Dense triangle-free graphs are four-colourable: A solution to the Erdős-Simonovits problem

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(joint work with Stéphan Thomassé)

In 1972, Erdős and Simonovits conjectured that triangle-free graphs with n vertices and minimum degree $\delta > n/3$ are 3-colourable. Hajnal gave a construction based on Kneser graphs, showing that for any $c < 1/3$ the chromatic number can be arbitrarily large among the triangle-free graphs with $\delta > cn$. In 1982 Häggkvist found a counterexample based on the 4-chromatic Grötzsch graph. Jin proved in 1993, that the original statement is true if $n/3$ is replaced by $10n/29$ which is sharp. Moreover he conjectured that the chromatic number can be arbitrarily large for triangle-free graphs with $\delta > n/3$. Recently, Brandt showed that 4 is an upper bound for the chromatic number of maximal triangle-free r-regular graphs with $r > n/3$, Thomassen proved that for any $c > 1/3$ the chromatic number of every triangle-free graph with $\delta > cn$ is bounded by a constant depending on c, and Łuczak improved this statement, showing that every such graph is homomorphic with a triangle-free graph on constantly many vertices, applying the Regularity Lemma.

We prove the original problem with chromatic number 4 in place of 3 by determining the homomorphically minimal subgraphs of the maximal triangle-free graphs with $\delta > n/3$. This class turns out to be a minor extension of a class described by Brandt and Pisanski in 1997.