1, 2, 3$?

PROBLEM 5 (15%)

Let G be the context-free grammar $G = (\{S, A, B\}, \{a, b\}, R, S)$, where R consists of the following rules:

$$\begin{aligned} S &\rightarrow bA|Aa|A|B \\ A &\rightarrow AA|aS|S|a|\epsilon \\ B &\rightarrow aBB|bS|b|A \end{aligned}$$

Here ϵ denotes the empty string.

Show how to convert G into an equivalent context-free grammar which is in Chomsky normal form. Note that you must show all the steps of the conversion and briefly explain what you do.

PROBLEM 6 (20%)

Let k be a fixed positive integer. Recall that a **k-head** Turing machine is like a standard Turing machine, except that it has k reading heads instead of one. Hence, at any step the machine collects the symbols under its k heads, updates those cells, moves each of the heads right, left or lets it stay where it was and then enters a new state from which the cycle starts again. A typical transition (where we assumed $k = 3$) will look like this: $\delta(q_i, a_1, a_2, a_3) = (q_j, b_1, b_2, b_3, R, L, S)$, meaning that the machine, in state q_i with the ith head reading the character a_i, $i = 1, 2, 3$ the machine will change a_i to b_i, $i = 1, 2, 3$, move the first head right, the second left and letting the third stay at the same position, respectivly and then enter state q_j. By definition, all k-head Turing machines start with the input w on the $|w|$ leftmost positions of its tape and all k heads reading the leftmost symbol. Below you may assume that the Turing machine knows when two or more of its heads read the same cell and then has a strategi for changing the cell based on which heads are over that cell.

In the questions below you are not supposed to give transition tables or similar, but you must describe, in words, the most important steps taken by the machine. For example one step could be: move the third head to the rightmost end of the current tape (this means to the rightmost character which is not a blank).

Question a.

Describe in words (by describing important steps) how a 3-head deterministic Turing machine can recognize the language $L = \{a^p b^p c^p | p \geq 0\}$ and then do the same for the language $L' = \{a^p b^p c^{2p} | p \geq 0\}$.

Question b.

An r-tape-k-head Turing machine is a Turing machine with r-tapes and k-heads on each of these. Argue, by describing the important steps of M', that for every 3-tape-2-head Turing machine M (that is, there are 2 heads on each tape so 6 in total) there exists a deterministic one-tape-one-head (that is, a standard) Turing machine M' such that $L(M) = L(M')$.