Concepts and definitions

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Graphs

- A **graph** $G = (V(G), E(G))$ consists of a finite set of vertices $V(G)$ and a set of edges $E(G)$ – for short denoted $G = (V, E)$.
- E is a subset of $\{(v, w)|v, w \in V\}$. The number of vertices resp. edges in G is denoted n resp. m: $|V(G)| = n$ and $|E(G)| = m$.

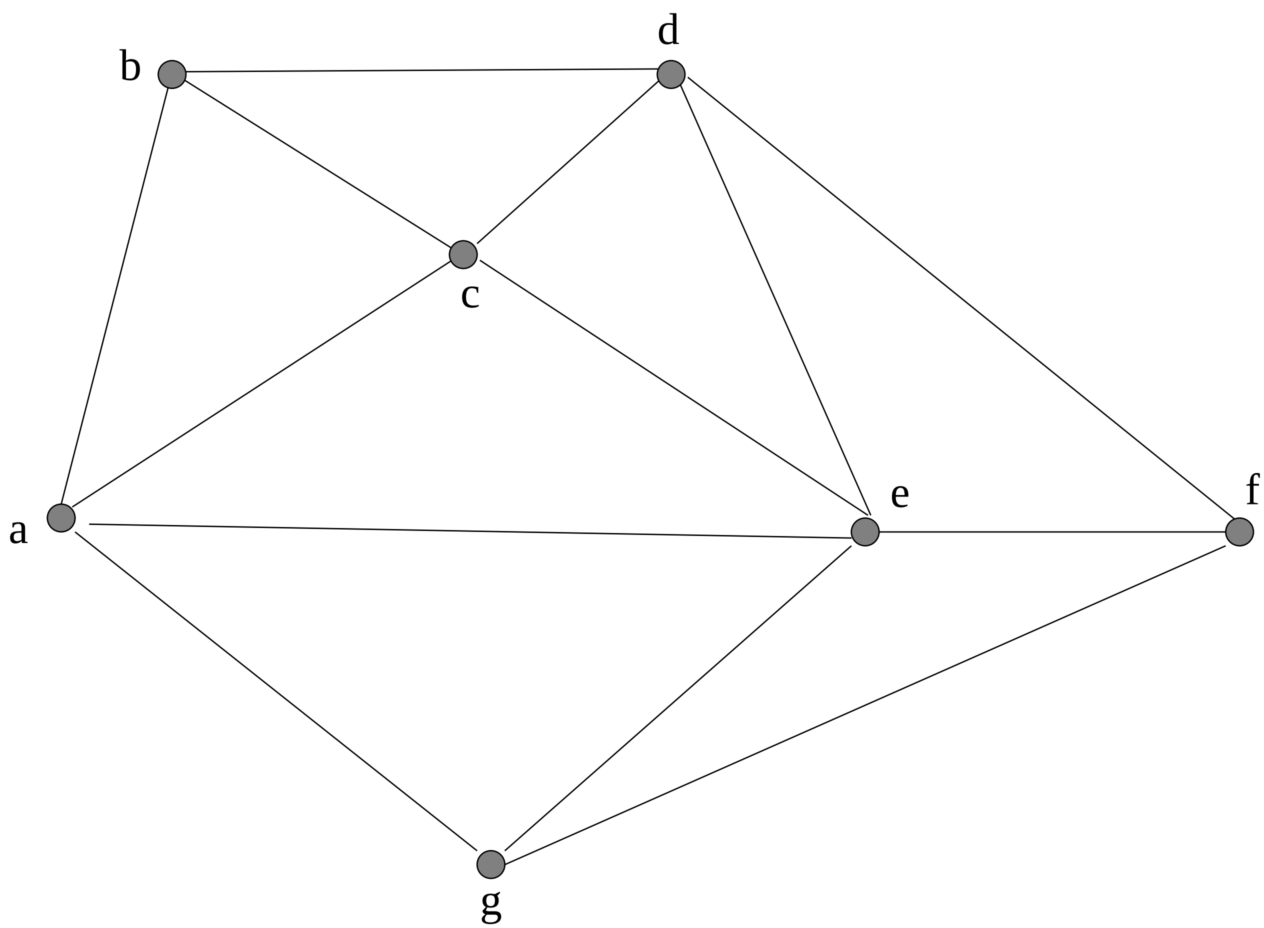
- An edge $(v, w) \in E$ is **incident with** the vertices v and w and v and w are called **neighbors** (**adjacent**).
- In a **complete** (or **fully connected**) graph all possible pairs of vertices are connected with an edge, i.e. $E = \{(v, w) | v, w \in V\}$.

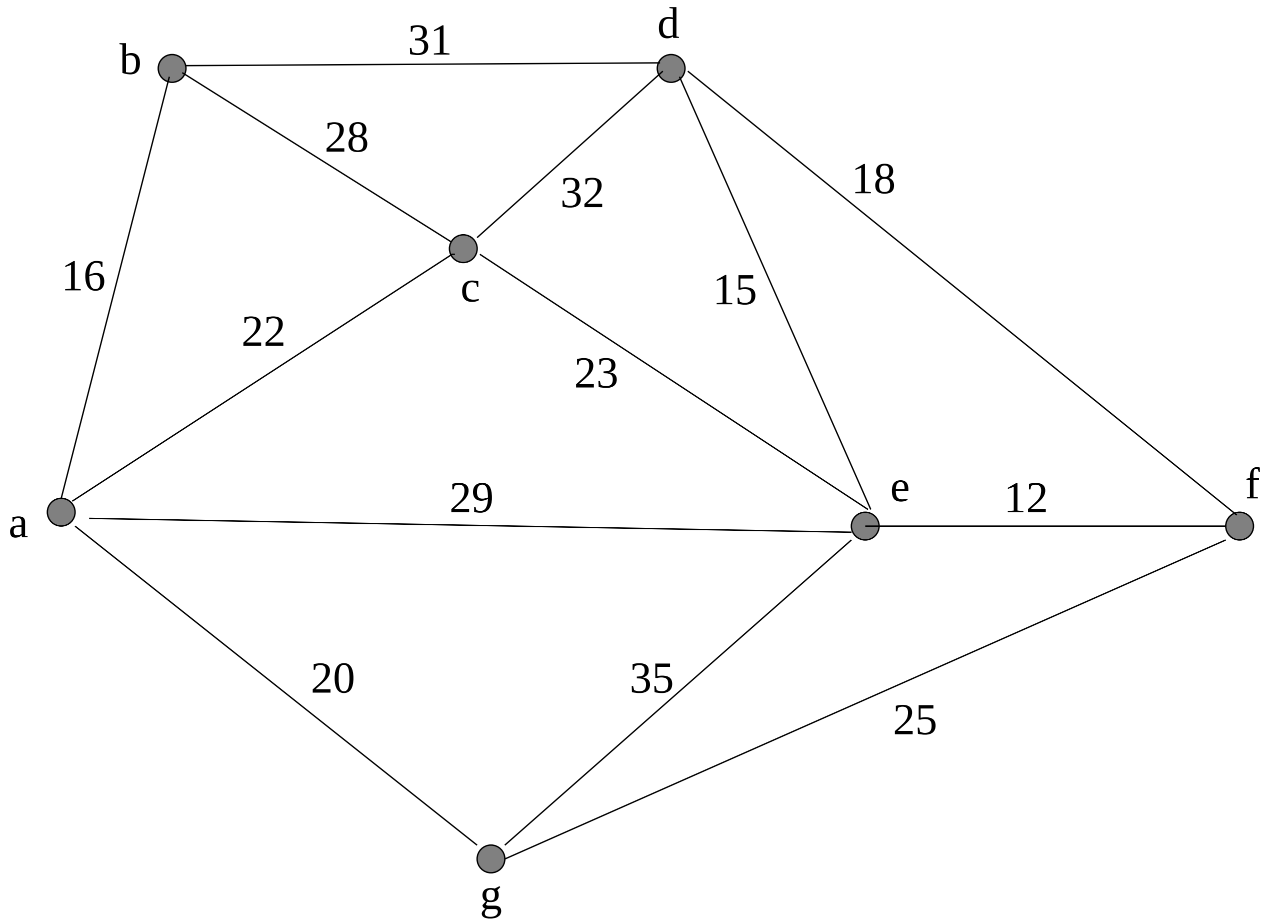
- A **subgraph** H of G has $V(H) \subset V(G)$ and $E(H) \subset E(G)$. $E(H) \subset \{(v,w)|v,w \in V(H)\}$ must also hold.
- When $A \subset E$, $G \setminus A$ is given by $V(G \setminus A) = V(G)$ and $E(G \setminus A) = E \setminus A$.
- When $B \subset V$, $G \setminus B$ (or $G[V \setminus B]$) is given by $V(G \setminus B) = V(G) \setminus B$ and $E(G \setminus B) = E \setminus \{(v,w)|v \in B \vee w \in B\}$. $G \setminus B$ is also called the subgraph induced by $V \setminus B$
- The subgraph H of G is **spanning** if $V(H) = V(G)$

- A **path** P in G is a sequence $v_0, e_1, v_1, e_2, ..., e_k, v_k$, in which each v_i is a vertex, each e_i an edge, and where $e_i = (v_{i-1}, v_i), I = 1, ..., k$. P is called a path from v_0 to v_k or a (v_0, v_k)-path.
- P is **closed** if $v_0 = v_k$, **edge-simple**, if all e_i are different, and **simple**, if all v_i are different.
- A **circuit** C is a path, which is closed and where $v_0, ..., v_{k-1}$ are all different.

- G is **connected**, if an (i, j)-path exists for all pairs (i, j) of vertices. If G is connected, $v \in V$, and $G \setminus v$ is not connected, v is called a **cut-vertex**.
- A circuit-free graph G is called a **forrest**; if G is also connected G is a **tree**.

Graph examples




b
31
d
28
32
18
16
c
15
22
23
a
29
e
12
f
20
35
25
g

Digraphs

- A **digraph**, $G = (V(G), E(G))$ consists of a finite set of vertices $V(G)$ and a set of edges $E(G)$ - often denoted $G = (V, E)$. E is a subset of $\{(v, w)|v, w \in V\}$. Each edge has a **startvertex** (**tail** - $t(v, w) = v$) and an **endvertex** (**head** - $h(v, w) = w$). The number of vertices resp. edges in G is denoted n resp. m : $|V(G)| = n$ and $|E(G)| = m$.

- A edge $(v, w) \in E$ is **incident with** the vertices v and w, and w is called a **neighbor** to v and they are called **adjacent**.

- In a **complete** (or **fully connected**) graph all edges are "present", i.e $E = \{(v,w) | v, w \in V\}$.
- Any digraph has a **underlying graph**, which is found by neglecting the direction of the edges. When graph-terminology is used for digraphs, these relates to the underlying graph.

- A **path** P in the digraph G is a sequence $v_0, e_1, v_1, e_2, ..., e_k, v_k$, in which each v_i is a vertex, each e_i an edge, and where e_i is either (v_{i-1}, v_i) or (v_i, v_{i-1}), $I = 1, ..., k$. If e_i is equal to (v_{i-1}, v_i), e_i is called **forward**, otherwise e_i is **backward**. P is called a **directed** path, if all edges are forward.

- A **circuit** C is a closed path in which all $v_0, ...v_{k-1}$ all are different. A **dicircuit** is a circuit, in which all edges are forward.

- G is **strongly connected**, if there exists an (i, j)-dipath for all pairs (i, j) of vertices.
- For any digraph G, the vertices can be partitioned into equivalence classes. The equivalence class for v contains v itself and all w, for which both a (v, w)-dipath and a (w, v)-dipath exist.
- The equivalence classes are called **strongly connected components**.

Cut

Consider a given graph, $G = (V, E)$ and $A \in V$. $\delta(\mathbf{A})$ is the set of edges connecting one vertex in A to one vertex in $V \setminus A$. $\delta(\mathbf{A})$ is called the **cut** generated by A:

$$\delta(A) = \{(v, w) \in E | v \in A \wedge w \in V \setminus A\}$$

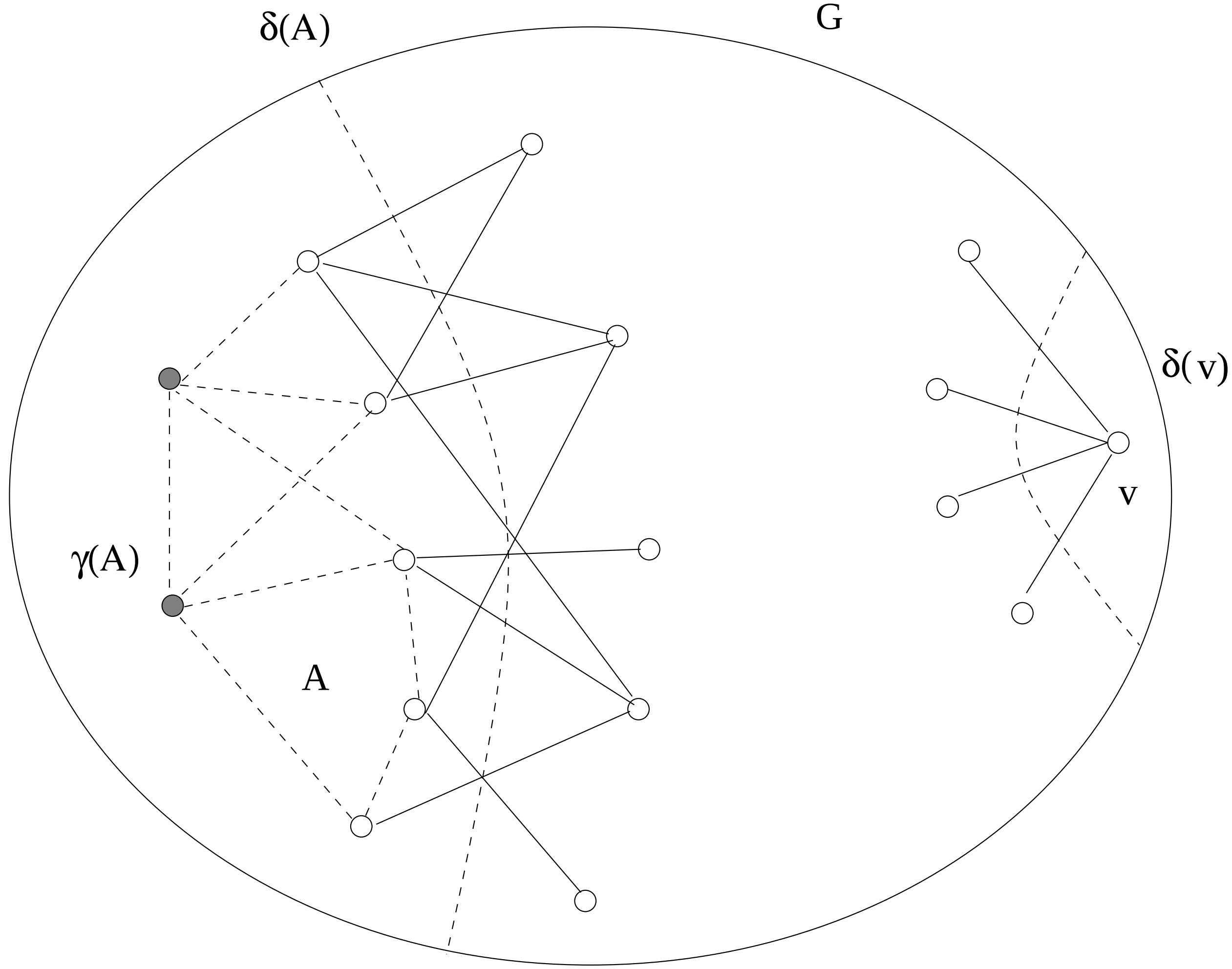
For $v \in V$, $\delta(\mathbf{v}) = \{(v, w) \in E | w \in V \setminus v\}$ i.e. $\delta(\mathbf{v})$ consists of all edges with v as one of the vertices.

$\gamma(\mathbf{A})$ is the set of edges connecting two vertices A:

$$\gamma(A) = \{(v,w) \in E | v,w \in A\}$$

The sum over a set A of the values $x(a)$ is denoted $x(A)$:

$$sum_{a \in A} x(a) = x(A)$$


δ(A)
G
δ(v)
v
γ(A)
A

Big-O notation

Background: Some functions grow much faster than others.

$$lim_{n \to \infty} \frac{a^n}{n^k} = \infty$$

for all $a > 1, k > 1$.

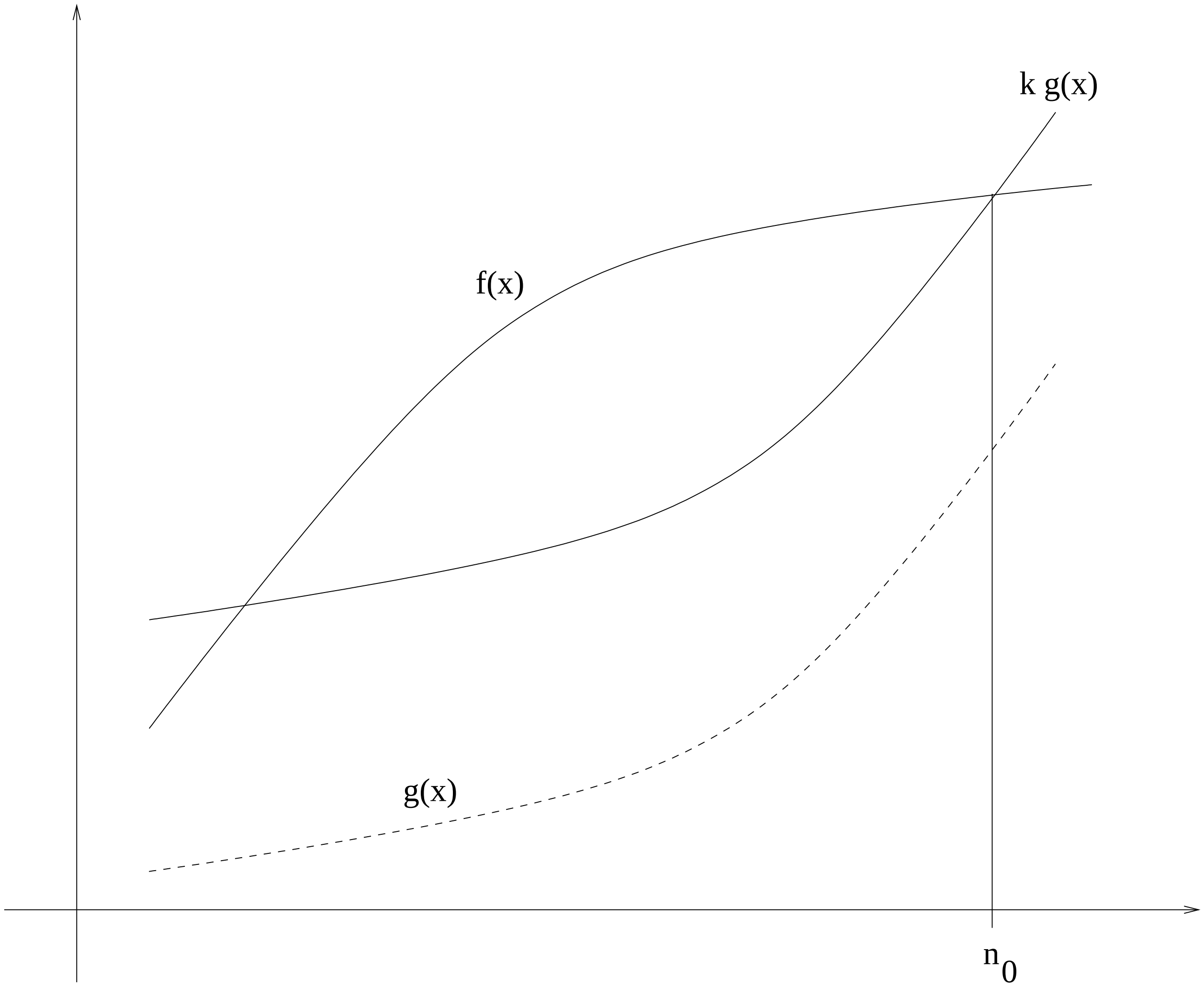
"Any exponential function grows faster than any polynomial function". But what is the exact meaning of "f grows faster than g" ?

Consider two functions f and g: $f : \mathcal{N} \rightarrow \mathcal{N}$ and $g : \mathcal{N} \rightarrow \mathcal{N}$. Then f grows **at most** as fast as g iff:

$$\exists k \; \exists n_0 : \forall n \geq n_0 : f(n) \leq kg(n)$$

This is denoted "f is $O(g(\cdot))$" or "$f \in O(g(\cdot))$"

Picture ?



f grows **at most** as fast as g.

The opposite: Not "f grows **at most** as fast as g" - is denoted $f \in \Omega(g(\cdot))$:

$$\neg(\exists k\ \exists n_0 : \forall n \geq n_0 : f(n) \leq kg(n)) \Leftrightarrow$$

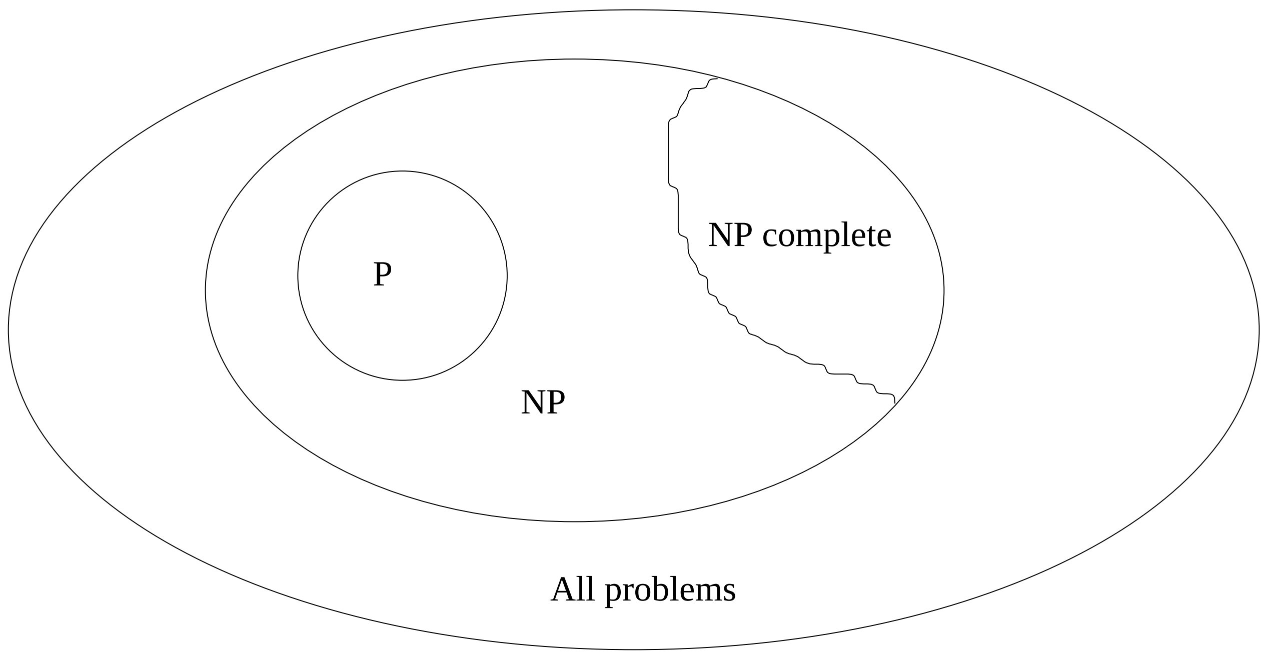
$$\forall k\ \forall n_0\ \exists n : n \geq n_0 \wedge f(n) > kg(n)$$

"The graph for $f(\cdot)$ is above that for $kg(\cdot)$ for infinitely many values of n, no matter how k is chosen". What if f and g grow equally fast ?

$$f \in \Theta(g(\cdot))$$

"f is $O(g(\cdot))$ and g is $O(f(\cdot))$"

Subdivision of the “world of Problems” into those, for which good (polynomial) algorithms have been identified, and those, for which such algorithms have not yet been found (or do not exist).



Growth of functions.

	10	50	100	300	1000
$5N$	50	250	500	1500	5000
$Nlog_2N$	33	282	665	2469	9966
N^2	100	2500	10.000	90.000	7 dig.
N^3	1000	125.000	7 dig.	8 dig.	10 dig.
2^n	1024	16 dig.	31 dig.	91 dig.	302 dig.
$N!$	7 dig.	65 dig.	161 dig.	623 dig.	"incompreh."
N^N	11 dig.	85 dig.	201 dig.	744 dig.	"incompreh."

How large is the number of possibilities when the growth rate is ... ? From Harel: *Algorithmics - the Spirit of Computing*.