Institut for Matematik og Datalogi
Syddansk Universitet

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JBJ

DM553/MM850 – Spring 2024 – Weekly Note 7

Stuff covered in Week 10

- Notes on undecidability from the coursepage.
- Rice's theorem.
- Part of Sections 7.2 and 7.3 on the complexity classes P and NP.

Key points

- If a nontrivial language L concerns a single Turing machine and membership in the language is determined only by the language of the Turing machine, then Rice's theorem says that L is undecidable, provided that some but not recognizable languages have the property described for L. More precisely: nontrivial properties of languages of Turing machines are undecidable. Here a property $\mathcal{P}$ is non-trivial if there exist two Turing machines M_1, M_2 so that $L(M_1)$ has the property while $L(M_2)$ does not. Examples of such properties that are undecidable by Rice's theorem are:

  1. The property that the language of M is regular. Here we can take M_1, M_2 such that $L(M_1) = \Sigma^*$ and $L(M_2) = \{a^n b^n | n \geq 0\}$.
  2. The property that the language of M is empty. Here we can take M_1 such that $L(M_1) = \emptyset$ and M_2 such that $L(M_2) \neq \emptyset$. E.g. M_2 could be the TM that accepts every string.
  3. The property that the language of M contains two strings of different lengths. Here we can use the TMs M_1, M_2 above to show that this is a non-trivial property of languages for TMs.

  In each case one has to argue that the TMs M_1 and M_2 exist. Note that it has to be properties of $L(M)$ for a TM M and **not** a property saying what M does to its tape, states etc. Examples of such properties where Rice's theorem can **not** be used are:

  1. The property that if TM M is started on the empty string it will eventually halt and have the string w on its tape. One such example is January 2000 Problem 4(b). Note that we also cannot use Rice's theorem directly on January 2000 Problem 4(a) since M may stop and reject the string 'dm17' so the question is NOT about the language of a TM. Note however that we may use Rice's theorem in the following way: Every TM M is equivalent (can be transformed into a TM

with that property by an algorithm and hence by a TM) to a TM M' which halts on exactly those strings which it accepts: just let M' simulate M and loop if M wanted to reject the input. Now apply Rice's theorem to languages of Turing machines M' in the sense that now the property above (stopping on 'dm17') IS a property about the language of a Turing machine.

2. The property that if TM M is started on the empty string it will run through all of its states, except one (it cannot use both q_{accept} and q_{reject}) before eventually halting. This is undecidable, as we show in the notes on (un)decidability and Video Lecture 14, but we cannot use Rice's theorem to prove it.
3. The property that if TMs M_1 and M_2 are started on w, then both will accept w (that is $w \in L(M_1) \cap L(M_2)$). Here the problem is that we are talking about the language of two Turing machines, not one, so we cannot use Rice's theorem to show that this problem is undecidable, which it is.

New material in Week 11

- Sipser Sections 7.2 and 7.3. (Video 15)
- I will also cover most of Sipser Section 7.4 and Cormen section 34.3 pages 1067-1069 (the rest of Section 34.3 and Cormen Section 34.4 is not pensum!) (Video 16)

Exercises in Week 11

- You will discuss selected problems from the first set of exam problems.
- 5.10 (5.7 in 3rd edition). Hint to show that $A \leq_M A_{TM}$ when A is recognizable use the universal Turing machine together with a TM M_A such that $A = L(M_A)$.
- October 2011, Problem 5
- January 2000, Problem 4
- January 2002, Problem 6
- January 2003, problem 5
- Sipser 7.5,7.6,7.7,7.8,7.9,7.10.7.12 Same numbers in both editions of Sipser.
- If there is more time (and if Magnus has corrected the first set of problems), you can discuss solutions to the first set of exam problems.