Stability method and the exact solution of the Erdős-T. Sós conjecture

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Many extremal graph conjectures turned out to be solvable, but many degenerate ones (where the excluded graph is bipartite) are still very hopeless. We know the approximate edge-density only in a very few cases.

Embedding a fixed k-vertex tree T_k into an n-vertex graph G_n turned out to be one of the most difficult problems of the solvable ones. In my lecture I shall discuss the following beautiful conjecture.

Conjecture 1 (Erdős-T. Sós conjecture). *If T_k is a fixed tree of k vertices, then every graph G_n of n vertices and*

$$e(G_n) > \frac{1}{2}(k-2)n \tag{1}$$

edges contains T_k.

Our main result is that

Theorem 1 (Ajtai-Komlós-Simonovits-Szemerédi). *There exists an integer k_0 for which, if $k > k_0$ then Conjecture 1 holds.*

I will sketch the proof of the Erdős-Sós conjecture. In the first part of the proof a weakened Erdős-T. Sós conjecture is proved, according to which for every $\eta > 0$ there exists an integer $n_0(\eta)$ such that if $n, k > n_0(\eta)$ and a graph G on n vertices contains no T_k then

$$e(G_n) \leq \frac{1}{2}(k-2)n + \eta n.$$

That proof, combined with some stability methods shows that in most cases either we know that $T_k \subseteq G_n$ even under the weaker condition (1) or we can prove that the structure of G_n is very near to the conjectured extremal graphs: it is the union of small **complete blocks** or some **almost complete bipartite** graphs. Then, for $k > k_0$, applying some elementary arguments, we can embed T_k into G_n using only (1).

This is a joint work with Miklós Ajtai, János Komlós, and Endre Szemerédi. It is strongly connected to the solution of the Loebl-Komlós-Sós conjecture, by Hladký, Komlós, Piguet, Simonovits, Maya Stein, and Endre Szemerédi (see e.g. Arxiv): while in the Erdős-Sós Conjecture we assume that the average degree is large, in the Loebl-Komlós-Sós Conjecture the median degree is assumed to be large to ensure a subtree T_k in G_n.