Total domination in graphs

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(partially joint work with S. Thomasse and M. Henning)

A total dominating set S in a graph $G = (V(G), E(G))$ is a set of vertices such that every vertex in G is adjacent to a vertex in S. In other words $\forall x \in V(G)\ \exists s \in S$: $xs \in E(G)$.

The minimum size of a total dominating set, $\gamma_t(G)$, in a graph, G, is well studied. We will talk about the following new bounds, where $\delta(G)$ is the minimum degree of G and $\Delta(G)$ is the maximum degree in G:

(a): $\gamma_t(G) \leq \frac{3}{7}|V(G)|$, when $\delta(G) \geq 4$.

(b): $\gamma_t(G) \leq |V(G)| - \frac{2|E(G)|}{\Delta(G)+2\sqrt{\Delta(G)}}$, when $\Delta(G) \geq 4$.

In fact we can improve (a) above, if we exclude one specific graph, G_{14}, on 14 vertices. In this case we can obtain the following bound.

(c): $\gamma_t(G) \leq (\frac{3}{7} - \frac{1}{5943})|V(G)|$, when $\delta(G) \geq 4$ and G contains no connected components isomorphic to G_{14}.

(b) above generalises a result by M. Henning. Using this result we can show that the following holds, where $\alpha'(G)$ denotes the size of a maximum matching in a graph G.

(d): If G is a k-regular graph with $k \geq 3$, then $\gamma_t(G) \leq \alpha'(G)$.

There are however examples of (non-regular) graphs, G, with arbitrary high minimum degree where $\gamma_t(G) > \alpha'(G)$. So (d) cannot be extended in this way. We also mention new (best possible) lower bounds on the size of a maximum matching in connected regular graphs. These results can be used to shorten the proofs of (d).

We will also mention other related results and open problems. Many of the results mentioned in this talk have been obtained by observing that a total dominating set in a graph G is also a transversal in the hypergraph $H(G)$ on the same vertex-set as G and with edge-set $\{N(x)|x \in V(G)\}$. This allows us to use hypergraph techniques in order to obtain the above results.