Critical Graphs and Hypergraphs

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The talk discusses results and open problems related to the structure of colour critical graphs and hypergraphs. The *chromatic number* $\chi(H)$ of a hypergraph H is the least integer k, for which there exists a colouring of the vertices of H such that no edge is monochromatic. This colouring concept for hypergraphs is due to P. Erdős and A. Hajnal and generalizes the classical colouring concept for graphs (2-uniform hypergraphs). A hypergraph H is *critical* or *k-critical* if $\chi(H) = k$, but $\chi(H') < k$ whenever H' is a proper subhypergraph of H. Critical graphs were first defined and investigated by G. A. Dirac in the 1950s, and critical hypergraphs were introduced by L. Lovász in 1968. That critical hypergraphs form a useful concept relies on the fact that many problems concerning the chromatic number of hypergraphs can be reduced to critical hypergraphs, and critical hypergraphs have stronger structural properties than hypergraphs in general.

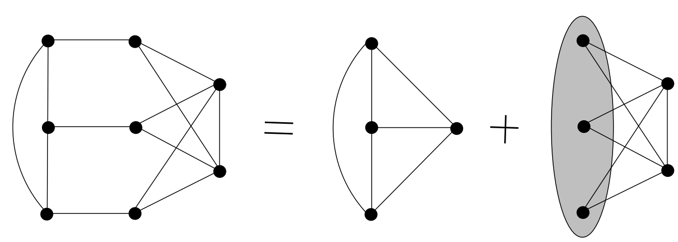

Figure 1: A decomposition of a 4-critical graph involving hypergraphs.

In 1970 Bjarne Toft submitted a Ph. D thesis entitled *Some Contribution to the Theory of Colour-Critical Graphs* to the University of London. In this thesis Toft continued the study of critical graphs started by Dirac, Gallai, Hajós and Ore; the thesis also contains a whole chapter on the structure of critical hypergraphs. In general, critical hypergraphs might be expected to behave much more chaotical than critical graphs. However, as first noticed by Toft, if $k \geq 4$, then k-critical hypergraphs can be transformed into k-critical graphs. Thus if the structure of all k-critical graphs ($k \geq 4$) were known, then the structure of all k-critical hypergraphs would be known.

On the other hand, results on critical hypergraphs in some cases provide results for critical graphs that it would be more difficult to obtain otherwise. With respect to vertex coloring it is advantageous to study hypergraphs, even if one is primarily interested in graphs. In particular, for constructing critical graphs, it has proved useful to involve critical hypergraphs in such construction.