



Degree constrained 2-partitions of semicomplete digraphs[☆]

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ABSTRACT

A 2-partition of a digraph D is a partition (V_1, V_2) of $V(D)$ into two disjoint non-empty sets V_1 and V_2 such that $V_1 \cup V_2 = V(D)$. A semicomplete digraph is a digraph with no pair of non-adjacent vertices. We consider the complexity of deciding whether a given semicomplete digraph has a 2-partition such that each part of the partition induces a (semicomplete) digraph with some specified property. In [4] and [5] Bang-Jensen, Cohen and Havet determined the complexity of 120 such 2-partition problems for general digraphs. Several of these problems are NP-complete for general digraphs and thus it is natural to ask whether this is still the case for well-structured classes of digraphs, such as semicomplete digraphs. This is the main topic of the paper. More specifically, we consider 2-partition problems where the set of properties are minimum out-, minimum in- or minimum semi-degree. Among other results we prove the following:

- For all integers $k_1, k_2 \geq 1$ and $k_1 + k_2 \geq 3$ it is NP-complete to decide whether a given digraph D has a 2-partition (V_1, V_2) such that $D\langle V_i \rangle$ has out-degree at least k_i for $i = 1, 2$.
- For every fixed choice of integers $\alpha, k_1, k_2 \geq 1$ there exists a polynomial algorithm for deciding whether a given digraph of independence number at most α has a 2-partition (V_1, V_2) such that $D\langle V_i \rangle$ has out-degree at least k_i for $i = 1, 2$.
- For every fixed integer $k \geq 1$ there exists a polynomial algorithm for deciding whether a given semicomplete digraph has a 2-partition (V_1, V_2) such that $D\langle V_1 \rangle$ has out-degree at least one and $D\langle V_2 \rangle$ has in-degree at least k .
- It is NP-complete to decide whether a given semicomplete digraph D has a 2-partition (V_1, V_2) such that $D\langle V_i \rangle$ is a strong tournament.

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1. Introduction

A **2-partition** of a (di)graph G is a partition (V_1, V_2) of $V(G)$ into two disjoint non-empty sets. Let $\mathcal{P}_1, \mathcal{P}_2$ be two graph properties, then a $(\mathcal{P}_1, \mathcal{P}_2)$ -**partition** is a 2-partition (V_1, V_2) where V_1 induces a graph with property \mathcal{P}_1 and V_2 a graph with property \mathcal{P}_2 . For example a $(\delta^+ \geq k, \delta^+ \geq k)$ -partition is a 2-partition of a digraph where each partition induces a subdigraph with minimum out-degree at least k . It is natural to ask when properties such as high (edge)-connectivity or

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minimum degree can be maintained by both parts of some 2-partition of a (di)graph G . As an example of this, Alon [1] and independently Stiebitz [14] posed the following problem.

Problem 1.1. [1,14] *Does there exist a function $h(k_1, k_2)$ such that every digraph $D = (V, A)$ with minimum out-degree $h(k_1, k_2)$ has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition?*

It is easy to see that the answer to Problem 1.1 is yes if and only if there exists a function $h'(k_1, k_2) \geq k_1 + k_2 + 1$ such that every digraph with minimum out-degree $h'(k_1, k_2)$ contains disjoint induced subdigraphs D_1, D_2 such that D_1 has minimum out-degree at least k_1 and D_2 has minimum out-degree at least k_2 . The lower bound comes from the complete digraph on $k_1 + k_2 + 1$ vertices in which every vertex has out-degree $k_1 + k_2$. Clearly this does not have the desired 2-partition since there will be too few vertices in one of the sets of any 2-partition.

It is known that high minimum out-degree guarantees many disjoint cycles in a digraph [1,15], in particular minimum out-degree 3 is enough to guarantee two disjoint cycles. Using this, one can easily show that $h(1, 1) = 3$ (see the beginning of Section 3). Already the existence of $h(1, 2)$ is open and we will show in Theorem 3.1 that it is NP-complete to decide if a digraph has a $(\delta^+ \geq 1, \delta^+ \geq 2)$ -partition.

The following two problems are natural variations of Problem 1.1. For definitions of semi-degree etc, see the next section.

Problem 1.2. *Do there exist functions $w_1(k_1, k_2), w_2(k_1, k_2)$ such that every digraph $D = (V, A)$ with minimum out-degree $\delta^+(D) \geq w_1(k_1, k_2)$ and minimum in-degree $\delta^-(D) \geq w_2(k_1, k_2)$ has a $(\delta^+ \geq k_1, \delta^- \geq k_2)$ -partition.*

Problem 1.3. *Does there exist a function $z(k_1, k_2)$ such that every digraph $D = (V, A)$ with minimum semi-degree $\delta^0(D) \geq z(k_1, k_2)$ has a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$ -partition.*

In [10] Lichiardopol answered Problems 1.1 to 1.3 in the affirmative for tournaments. It can easily be seen that his proofs can be generalized to semicomplete digraphs.

Theorem 1.4. [10] *Let T be a tournament (semicomplete digraph) with minimum out-degree at least $\frac{k_1^2 + 3k_1 + 2}{2} + k_2$, then T has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition.*

Theorem 1.5. [10] *Let T be a tournament (semicomplete digraph) with minimum semi-degree at least $k_1^2 + 3k_1 + 2 + k_2$, then T has a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$ -partition.*

In [4,5] Bang-Jensen, Havet and Cohen determined the complexity of 120 partition problems for general digraphs. Among these are the following.

Theorem 1.6. [5] *The following 3 decision problems are NP-complete for general digraphs:*

- *deciding whether D has a $(\delta^+ \geq 1, \delta^- \geq 1)$ -partition,*
- *deciding whether D has a $(\delta^0 \geq 1, \delta^- \geq 1)$ -partition and*
- *deciding whether D has a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition.*

We show in the beginning of Section 3 that the $(\delta^+ \geq 1, \delta^+ \geq 1)$ -partition problem is polynomially solvable for general digraphs. From these results two natural questions emerge. What is the complexity of the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition problem when $k_1 + k_2 \geq 3$ and what is the complexity of the three problems in Theorem 1.6 if we restrict the input to a well-structured class of digraphs such as semicomplete digraphs? This paper will answer these questions as well as several related ones. In Section 3 we prove that as soon as $k_1 + k_2 \geq 3$ the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition problem becomes NP-complete for general digraphs. Then we prove that for all fixed pairs of integers $k_1, k_2 \geq 1$ there exists a polynomial algorithm for the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition problem for semicomplete digraphs. In Section 4 we prove that all three problems from Theorem 1.6 are polynomially solvable for semicomplete digraphs and in Section 5 we prove that for every fixed integer $k \geq 1$ the $(\delta^+ \geq 1, \delta^- \geq k)$ -problem is polynomially solvable for semicomplete digraphs. In Section 6 we prove that, even for semicomplete digraphs, if we require several properties for each part of a 2-partition, we may obtain problems that are NP-complete, by showing that it is NP-complete to decide whether a given semicomplete digraph D has a 2-partition (V_1, V_2) so that $D(V_i)$ is a strong tournament for $i = 1, 2$. Finally in Section 7 we conclude with some remarks and open problems. In particular we outline why one of our proofs generalizes to digraphs of bounded independence number.

2. Notation, definitions and preliminary results

Notation follows [3] all digraphs considered have neither loops nor parallel arcs. We use the shorthand notation $[k]$ for the set $\{1, 2, \dots, k\}$ and $[i, k]$ for the set $\{i, i + 1, \dots, k\}$. Let $D = (V, A)$ be a digraph with vertex set V and arc set

A and let $v \in V$. If $xy \in A$ is an arc, then we say that x **dominates** y . The **in-degree** of v , denoted by $d_D^-(v)$, is the number of arcs from $V - v$ to v . Similarly the **out-degree**, denoted by $d_D^+(v)$, is the number of arcs from v to $V - v$. Furthermore $N^+(v)$ is the out-neighbours of v , for a set $X \subseteq V$, $N^+[X]$ is the union of the set X and all out-neighbours of the vertices of X and $N^+(X) = N^+[X] - X$ denotes the set of out-neighbours of X that do not belong to X . Definitions for the in-neighbours of vertices and sets are similar. Finally the **minimum out-degree**, respectively **minimum in-degree** of a digraph D is denoted by $\delta^+(D)$, respectively $\delta^-(D)$ and the **minimum semi-degree** of D , denoted by $\delta^0(D)$ is defined as $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$.

The **subdigraph induced** by a set of vertices X in a digraph D , denoted $D\langle X \rangle$, is the digraph with vertex set X and which contains those arcs from D that have both end-vertices in X .

A **path** of a digraph is a sequence of distinct vertices x_1, x_2, \dots, x_l such that $x_i x_{i+1}$ is an arc for every $i \in [l - 1]$. A **cycle** is defined as a path except that $x_1 = x_l$. If C is a cycle of k vertices we say that C is a **k -cycle**. A digraph D is **acyclic** if it does not contain any cycles and a **feedback vertex set** of D is a set $Z \subseteq V$ such that $D - Z$ is acyclic.

A **strong component** of a digraph $D = (V, A)$ is a maximal induced subdigraph $D\langle X \rangle$ with the property that there exists a path from u to v for every ordered pair of vertices $u, v \in X$. A digraph $D = (V, A)$ is **strongly connected** or just **strong** if it has exactly one strong component. If D is not strongly connected, then we can order its strong components D_1, \dots, D_k , $k \geq 2$ such that there is no arc in D which goes from a vertex in D_j to a vertex in D_i where $i < j$. A strong component is **trivial** if it consists on just one vertex. A digraph on at least $k + 1$ vertices is **k -strong** if the digraph $D - X$, obtained by deleting all vertices of X and their incident arcs, remains strongly connected for every subset $X \subseteq V$ with $|X| \leq k - 1$. Furthermore if $Y \subset V$ such that $D(V - Y)$ is not strong, then Y is called a **separator** of D and it is a **minimal separator** if $D - Y'$ is strong for every proper subset Y' of Y .

A **complete digraph** is a digraph in which every pair of distinct vertices induce a directed 2-cycle. A **semicomplete digraph** is a digraph where there is an arc between every pair of vertices and a **tournament** is a semicomplete digraph without 2-cycles. A **transitive tournament** or **acyclic tournament**, is a tournament without any cycles. For such tournaments there exists a unique ordering of the vertices v_1, \dots, v_n such that $v_i v_j$ is an arc if and only if $i < j$. We call the vertex v_1 (v_n) the **source** (**sink**) of T . It is easy to see that for non-strong semicomplete digraphs there exists a unique ordering of its strong components D_1, \dots, D_r such that each vertex of D_i dominates all vertices of D_j if and only if $i < j$. Strong semicomplete digraphs have many cycles as indicated by the following classical result of Moon.¹

Theorem 2.1. [12] Every vertex of a strong semicomplete digraph on n vertices is contained in a k -cycle for every $k \in [3, n]$.

Below we let $D = (V, A)$ be a given digraph and let k be a fixed integer. D is said to be **out-critical** (with respect to k) if $\delta^+(D) = k$ and no subset of its vertices can be removed without decreasing the minimum out-degree of the resulting digraph. Let D be a digraph with minimum out-degree at least k and let X be a subset of its vertices. A set $X' \subseteq V$ is called **X -out-critical** if $X \subseteq X'$, $\delta^+(D\langle X' \rangle) \geq k$ and $\delta^+(D\langle X' - Z \rangle) < k$ for every $\emptyset \neq Z \subseteq X' - X$. Note that if $\delta^+(D\langle X \rangle) \geq k$, then X is the only X -out-critical set in D . By definition, a digraph of minimum out-degree at least k contains at least one X -out-critical set for every subset X of vertices (including the empty set).

3. The complexity of the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition problem

As mentioned in the introduction, the $(\delta^+ \geq 1, \delta^+ \geq 1)$ -partition problem is polynomial for general digraphs. In fact, such a partition exists if and only if D has two vertex disjoint cycles. One direction is clear and if we have a pair of disjoint cycles C_1, C_2 in D , then put all vertices with a directed path to $V(C_1)$ in $D(V - V(C_2))$ together with $V(C_1)$ and the rest together with $V(C_2)$. By a result of McCuaig [11] one can test the existence of two vertex disjoint cycles, and find such a pair if they exist, in a given digraph in polynomial time, implying that $(\delta^+ \geq 1, \delta^+ \geq 1)$ -partition is polynomial for D .

However already for $k_1 + k_2 \geq 3$ the problem becomes NP-complete.

Theorem 3.1. Let k_1, k_2 be positive integers such that $k_1 + k_2 \geq 3$. It is NP-complete to decide whether a given input digraph $D = (V, A)$ has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition.

Proof. Without loss of generality we have $k_1 \leq k_2$. Let $\mathcal{F} = C_1 \wedge C_2 \wedge \dots \wedge C_m$ be an instance of monotone 1-IN-3-SAT with boolean variables x_1, x_2, \dots, x_n (monotone means that no clauses contain negated variables). That is, we seek a truth assignment $t : \{x_1, \dots, x_n\} \rightarrow \{T, F\}^n$ such that each clause C_i will have exactly one variable true (T). This problem is NP-complete [16]. It is easy to see that given an instance \mathcal{F} of monotone 1-IN-3-SAT we can extend it by adding clauses to an equivalent instance \mathcal{F}' in which each variable occurs in at least k_2 clauses. So below we assume that already \mathcal{F} has this property. We construct the digraph $D = D(\mathcal{F})$ from \mathcal{F} as follows: First we take m disjoint copies $H_1^{(1)}, \dots, H_m^{(1)}$ of the complete digraph on k_1 vertices and m copies $H_1^{(2)}, \dots, H_m^{(2)}$ of the complete digraph on k_2 vertices all disjoint and disjoint from the first complete digraphs. Then we add the following new vertices $\{c_i^{(1)} \mid i \in [m]\} \cup \{c_i^{(2)} \mid i \in [m]\} \cup \{w_i \mid i \in [m]\} \cup \{z_i \mid i \in$

¹ As every strong semicomplete digraph contains a spanning strong tournament, Moon's original theorem also holds for semicomplete digraphs.

$[m] \cup \{v_j | j \in [n]\}$. The vertices $c_i^{(1)}, c_i^{(2)}, w_i, z_i$ as well as the two complete digraphs $H_i^{(1)}, H_i^{(2)}$ are associated with the clause C_i for each $i \in [m]$ and the vertex v_j is associated with the variable x_j for each $j \in [n]$. Now we add the following arcs:

- $\{w_i c_i^{(1)} | i \in [m]\} \cup \{c_i^{(2)} z_i | i \in [m]\}$.
- $\max\{k_1 - 2, 0\}$ non-parallel arcs from w_i to $V(H_i^{(1)})$ and an arc from each vertex of $V(H_i^{(1)})$ to w_i .
- $k_1 - 1$ non-parallel arcs from $c_i^{(1)}$ to $V(H_i^{(1)})$.
- $k_2 - 2$ non-parallel arcs from z_i to $V(H_i^{(2)})$.
- $k_2 - 2$ non-parallel arcs from $c_i^{(2)}$ to $V(H_i^{(2)})$ and an arc from each vertex of $V(H_i^{(2)})$ to $c_i^{(2)}$.
- The arcs of the cycle $c_1^{(2)} c_2^{(2)} \dots c_m^{(2)} c_1^{(2)}$.
- If $k_1 > 1$ then add the arcs of the cycle $w_1 w_2 \dots w_m w_1$.
- for each $i \in [m]$, if C_i is given by $C_i = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$, then we add the 12 arcs

$$c_i^{(1)} v_{i_1}, c_i^{(1)} v_{i_2}, c_i^{(1)} v_{i_3}, v_{i_1} c_i^{(1)}, v_{i_2} c_i^{(1)}, v_{i_3} c_i^{(1)}, v_{i_1} c_i^{(2)}, v_{i_2} c_i^{(2)}, v_{i_3} c_i^{(2)}, z_i v_{i_1}, z_i v_{i_2}, z_i v_{i_3}.$$

Clearly D can be constructed in polynomial time, given \mathcal{F} . We claim that there exists a truth assignment to x_1, \dots, x_n such that each clause C_i contains exactly one true literal if and only if D has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition.

We first note some properties of any $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition (V_1, V_2) of D . As all the vertices $w_i, i \in [m]$ have out-degree exactly k_1 in D , they, the vertices $c_i^{(1)}, i \in [m]$ and all the vertices of $H_1^{(1)}, \dots, H_m^{(1)}$ must all belong to the same set V_q and if $k_1 < k_2$ this must be V_1 . Similarly, since each vertex $c_i^{(2)}$ has out-degree exactly k_2 in D , all the vertices $c_i^{(2)}, i \in [m]$, all the vertices $z_i, i \in [m]$ and all vertices of the digraphs $H_1^{(2)}, \dots, H_m^{(2)}$ must belong to the same set V_p . It is easy to see that we must have $p \neq q$ as otherwise only the variable vertices v_j can be in the set V_{3-q} but there are no arcs between the variable vertices. Now the fact that the vertices $z_i, c_i^{(1)}$ belong to the different sets of the partition and the fact that z_i has exactly $k_2 + 1$ out-neighbours in D implies that exactly two of the vertices $v_{i_1}, v_{i_2}, v_{i_3}$ must belong to V_p to give z_i out-degree k_2 in $D(V_p)$ and the last will belong to V_q to give $c_i^{(1)}$ out-degree k_1 in $D(V_q)$.

Now we can finish the proof easily. Suppose first that D has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition (V_1, V_2) . If $k_1 < k_2$ we have all $c_i^{(1)}$'s in V_1 . If $k_1 = k_2$ then by renaming if necessary, we again have that V_1 contains all $c_i^{(1)}$'s and in both cases each such vertex has exactly one of its neighbours among the variable vertices in V_1 . Thus if we assign the value true to $x_r, r \in [n]$ if the vertex v_r is in V_1 , then we obtain a truth assignment that sets exactly one literal true for each clause. Conversely, given a truth assignment ϕ that sets exactly one literal true for each clause, we obtain the desired partition by letting V_1 consist of all vertices of $H_1^{(1)}, \dots, H_m^{(1)}$, all vertices $c_i^{(1)}, i \in [m]$, all vertices $w_i, i \in [m]$ and all those variable vertices v_r for which the corresponding variable x_r is set true by ϕ . It follows from the observations above and the fact that each variable vertex $v_h, h \in [n]$ has at least k_2 out-neighbours in each of the sets $\{c_1^{(1)}, \dots, c_m^{(1)}\}, \{c_1^{(2)}, \dots, c_m^{(2)}\}$ that $(V_1, V - V_1)$ is a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition of V . \square

As mentioned earlier in Theorem 1.4, Lichiardopol proved in [10] that $\delta^+(T) \geq \frac{k_1^2 + 3k_1 + 2}{2} + k_2$ is sufficient to guarantee a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition for tournaments. This is also true for semicomplete digraphs. We now describe a polynomial algorithm to find such a partition if one exists. We need a few lemmas. For fixed integers k_1, k_2 a vertex v of a semicomplete digraph D is said to be **out-dangerous** (with respect to k_1 and k_2) if $d^+(v) < (k_1 + k_2) - 1$.

Lemma 3.2. *Let k_1, k_2 be fixed integers and let S be a semicomplete digraph. Then the number of out-dangerous vertices of S is at most $2(k_1 + k_2) - 3$.*

Proof. Let X be the set of out-dangerous vertices of S . Then the number of arcs in the semicomplete digraph $S(X)$ is at most $|X|(k_1 + k_2 - 2)$ and at least $\frac{|X|(|X|-1)}{2}$ implying that $|X| \leq 2(k_1 + k_2) - 3$. \square

In [10] it was shown that for every fixed k any out-critical set is of bounded size. With a small modification of this proof we can bound the size of X -out-critical sets for any fixed set X .

Lemma 3.3. *Let S be a semicomplete digraph with minimum out-degree at least k and let $X \subseteq V(S)$. Then every X -out-critical set X' of S will have size at most $\frac{k^2 + 3k + 2}{2} + |X|$.*

Proof. Suppose that for some set $X \subset V$ there is an X -out-critical set X' of size at least $\frac{k^2 + 3k + 2}{2} + |X| + 1$. Consider the semicomplete digraph $S' = S(X')$. Let M be the set of vertices that have out-degree exactly k in S' and let $m = |M|$.

As each $v \in M$ has out-degree k in the semicomplete digraph $S'(M)$ we have

$$|N_{S'}^+[M]| \leq m + mk - \frac{m(m-1)}{2} = -\frac{m^2}{2} + \left(\frac{3}{2} + k\right)m =: P(m).$$

Now $P(m)$ has global maximum at $(3/2 + k)$ and maximum for m integer at $k + 1$ and $k + 2$ with $P(k + 1) = P(k + 2) = \frac{k^2 + 3k + 2}{2}$. Hence as $|X'| > \frac{k^2 + 3k + 2}{2} + |X|$ there exists a vertex $u \in X' - (N_{S'}^+[M] \cup X)$ such that $\delta^+(S'(X' - u)) \geq k$. But then the set $Z = \{u\}$ is contained in $X' - X$ and $\delta^+(S(X' - Z)) \geq k$, contradicting the fact that X' is an X -out-critical set in S . \square

We are now ready to prove the existence of a polynomial algorithm for deciding whether a given semicomplete digraph has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition.

Theorem 3.4. *For every fixed pair of integers k_1, k_2 there exists a polynomial algorithm that either constructs a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition of a given semicomplete digraph S or correctly outputs that none exist.*

Proof. Let (O_1, O_2) be a given partition of the out-dangerous vertices of S . Let $X \subseteq V - O_2$ be a set containing O_1 such that $|X| \leq \frac{k_1^2 + 3k_1 + 2}{2} + |O_1|$ and $\delta^+(S(X)) \geq k_1$ (if no such set exists, we stop considering the pair (O_1, O_2)). The following subalgorithm \mathcal{B} will decide whether there exists a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition (V_1, V_2) with $X \subseteq V_1, O_2 \subseteq V_2$: Starting from the partition $(V_1, V_2) = (X, V - X)$, and moving one vertex at a time, the algorithm will move vertices of $V_2 - O_2$ which have $d_{S(V_2)}^+(v) < k_2$ to V_1 . If, at any time, this results in a vertex $v \in O_2$ having $d_{S(V_2)}^+(v) < k_2$, or $V_2 = \emptyset$, then there is no $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition with $X \subseteq V_1$ and $O_2 \subseteq V_2$ and the algorithm \mathcal{B} terminates. Otherwise \mathcal{B} will terminate with $O_2 \subseteq V_2 \neq \emptyset$ and hence it has found an $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition (V_1, V_2) with $O_i \subseteq V_i, i = 1, 2$.

The correctness of \mathcal{B} follows from the fact that we only move vertices that are not dangerous and each such vertex has at least $k_1 + k_2 - 1$ out-neighbours in S . Hence, as the vertex that we move does not have k_2 out-neighbours in V_2 , it must have at least k_1 out-neighbours in V_1 , so $\delta^+(S(V_1)) \geq k_1$ will hold throughout the execution of \mathcal{B} .

By Lemma 3.2, the number of out-dangerous vertices is at most $2(k_1 + k_2) - 3$ and hence the number of (O_1, O_2) -partitions of the set of out-dangerous vertices is at most $2^{2(k_1 + k_2) - 3}$ which is a constant because k_1, k_2 are fixed. Furthermore, by Lemma 3.3, the size of every O_1 -critical set is also bounded by a function of k_1 and hence for each (O_1, O_2) -partition there is only a polynomial number of O_1 -critical sets that are disjoint from O_2 . Thus we obtain the desired polynomial time algorithm by running the subalgorithm \mathcal{B} for all possible partitions (O_1, O_2) of the out-dangerous vertices and all possible choices of sets X with $O_1 \subseteq X$ and $|X| \leq \frac{k_1^2 + 3k_1 + 2}{2} + |O_1|$. Note that we do not need to check whether X is O_1 -out-critical, we just check all possible supersets of O_1 of size at most $\frac{k_1^2 + 3k_1 + 2}{2} + |O_1|$. \square

As k_1, k_2 are fixed, the running time of the algorithm above is $O(n^{g(k_1, k_2)})$ for some (quadratic) polynomial g . We made no attempt to improve the running time above and it is natural to ask whether there exists an FPT algorithm for the problem.

Problem 3.5. *Does there exist a function $f(k_1, k_2)$ and a constant c such that one can decide, for a given semicomplete digraph S and pair of integers k_1, k_2 whether S has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition in time $O(f(k_1, k_2)n^c)$?*

We saw above that we could solve the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition problem in polynomial time using the fact that the number of out-dangerous vertices are bounded for semicomplete digraphs. It is natural to ask whether a similar approach can be used for the $(\delta^+ \geq k_1, \delta^- \geq k_2)$ -, $(\delta^0 \geq k_1, \delta^- \geq k_2)$ - and $(\delta^0 \geq k_1, \delta^0 \geq k_2)$ -partition problem. This however is not the case. There is no natural way to define dangerous vertices in these cases, as it depends on whether we consider a vertex with respect to the first property or with respect to the second property. Given a partition (V_1, V_2) of a digraph D where $\delta^+(D(V_1)) \geq k_1$ and $\delta^-(D(V_2)) < k_2$, a vertex with in-degree less than k_2 in V_2 might not have k_1 out-neighbours in V_1 and hence cannot be moved directly to V_1 without decreasing the minimum out-degree of V_1 .

4. 2-partitions where both constants are one

In this section we will consider the three problems from Theorem 1.6 and prove that there is a polynomial algorithm for each of these when the input is a semicomplete digraph. It is clear that for all three problems the existence of a pair of disjoint cycles is a necessary condition. It is easy to check whether a semicomplete digraph has a pair of disjoint cycles: D has such cycles if and only if it has disjoint cycles C_1, C_2 where $|V(C_i)| \leq 3$ for $i \in [2]$ (every cycle of length 4 or more in a semicomplete digraph induces a semicomplete digraph with a shorter cycle).

Lemma 4.1. *A semicomplete digraph S with $\delta^-(S) \geq 1$ has a $(\delta^0 \geq 1, \delta^- \geq 1)$ -partition if and only if it has a pair of disjoint cycles. Furthermore, such a partition can be found in polynomial time when it exists.*

Proof. Let S be a semicomplete digraph with $\delta^-(S) \geq 1$ and disjoint cycles C_1 and C_2 . If $S_1 = S(V - V(C_1))$ has minimum in-degree at least 1 then $(V(C_1), V - V(C_1))$ is a $(\delta^0 \geq 1, \delta^- \geq 1)$ -partition so let x_1 be a vertex in $V - C_1$ with in-degree 0 in S_1 . Similarly either $(V(C_2), V - V(C_2))$ is a $(\delta^0 \geq 1, \delta^- \geq 1)$ -partition or there is a vertex x_2 in $S_2 = S(V - C_2)$ with in-degree 0. As $\delta^-(S) \geq 1$ the vertex x_i must have an in-neighbour on C_i for $i = 1, 2$, implying that $x_1 \neq x_2$. It follows from the choice of x_1, x_2 above that $x_1, x_2 \notin V(C_1) \cup V(C_2)$ and hence both have in-degree 0 in the semicomplete digraph $S(V - V(C_1) - V(C_2))$, contradiction. It follows from the proof that it will always be the case that $(V(C_i), V - V(C_i))$ is a $(\delta^0 \geq 1, \delta^- \geq 1)$ -partition for $i = 1$ or 2 , implying that we can find the desired partition in polynomial time. \square

Theorem 4.2. *A semicomplete digraph S has a $(\delta^+ \geq 1, \delta^- \geq 1)$ -partition if and only if it has a pair of disjoint cycles. Furthermore, such a partition can be found in polynomial time when it exists.*

Proof. We prove by induction on the number of vertices that the presence of two disjoint cycles guarantee the existence of the desired partition. In the base case S consists of two disjoint induced cycles so they each have length at most 3 and they themselves form the desired partition. Hence we proceed to the induction step and assume that S is a semicomplete digraph with disjoint cycles C_1 and C_2 . By Lemma 4.1, it suffices to consider the case when S contains a vertex x of in-degree 0. Clearly $S - x$ also has two disjoint cycles (as x is not on any cycle) so by induction it has a $(\delta^+ \geq 1, \delta^- \geq 1)$ -partition (V'_1, V'_2) and now $(V'_1 + x, V_2)$ is the desired partition of S . From the argument above it is easy to obtain a polynomial algorithm to find the desired partition: continue to remove vertices of in-degree 0 until the remaining semicomplete digraph S' has $\delta^-(S') \geq 1$. Then find a $(\delta^0 \geq 1, \delta^- \geq 1)$ -partition (V'_1, V'_2) of S' using the algorithm from Lemma 4.1 and finally return the $(\delta^+ \geq 1, \delta^- \geq 1)$ -partition $(V - V'_2, V'_2)$. \square

Theorem 4.3. *There exists a polynomial algorithm that either finds a $(\delta^0 \geq 1, \delta^- \geq 1)$ -partition of a semicomplete digraph or correctly outputs that none exist.*

Proof. If D has a vertex of in-degree 0, then it has no $(\delta^0 \geq 1, \delta^- \geq 1)$ -partition and otherwise it follows from Lemma 4.1 that the partition exists and can be found in polynomial time. \square

For the $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition problem the existence of disjoint cycles is not sufficient in general, but the existence of a pair of complementary cycles is. Two cycles C_1, C_2 of a digraph D are **complementary** if they are disjoint and cover all vertices of D . Reid [13] proved that every 2-strong tournament on at least 8 vertices has a pair of complementary cycles. In [7] Guo and Volkmann proved that every 2-strong semicomplete digraph of at least 8 vertices has a pair of complementary cycles. Bang-Jensen and Nielsen [6] proved that checking the existence of complementary cycles of semicomplete digraphs and finding such a pair if they exist can be done in polynomial time. Notice that if C_1, C_2 is a pair of complementary cycles of a semicomplete digraph, then C_1 (or C_2) is allowed to be a 2-cycle. Now we are ready to prove the following.

Theorem 4.4. *There exists a polynomial algorithm that either finds a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition of a semicomplete digraph or correctly outputs that none exist.*

Proof. Suppose first that S is not strong and let D_1, \dots, D_r , $r \geq 2$, be the strong components. If D_1 or D_r is a trivial component, then clearly there is no $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition, so we may assume that $\min\{|D_1|, |D_r|\} \geq 2$ and that $r \geq 3$ or we are done. If $|D_i| \geq 2$ for some $i \in [2, r - 1]$, then $(V(D_i), V - V(D_i))$ is a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition. Hence we may assume that there are distinct vertices d_2, \dots, d_{r-1} such that $D_i = \{d_i\}$ for $i \in [2, r - 1]$. Now it is easy to see that there is a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition if and only if at least one of the following holds:

- D_1 has a $(\delta^- \geq 1, \delta^- \geq 1)$ -partition and D_r has a $(\delta^+ \geq 1, \delta^+ \geq 1)$ -partition.
- D_1 has a $(\delta^0 \geq 1, \delta^- \geq 1)$ -partition.
- D_r has a $(\delta^+ \geq 1, \delta^0 \geq 1)$ -partition.

For each of these problems we already established polynomial algorithms so from now on we may assume that S is strong. If $n < 8$ we just check all possible partitions, so assume $n \geq 8$. First check whether S has a pair of complementary cycles, using the algorithm of [6] and output the 2-partition induced by these if they exist. Hence we may now assume that S does not contain a pair of complementary cycles and that it is not 2-strong by the aforementioned results of [7,13]. Note that, because every strong semicomplete digraph is Hamiltonian (by Theorem 2.1), the fact that S has no pair of complementary cycles implies that S must have at least 3 disjoint cycles if it has a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition.

Let x be a separating vertex of S and let D_1, \dots, D_r , $r \geq 2$ be the strong components of $S - x$. As S is strong the vertex x dominates at least one vertex in D_1 and is dominated by at least one vertex in D_r . Now it is easy to either find a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition or deduce the following (in each case, if the claim does not hold, then a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition can be constructed easily):

- (i) If $r \geq 3$ then there are distinct vertices d_2, \dots, d_{r-1} such that $D_i = \{d_i\}$ for all $i \in [2, r - 1]$.

- (ii) If D_1 is non-trivial, then x only dominates vertices y of D_1 that are separators of D_1 and only the initial strong component of $D_1 - y$ (denoted D_{11}) can be non-trivial. Furthermore, either $r = 2$ or $xd_2 \notin A$, implying that $d_2x \in A$.
- (iii) Similarly, if D_r is non-trivial, then x is only dominated by vertices z of D_r that are separators of D_r and only the terminal strong component of $D_r - z$ (denoted D_{rs}) can be non-trivial. Furthermore, either $r = 2$ or $d_{r-1}x \notin A$, implying that $xd_{r-1} \in A$.

Case 1) $r = 2$ and $|D_1|, |D_2| \geq 2$.

If x has arcs to and from D_i for $i = 1$ or $i = 2$ then $(V(D_i) \cup \{x\}, V(D_{3-i}))$ is a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition so we can assume that x dominates all vertices of D_1 and is dominated by all vertices of D_2 . As $n \geq 8$ we have $\max\{|D_1|, |D_2|\} \geq 4$, so, by Theorem 2.1, for some $i \in [2]$ there exists a vertex $v \in D_i$ such that $D_i - v$ is strong. Now we see that (V_1, V_2) is a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition if we let $V_1 = V(D_i) - v$ and $V_2 = V - V_1$.

Case 2a) $r \geq 2$ and $|D_1| = |D_r| = 1$. In this case the vertex x is on all cycles of S so it is a ‘no’-instance.

Case 2b) $r \geq 2$, $\min\{|D_1|, |D_2|\} = 1$ and $\max\{|D_1|, |D_r|\} \geq 2$.

By reversing all arcs if necessary, we may assume $D_1 = \{d_1\}$ and $|D_r| \geq 2$. Assume that (V_1, V_2) is a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition of S with $x \in V_1$. Then as x is the only in-neighbour of d_1 , d_1 also belongs to V_1 . Continuing this way we see that $\{x, d_1, \dots, d_{r-1}\} \subseteq V_1$. As $xd_{r-1} \in A$ any $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition (V_1, V_2) will have vertices of D_r in both V_1 and V_2 , and $V_2 \subset D_r$. Suppose first that x has no in-neighbour among $\{d_1, d_2, \dots, d_{r-1}\}$. Then it is easy to see that the semi-complete digraph S' obtained by deleting d_1, \dots, d_{r-1} and adding an arc from x to each vertex of D_r will have a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition if and only if S does. Thus we can solve the problem by calling the algorithm recursively on S' . Hence we may assume that x has at least one in-neighbour among $\{d_1, d_2, \dots, d_{r-1}\}$. Note that if D_r does not have two disjoint cycles, then D has no set of 3-disjoint cycles and hence is a no-instance as we already know it has no pair of complementary cycles. Thus we may assume that D_r has a pair of disjoint cycles and now it follows from Theorem 4.3 (applied to D_r with all arcs reversed) that, in polynomial time we can find a $(\delta^+ \geq 1, \delta^0 \geq 1)$ -partition (V'_1, V'_2) of D_r . Now it is easy to check that $(V - V'_2, V'_2)$ is a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition of S .

Case 3) $r > 2$ and $|D_1|, |D_r| \geq 2$.

If there are indices $1 < i < j < r$ so that $xd_i, d_jx \in A$, then $(\{x, d_i, d_{i+1}, \dots, d_j\}, V - \{x, d_i, d_{i+1}, \dots, d_j\})$ is a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition. Hence, by (ii) and (iii) we may assume that there is an index $2 \leq f < r - 1$ such that x has no arc to $\{d_2, \dots, d_f\}$ and $\{d_{f+1}, \dots, d_{r-1}\}$ has no arc to x . Fix a vertex $y \in V(D_1)$ such that $xy \in A$ and a vertex $z \in V(D_r)$ such that $zx \in A$. By (ii) and (iii) S contains the 3-cycles $C_3 = \{x, y, d_2\}$ and $C'_3 = \{x, d_{r-1}, z\}$. If the initial component D_{11} of $D_1 - y$ satisfies $|D_{11}| \geq 2$, then we have the $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition $(V(C_3), V - V(C_3))$. Similarly if the terminal component D_{rs} of $D_r - z$ satisfies $|D_{rs}| \geq 2$, then we have a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition $(V(C'_3), V - V(C'_3))$. So assume $|D_{11}| = |D_{rs}| = 1$ and hence by (ii) and (iii) that the strong components of $D_1 - y$, respectively $D_r - z$ all have size one. Denote the vertices of these by d_{11}, \dots, d_{1p} and d_{r1}, \dots, d_{rs} , respectively. Since $D_1 - d_{1j}$ is strong when $j \notin \{1, p\}$, it follows from (ii) that the only possible out-neighbours of x in $V(D_1) - y$ are d_{11} and d_{1p} . Similarly, (iii) implies that the only possible in-neighbours of x in $V(D_r) - z$ are d_{r1} and d_{rs} . If $xd_{1p} \in A$ and $d_{1j}y \in A$ for some $1 \leq j < p$, then $(\{x, d_{1p}, d_2\}, V - \{x, d_{1p}, d_2\})$ is a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition. If $xd_{11} \in A$ and $yd_{1i} \in A$ for some $1 < i \leq p$, then again we easily get a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition. So either we find the desired partition or conclude that following holds: if $xd_{1p} \in A$ then d_{1p} is the only in-neighbour of y in D_1 and if $xd_{11} \in A$, then d_{11} is the only out-neighbour of y in D_1 . By similar observations we either find the desired partition or conclude that if $d_{r1}x \in A$, then d_{r1} is the only out-neighbour of z in D_r and if $d_{rs}x \in A$, then d_{rs} is the only in-neighbour of z in D_r . This implies that S does not have 3 disjoint cycles, because $D - \{x, y, z\}$ is acyclic and none of $D - D_1, D - D_r$ have two disjoint cycles. But then it cannot have a $(\delta^0 \geq 1, \delta^0 \geq 1)$ -partition, since we have already assumed that S has no pair of complementary cycles.

This completes the description of the algorithm. For Case 2b, notice that each of the recursive calls (if any) will be on semicomplete digraphs which have at least 2 vertices less than the current one. Hence in at most $O(n)$ calls the algorithm will terminate and thus the algorithm runs in polynomial time. \square

5. 2-partitions when one constant is 1 and the other at least 2

When one of the two sides of the partition must have in-degree or out-degree at least k for some $k \geq 2$, we need more work to establish a polynomial algorithm. We begin with a Lemma which could be of independent interest. Below we use the shorthand notation $d_X^+(v)$ ($d_X^-(v)$) for $d_{D(X)}^+(v)$ ($d_{D(X)}^-(v)$), where X is a subset of the vertices of D and $v \in X$.

Lemma 5.1. *Let $k \geq 1$ be a fixed integer. Then there exists a polynomial algorithm for the following problem: let $S = (V, A)$ be a semicomplete digraph and X_1, X_2 disjoint subsets of V such that*

- (a) $V - X_1 - X_2$ induces a transitive tournament.
- (b) If there is a vertex v of X_1 such that $d_{X_1}^+(v) = 0$ then v is dominated by at most $k - 1$ vertices of $V - X_1 - X_2$.

decide whether S has a $(\delta^+ \geq 1, \delta^- \geq k)$ -partition (V_1, V_2) with $X_i \subset V_i$ for $i \in [2]$ and find such a partition when it exists.

Proof. We start by setting $V'_1 = X_1$ and $V'_2 = X_2$. Throughout this proof we let $Z = V - V'_1 - V'_2$ and say that a $(\delta^+ \geq 1, \delta^- \geq k)$ -partition (V_1, V_2) is **good** if $V'_i \subset V_i$ where V'_1, V'_2 are the current sets obtained by the algorithm. The algorithm will move vertices of Z to V'_1 and V'_2 until we either find such a good partition or we can conclude that none exists.

Assume first that there is a vertex $u \in V'_1 \cup Z$ ($u \in V'_2 \cup Z$) such that $d^+_{V'_1 \cup Z}(u) = 0$ ($d^-_{V'_2 \cup Z}(u) < k$). If $u \in V'_1$ then clearly there is no good $(\delta^+ \geq 1, \delta^- \geq k)$ -partition and we can stop. If $u \in Z$ then there can only exist a good $(\delta^+ \geq 1, \delta^- \geq k)$ -partition if $u \in V'_{3-i}$ and we may move u to V'_{3-i} . Continue moving vertices from the current Z which are forced into one of the sides of any good partition (as above) until we either have $Z = \emptyset$, in which case we just check whether (V'_1, V'_2) is a $(\delta^+ \geq 1, \delta^- \geq k)$ -partition, or we have $Z \neq \emptyset$, $V'_1 \cup Z$ induces a semicomplete digraph with $\delta^+ \geq 1$ and $V'_2 \cup Z$ induces a semicomplete digraph with $\delta^- \geq k$. If $(V'_1, V'_2 \cup Z)$ is a $(\delta^+ \geq 1, \delta^- \geq k)$ -partition we are done so let v be the unique vertex of out-degree zero in $S(V'_1)$.

So far we have only moved a vertex to $V'_i - X_i$ if it was forced to belong to that set in any good partition. Assume now that there exists a good $(\delta^+ \geq 1, \delta^- \geq k)$ -partition (V_1, V_2) and let $W = Z \cap V_1$. Furthermore let w_1, \dots, w_m be the unique acyclic ordering of the vertices of W . Then w_m must dominate at least one vertex of V'_1 and the vertex v of V'_1 must dominate a vertex of W . Let w_j be the vertex of W with the highest index such that v dominates w_j . If $j > 1$ then $(V_1 \setminus \{w_1, \dots, w_{j-1}\}, V_2 \cup \{w_1, \dots, w_{j-1}\})$ is also a good $(\delta^+ \geq 1, \delta^- \geq k)$ -partition since $\delta^-(S(V'_2 \cup (Z - W))) \geq k$ and w_1, \dots, w_{j-1} have no in-neighbours among w_j, \dots, w_m . Hence we can restrict the search for a good $(\delta^+ \geq 1, \delta^- \geq k)$ -partition (V_1, V_2) to one where v only dominates the source of the transitive tournament $V_1 \cap Z$. Furthermore v is either the (original) sink of X_1 or a vertex added to V'_1 because it had less than k in-neighbours in $V'_2 \cup Z$. In any case, by (b), the vertex v is dominated by at most $k - 1$ vertices of Z and hence for any subset of Z of size at least $k + 1$, v will dominate at least two vertices. This implies that if there is any good partition (V_1, V_2) then there is one where $|V_1| \leq |V'_1| + k$. We can check for such a partition by looking at all possible subsets W of Z of size at most k with the further condition that v only dominates the unique vertex of in-degree 0 in $S(W)$. There are $O(n^{k+1})$ such subsets, implying that our algorithm is polynomial. \square

Theorem 5.2. *There exists a polynomial algorithm that either finds a $(\delta^+ \geq 1, \delta^- \geq 2)$ -partition of a semicomplete digraph S or correctly outputs that none exist.*

Proof. Below we make no attempt to optimize the running time. There are $O(n^3)$ cycles of length at most 3 in S . Let h denote the number of such cycles and order them as C_1, \dots, C_h . For each $i \in [h]$ or until we find a solution we proceed as follows. Start by letting $V_1 = V(C_i)$, where C_i is the next cycle to consider, and let $V_2 = V - V_1$. Now move vertices of in-degree at most one in $D(V_2)$ to V_1 until either $V_2 = \emptyset$ in which case there is no $(\delta^+ \geq 1, \delta^- \geq 2)$ -partition (V_1, V_2) with $C_i \subseteq V_1$ (and we go to the next cycle C_{i+1}) or V_2 induces a semicomplete digraph with minimum in-degree at least 2. If $\delta^+(S(V_1)) \geq 1$ then we have found a $(\delta^+ \geq 1, \delta^- \geq 2)$ -partition (V_1, V_2) , so assume that v has out-degree zero in $S(V_1)$.

Let B be the set of vertices in V_2 that have in-degree at most 4 in $S(V_2)$. If there exists a cycle C' of length at most 3 in $S(V_2)$ such that $V_2 - C'$ still induces a semicomplete digraph with minimum in-degree 2, then $(V_1 \cup C', V_2 - C')$ is a $(\delta^+ \geq 1, \delta^- \geq 2)$ -partition, because v has at most one in-neighbour on C' (as it was moved at some point) so it will dominate at least one vertex on C' . Hence we may assume that for each 3-cycle C' of $S(V_2)$ there is a vertex of B that has an in-neighbour in C' . But then it follows from Theorem 2.1 that every cycle of $S(V_2)$ contains an in-neighbour of B . This implies that $F = N^-[B]$ is a feedback vertex set of $S(V_2)$, that is, $S(V_2) - F$ is a transitive tournament.

If T has any $(\delta^+ \geq 1, \delta^- \geq 2)$ -partition $(\widehat{V}_1, \widehat{V}_2)$ with $V_1 \subset \widehat{V}_1$, then $F_i = \widehat{V}_i \cap F$, $i = 1, 2$, induces a partition of the vertices of F . Hence to find a $(\delta^+ \geq 1, \delta^- \geq 2)$ -partition we need only check if S has a partition $(\widehat{V}_1, \widehat{V}_2)$ where $V_1 \cup F_1 \subset \widehat{V}_1$ and $F_2 \subset \widehat{V}_2$ for every partition F_1, F_2 of F (possibly with $F_i = \emptyset$ for $i = 1$ or $i = 2$).

To realize that this can be done in polynomial time, notice that there are at most 9 vertices in B and since each of these has in-degree at most 4 in V_2 the size of F is at most² 45. Hence there are at most 2^{45} partitions of F to check. For each partition (F_1, F_2) of F we can use the algorithm of Lemma 5.1 with $X_1 = F_1 \cup V_1$ and $X_2 = F_2 \cup V_2$. If none of these partitions of F result is a solution, we move to the next cycle C_{i+1} . \square

With a bit more effort we can extend the theorem to any fixed lower bound on the in-degree in $S(V_2)$.

Theorem 5.3. *For every fixed integer $k \geq 1$ there exists a polynomial algorithm that either constructs a $(\delta^+ \geq 1, \delta^- \geq k)$ -partition of a semicomplete digraph S or correctly outputs that none exist.*

Proof. Again we order the set of cycles of length at most 3 as C_1, \dots, C_h , where $h \in O(n^3)$ and consider these one by one until we either find a solution or there are no more cycles to try. When considering C_i we start by letting $V_1 = C_i$ and

² This estimate is not precise. In fact $|F| \leq 30$, but the crude estimate suffices for our argument.

$V_2 = V - V_1$. Then we move vertices of in-degree less than k in $D(V_2)$ to V_1 until either $V_2 = \emptyset$, in which case there is no partition with $C_i \subseteq V_1$, or the process stops when V_2 induces a semicomplete digraph with minimum in-degree at least k . Now if V_1 induces a semicomplete digraph with minimum out-degree at least 1, we have found a $(\delta^+ \geq 1, \delta^- \geq k)$ -partition (V_1, V_2) , so assume that v has out-degree zero in $S(V_1)$.

Let $p = \lceil \frac{k}{2} \rceil$ and let B be the set of vertices of V_2 that have in-degree at most $k + 3p - 1$. If $S(V_2)$ has a collection of p disjoint cycles C'_1, \dots, C'_p , each of length at most 3, such that $S(V_2 - \cup_{i=1}^p V(C'_i))$ has minimum in-degree at least k , then we obtain a $(\delta^+ \geq 1, \delta^- \geq k)$ -partition by adding the vertices of C'_1, \dots, C'_p to V_1 and removing them from V_2 (as in the previous proof the vertex v will have at least one out-neighbour among the newly added vertices). The existence of such a collection of cycles can be determined by trying all subsets of V_2 on at most $3p$ vertices. Hence we may assume below that $Y = N^-[B]$ intersects all sets of p disjoint cycles in V_2 . Now there are at most $p - 1$ disjoint cycles of length at most 3 in $V_2 - Y$ and hence removing some subset Y' of at most $3(p - 1)$ extra vertices of $V_2 - Y$ we obtain a transitive tournament. Thus $F = Y \cup Y'$ is a feedback vertex set of $S(V_2)$.

The rest of the proof is similar to that of Theorem 5.2. The only difference is that instead of moving one cycle of length at most 3 we move sets of at most p disjoint cycles, each of which have length at most 3. If no good set of p cycles was found, then we found a small feedback vertex set F and we try each partition of the feedback vertex set F . To finish the proof we only need to argue that the size of F is bounded by a function in k . This follows from the following crude estimate which suffices for our needs: the in-degree of the vertices in B is at most $\frac{5k}{2} + 2$ so there are at most $5k + 5$ vertices in B and hence $|Y| \leq (5k + 5) + (5k + 5)(\frac{5k}{2} + 2) \leq \frac{25k^2 + 55k + 30}{2}$. We also have that $|Y'| \leq 3(p - 1) \leq \frac{3k}{2}$ so $|F| \leq \frac{25k^2 + 58k + 30}{2}$. \square

We cannot directly use the same approach if we want a $(\delta^0 \geq 1, \delta^- \geq k)$ -partition. This is because, after moving vertices that are forced to be in V_1 the semicomplete digraph $S(V_1)$ may have both a vertex v with out-degree 0 and another vertex v' with in-degree 0. For v we still know that it has at most $k - 1$ in-neighbours in V_2 , but for v' we have no control of its number of out-neighbours in V_2 , so we cannot guarantee that we will add an in-neighbour of v' when we add any set of p cycles of length at most 3.

Consider the case where we want a $(\delta^+ \geq k_1, \delta^- \geq k_2)$ -partition when both k_1 and k_2 are at least 2. The following would be natural generalization of the proof technique used above: first construct the (polynomial) list of all k_1 -out-critical subgraphs X_1, X_2, \dots, X_q . Then starting from $V_1 = X_i$ and $V_2 = V - X_i$ first move all vertices with in-degree less than k_2 in $S(V_2)$ to V_1 and then try to move a small (as a function of k_1) subset of V_2 to V_1 so that we obtain a solution. Unfortunately this approach does not work as, already for $k_1 = k_2 = 2$, there exist infinitely many tournaments with minimum in-degree 2 that contain no subtournament of minimum out-degree 2.

Despite the seeming need for new proof techniques, based on the evidence from the results of this paper, we believe that the following holds.

Conjecture 5.4. *For every pair of fixed integers $k_1, k_2 \geq 2$ there exist polynomial algorithms for deciding the following for a given semicomplete digraph S :*

- whether S has a $(\delta^+ \geq k_1, \delta^- \geq k_2)$ -partition,
- whether S has a $(\delta^+ \geq k_1, \delta^0 \geq k_2)$ -partition,
- whether S has a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$ -partition.

In the proof of Lemma 5.1 we used the fact that k is fixed to obtain a polynomial algorithm \mathcal{A} . If k is part of the input the running time of \mathcal{A} will no longer be polynomial in the size of the input.

Problem 5.5. *What is the complexity of the $(\delta^+ \geq 1, \delta^- \geq k)$ -partition problem when the input is a semicomplete digraph and a positive integer k ?*

6. 2-partitions of semicomplete digraphs into tournaments

We now show that even for semicomplete digraphs we may obtain very difficult 2-partition problems if we pose the extra condition that each part of the 2-partition must induce a tournament. It follows from the polynomial algorithm from [6] for finding complementary cycles in semicomplete digraphs that without the requirement that each V_i induces a tournament, the problem below is polynomially solvable.

Theorem 6.1. *It is NP-complete to decide whether a given semicomplete digraph D has a 2-partition (V_1, V_2) such that $D(V_i)$ is a strong tournament.*

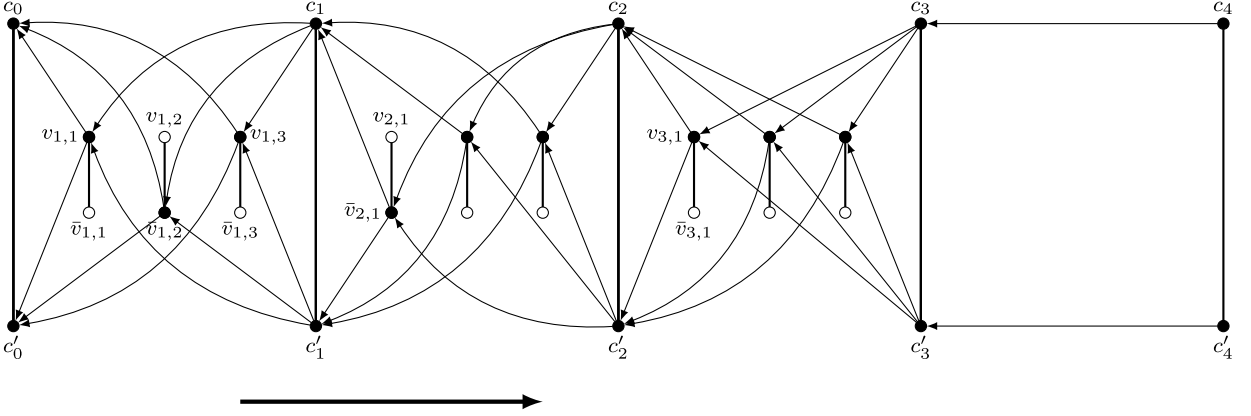


Fig. 1. An example of the digraph $D = D(\mathcal{F})$ where $\mathcal{F} = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$. For clarity only the most important arcs are shown. The big arrow indicates that, except for the 2-cycles between clause vertices and between vertices corresponding to literals over the same variable, all arcs not shown go from left to right.

Proof. ³

Let $\mathcal{F} = C_1 \wedge C_2 \wedge \dots \wedge C_m$ be an instance of not-all-equal 3-SAT (NAE-3-SAT) over the set of n boolean variables x_1, \dots, x_n . That is, we seek a truth assignment $t : \{x_1, x_2, \dots, x_n\} \rightarrow \{T, F\}^n$ so that each clause has at least one true literal and at least one false literal. This problem is NP-complete [16].

We construct a semicomplete digraph $D = D(\mathcal{F})$ which has a 2-partition (V_1, V_2) such that $D(V_i)$ is a strong tournament if and only if \mathcal{F} is a ‘Yes’-instance of NAE-3-SAT.

The vertex set of D is given by

$$V(D) = \{c_0, c_1, \dots, c_{m+1}\} \cup \{c'_0, c'_1, \dots, c'_{m+1}\} \cup \bigcup_{j=1}^m \{v_{j,1}, \bar{v}_{j,1}, v_{j,2}, \bar{v}_{j,2}, \dots, v_{j,n}, \bar{v}_{j,n}\}$$

Here the vertices c_j, c'_j correspond to the clause C_j for $j \in [m]$ and the vertices $\{v_{1,1}, \dots, v_{m,i}\}$, respectively $\{\bar{v}_{1,1}, \dots, \bar{v}_{m,i}\}$ correspond to the literal x_i , respectively the literal \bar{x}_i .

We first define the arc set A' of a semicomplete digraph D' on the same vertex set as D and then describe how to obtain the arc set A of D by reversing certain arcs that will correspond closely to the clauses of \mathcal{F} .

The arc set $A' = A(D')$ is defined as follows:

- For all $0 \leq j < j' \leq m + 1$, A' contains the arcs $c_j c_{j'}, c'_j c'_{j'}$, except when $j = m$ (and $j' = m + 1$) where we have $c_{m+1} c_m, c'_{m+1} c'_m \in A'$.
- There is a 2-cycle between c_i and c'_j for all $0 \leq i, j \leq m + 1$.
- For every $i \in [n]$ the vertices $\{v_{1,i}, \dots, v_{m,i}\} \cup \{\bar{v}_{1,i}, \dots, \bar{v}_{m,i}\}$ induce a complete bipartite digraph with bipartition $\{v_{1,i}, \dots, v_{m,i}\}, \{\bar{v}_{1,i}, \dots, \bar{v}_{m,i}\}$, that is, there is a 2-cycle between $v_{j,i}$ and $\bar{v}_{j',i}$ for all $j, j' \in [m], i \in [n]$.
- For all $j, j' \in [m]$ and all $i, i' \in [n]$ such that $j < j'$ and $i \neq i'$ or $j = j'$ and $i < i'$ A' contains the arcs $v_{j,i} v_{j',i'}, v_{j,i} \bar{v}_{j',i'}, \bar{v}_{j,i} v_{j',i'}, \bar{v}_{j,i} \bar{v}_{j',i}'$.
- For all $0 \leq j < j' \leq m$ and every $i \in [n]$ A' contains the arcs $c_j v_{j',i}, c_j \bar{v}_{j',i}, c'_j v_{j',i}, c'_j \bar{v}_{j',i}$.
- For all $1 \leq j \leq j' \leq m + 1$ and all $i \in [n]$, A' contains the arcs $v_{j,i} c'_{j'}, v_{j,i} c'_{j'}, \bar{v}_{j,i} c'_{j'}, \bar{v}_{j,i} c'_{j'}$.

Now we describe how to obtain D from $D' = (V, A')$ by performing $12m$ arc-reversals. For each $j \in [m]$; let $\ell_{i_1}, \ell_{i_2}, \ell_{i_3}$ be the literals of C_j and let $u_{j,1}, u_{j,2}, u_{j,3}$ be those three vertices of $\{v_{j,1}, \bar{v}_{j,1}, v_{j,2}, \bar{v}_{j,2}, \dots, v_{j,n}, \bar{v}_{j,n}\}$ which correspond to these literals (e.g. if $C_j = (x_1 \vee \bar{x}_5 \vee x_8)$ then $u_{j,1} = v_{j,1}, u_{j,2} = \bar{v}_{j,5}, u_{j,3} = v_{j,8}$). Now we reverse the 12 arcs between $\{c_{j-1}, c'_{j-1}, c_j, c'_j\}$ and $\{u_{j,1}, u_{j,2}, u_{j,3}\}$. This concludes the construction of D (Fig. 1).

It is easy to check that D is a semicomplete digraph. We first make some observations about 2-partitions (V_1, V_2) of $V(D)$ such that $D(V_i)$ is a tournament for $i \in [2]$.

- (a) The vertices $\{c_0, c_1, \dots, c_m, c_{m+1}\} \cup \{c'_0, c'_1, \dots, c'_m, c'_{m+1}\}$ induce a complete bipartite digraph which implies that we have $\{c_0, c_1, \dots, c_m, c_{m+1}\} \subset V_i$ and we have $\{c'_0, c'_1, \dots, c'_m, c'_{m+1}\} \subset V_{3-i}$ for $i = 1$ or $i = 2$.
- (b) for each $i \in [n]$, the vertices $\{v_{1,i}, \dots, v_{m,i}\} \cup \{\bar{v}_{1,i}, \dots, \bar{v}_{m,i}\}$ induce a complete bipartite digraph which implies that we have $\{v_{1,i}, \dots, v_{m,i}\} \subset V_p$ and $\{\bar{v}_{1,i}, \dots, \bar{v}_{m,i}\} \subset V_{3-p}$ for $p = 1$ or $p = 2$.

³ We would like to thank Anders Yeo for his help in correcting our first incomplete proof of Theorem 6.1.

- (c) If we delete any vertex c_j (c'_j), $j \in [m]$ from the set V_i which contains c_j (c'_j), then the resulting semicomplete digraph $D(V_i - c_j)$ ($D(V_i - c'_j)$) has no (c_p, c_q) -path when $m + 1 \geq p > j > q \geq 0$.

Suppose that $\phi : \{x_1, \dots, x_n\} \rightarrow \{T, F\}^n$ is a truth assignment such that each clause C_j , $j \in [m]$ has at least one true literal and at least one false literal. Define the 2-partition (V_1, V_2) so that V_1 consists of precisely the vertices $\{c_0, c_1, \dots, c_m, c_{m+1}\}$ and all those literal vertices which correspond to true literals. Because of the arcs we reversed when going from D' to D we have that $D(V_1)$ contains a cycle $H = c_{m+1}c_m u_{m,j_m} c_{m-1} u_{m-1,j_{m-1}} \dots c_1 u_{1,j_1} c_0 c_{m+1}$, where u_{j_q} is one of the vertices corresponding to a true literal of C_j , $j \in [m]$. If C_1 contains two true literals, then let u_{i,j'_1} be the other and add the path $c_1 u_{i,j'_1} c_0$ to H . Because c_0 dominates all vertices of $V_1 - V(H)$ and c_{m+1} is dominated by all of these, we see that $D(V_1)$ is strong. A similar argument shows that $D(V_2)$ is strong (because at least one literal of each clause is false under ϕ).

Suppose now that (V_1, V_2) is a 2-partition of $V(D)$ such that $D(V_i)$ is a strong tournament for $i \in [2]$. By (a) we can assume w.l.o.g. that $\{c_0, c_1, \dots, c_m, c_{m+1}\} \subset V_1$ and $\{c'_0, c'_1, \dots, c'_m, c'_{m+1}\} \subset V_2$. Now (c) implies that for each $j \in [m]$ V_1 must contain at least one and at most two of the vertices corresponding to the literals of C_j . Thus if we construct a truth assignment where variable x_i is true if and only if all the vertices (by (b)) $\{v_{1,i}, \dots, v_{m,i}\}$ are in V_1 , then we obtain a truth assignment which satisfies at least one literal per clause and also has at least one false literal per clause. Thus \mathcal{F} is a 'Yes'-instance of NAE-3-SAT. \square

7. Remarks and open problems

In this paper we have considered 2-partition problems on semicomplete digraphs. These are also the digraphs of independence number⁴ $\alpha = 1$. It is well-known and easy to show that for digraphs with bounded independence number $\alpha \leq r$ we also have that the number of vertices of in-, out- or semi-degree at most k is bounded by a function $g(k, r)$. In particular, it follows from Turan's theorem that in a digraph with independence number at most α there are at most $\alpha(2k + 1)$ vertices of out-degree at most k . Furthermore, it is easy to check that if a digraph D with independence number at most r has a cycle C , then $D(V(C))$ contains a cycle of length at most $2r + 1$. Using these observations it is not hard to see that we can extend Theorem 3.4 to the following. We leave the details to the interested reader.

Theorem 7.1. *For every choice of positive integers r, k_1, k_2 there exists a polynomial algorithm that either constructs a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$ -partition of a given digraph with independence number at most r or correctly outputs that none exist.*

We cannot directly extend our proof of Theorem 5.3 to digraphs of bounded independence number: in the proof of Lemma 5.1 we use the fact that if we take any set of $k + 1$ vertices from Z , the vertex v will dominate at least two of these. The corresponding property does not necessarily hold even if we take a set of some $f(k)$ vertices from Z when we have independence number at most α .

Conjecture 7.2. *For every choice of positive integers r, k there exists a polynomial algorithm that either constructs a $(\delta^+ \geq 1, \delta^- \geq k)$ -partition of a given digraph with independence number at most r or correctly outputs that none exist.*

Problem 7.3. *Determine the complexity every pair of fixed integers $r, k_1, k_2 \geq 1$ of deciding the following for a given digraph D with independence number r :*

- whether D has a $(\delta^+ \geq k_1, \delta^- \geq k_2)$ -partition,
- whether D has a $(\delta^+ \geq k_1, \delta^0 \geq k_2)$ -partition,
- whether D has a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$ -partition.

Kühn et al. proved the following result about 2-partitions and tournaments into highly connected tournaments. We formulate it for semicomplete digraphs, since every $3r - 2$ strong semicomplete digraph contains an r -strong spanning tournament, see e.g. [3, Theorem 11.10.4].

Theorem 7.4. [8] *There exists a constant c such that every ck^7 -strong semicomplete digraph S has a 2-partition (V_1, V_2) such that $S(V_i)$ is k -strong for $i = 1, 2$.*

Instead of demanding high strong connectivity inside each set of the partition (V_1, V_2) we may also ask for a partition of a semicomplete digraph S into strongly connected semicomplete digraphs S_1, S_2 , each of which have out-degree at least a specified number k_i , $i = 1, 2$. Note that, since every strong semicomplete digraph is Hamiltonian, when $k_1 = k_2 = 1$ we just ask for a pair of complementary cycles.

⁴ The independence number α denotes the maximum cardinality of set of vertices such that there are no arcs between vertices in the set.

Problem 7.5. Does there exist a function $g(k_1, k_2)$ such that every strong semicomplete digraph S with $\delta^+(S) \geq g(k_1, k_2)$ has a 2-partition (V_1, V_2) such that $S\langle V_i \rangle$ is strong and $\delta^+(S\langle V_i \rangle) \geq k_i$ for $i = 1, 2$?

It was shown in [9] that every tournament with minimum out-degree at least 3 has a 2-partition into two strong tournaments. As every mixed graph has an orientation whose out-degree at every vertex is at least half of the original out-degree, this implies that $g(1, 1)$ exists (it is at most 6). The problem is open for all values $k_1, k_2 \geq 1$ with $k_1 + k_2 \geq 3$.

The following related result of [2] shows that the bounds in Theorems 1.4 and 1.5 are far from being best possible when k becomes large.

Theorem 7.6. There exists an absolute constant c_1 so that every semicomplete digraph S with minimum out-degree at least $2k + c_1\sqrt{k}$ has a 2-partition (V_1, V_2) so that $\delta^+(S\langle V_i \rangle) \geq k$ for $i = 1, 2$.

Theorem 7.7. There exists an absolute constant c_2 so that every semicomplete digraph S with minimum semi-degree at least $2k + c_2\sqrt{k}$ has a 2-partition (V_1, V_2) so that $\delta^0(S\langle V_i \rangle) \geq k$ for $i = 1, 2$.

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