Edge Colourings of Graphs

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For the chromatic index $\chi'(G)$ of a (multi)graph G there are two natural lower bounds. On the one hand, $\chi'(G) \geq \Delta(G)$ where $\Delta(G)$ is the maximum degree of G. On the other hand, $\chi'(G) \geq W(G)$ where

$$W(G) = \max_{H \subseteq G} \left\lceil \frac{\mid E(H) \mid}{\lfloor \frac{1}{2} \mid V(H) \mid \rfloor} \right\rceil.$$

A graph G is called *elementary* if $\chi'(G) = W(G)$. Goldberg conjectured around 1970 that every graph G is elementary provided that $\chi(G) \geq \Delta(G) + 2$. For an integer $m \geq 3$, let $\mathcal{J}_m$ denote the class of all graphs G such that

$$\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}.$$

Shannon's theorem implies that $\mathcal{J}_3$ is empty. Furthermore, for every integer $m \geq 3$, we have $\mathcal{J}_m \subseteq \mathcal{J}_{m+1}$ and the class $\mathcal{J} = \bigcup_{m=3}^{\infty} \mathcal{J}_m$ consits of all graphs G such that $\chi'(G) \geq \Delta(G) + 2$.

A graph G is called *critical* if $\chi'(H) < \chi'(G)$ for every proper subgraph H of G. Jakobsen conjectured around 1975 that every critical graph in $\mathcal{J}_m$ has at most $m-2$ vertices provided that $m \geq 3$ is odd. Up to now this conjecture is known to be true only for $m \in \{5, 7, 9, 11\}$. In all these cases the proof of the statement that every graph in $\mathcal{J}_m$ has at most $m-2$ vertices is based on a proof of the seemingly more general statement that every graph in $\mathcal{J}_m$ is elementary. This was proved, independently, by Sørensen for $m = 5, 7$ (unpublished), by Andersen for $m = 5, 7$ in 1977, by Goldberg for $m = 5$ in 1973 and for $m = 9$ in 1984, by Nishizeki and Kashiwagi for $m = 11$ in 1990, and, by Tashkinov for $m = 11$ in 2001. We use an extension of Tashkinov's method to prove that every graph in $\mathcal{J}_{13}$ is elementary.