

## ARTICLE

# Completing orientations of partially oriented graphs

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**Abstract**

We initiate a general study of what we call orientation completion problems. For a fixed class  $C$  of oriented graphs, the orientation completion problem asks whether a given partially oriented graph  $P$  can be completed to an oriented graph in  $C$  by orienting the (nonoriented) edges in  $P$ . Orientation completion problems commonly generalize several existing problems including recognition of certain classes of graphs and digraphs as well as extending representations of certain geometrically representable graphs.

We study orientation completion problems for various classes of oriented graphs, including  $k$ -arc-strong oriented graphs,  $k$ -strong oriented graphs, quasi-transitive-oriented graphs, local tournaments, acyclic local tournaments, locally transitive tournaments, locally transitive local tournaments, in-tournaments, and oriented graphs that have directed cycle factors. We show that the orientation completion problem for each of these classes is either polynomial time solvable or NP-complete. We also show that some of the NP-complete problems become polynomial time solvable when the input-oriented graphs satisfy certain extra conditions. Our results imply that the representation extension problems for proper interval graphs and for proper circular arc graphs are polynomial time solvable. The latter generalizes a previous result.

**KEYWORDS**

friendly partial oriented graph, in-tournament, local tournament, locally transitive local tournament, NP-complete, orientation completion problem, partially oriented graph, polynomial time algorithm, proper circular arc graph, proper interval graph, recognition, representation extension

## 1 | INTRODUCTION

For a fixed class  $C$  of oriented graphs, the **orientation completion problem** asks whether a given partially oriented graph can be completed to an oriented graph in  $C$  by orienting the (nonoriented) edges. Orientation completion problems commonly generalize several existing problems including recognition of certain classes of graphs and digraphs as well as extending representations of certain geometrically representable graphs.

For the class of acyclic oriented graphs, the orientation completion problem is easily seen to be polynomial time solvable; a partially oriented graph can be completed to an acyclic-oriented graph if and only if it does not contain a directed cycle.

When the fixed class  $C$  consists of all transitive-oriented graphs, the underlying graphs of the oriented graphs in  $C$  are precisely the comparability graphs. Thus if the input partially oriented graph has no oriented edges the corresponding orientation completion problem just asks whether the input is a comparability graph. So the problem is just the **recognition problem** for comparability graphs, cf. [15]. In general the orientation completion problem for this  $C$  can be solved in polynomial time using Gallai's decomposition scheme for comparability graphs, cf. [9,14].

Suppose that  $C$  is the class of all strong oriented graphs. A result from [6] implies that a partially oriented graph can be completed to a strong oriented graph if and only if it has no bridge and no directed cut. Either a bridge or a directed cut in a partially oriented graph (if any exists) can be detected in polynomial time. Hence the orientation completion problem for  $C$  is polynomial time solvable. In contrast to this, the orientation completion problem for the class of  $k$ -strong oriented graphs is NP-complete for each  $k \geq 3$  (see Theorem 15).

Orientation completion problems also generalize the problem of extending partial proper interval representation of proper interval graphs, cf. [20]. A proper interval graph is the intersection graph of a family of intervals in a line where no interval is contained in another. Suppose that an induced subgraph  $H$  of a graph  $G$  is represented by a family of intervals where no interval contains another. The **representation extension problem** for proper interval graphs asks whether it is possible to obtain a proper interval representation of  $G$  that includes the one given for  $H$ . It is well-known that a proper interval representation of  $G$  corresponds to an acyclic local tournament orientation of  $G$ , cf. [16]. Thus the representation extension problem for proper interval graphs is just the orientation completion problem for the class of acyclic local tournaments where the partial orientation of  $G$  corresponds to an interval representation of  $H$ . The representation extension problem for proper interval graphs was shown to be polynomial time solvable, cf. [20]. We provide a short proof for this in Corollary 2.

In this article, we study orientation completion problems for various well-studied classes of oriented graphs but focusing on three of them: local tournaments, locally transitive local tournaments, and acyclic local tournaments. These three classes are nested; the class of local tournaments properly contains the class of locally transitive local tournaments which in turn properly contains the class of acyclic local tournaments. We show that the complexity of the orientation problems for the three classes alternates; while the orientation completion problem is polynomial time solvable for the class of local tournaments and for the class of acyclic local tournaments, it is NP-complete for the class of locally transitive local tournaments. In fact we show that the problem remains NP-complete for the class of locally transitive tournaments (i.e. complete locally transitive local tournaments). Since, as mentioned above, the orientation completion problem for acyclic local tournaments generalizes the representation extension problem for proper interval graphs, our result on the orientation completion problem for acyclic local tournaments generalizes a result from [20]. We also show that if the input oriented graphs are restricted to being friendly (see the definition in Section 4) then the orientation completion problem for locally transitive local tournaments is polynomial time solvable. The underlying graphs

of connected locally transitive local tournaments are precisely the proper circular arc graphs [1]. Our result on the restricted orientation completion problem for locally transitive local tournaments implies that the representation problem for proper circular arc graphs is solvable in polynomial time. For various other classes of oriented graphs, including  $k$ -arc-strong oriented graphs,  $k$ -strong oriented graphs, quasi-transitive oriented graphs, in-tournaments, and oriented graphs that have directed cycle factors, we show that the corresponding orientation completion problems are either polynomial time solvable or NP-complete.

The article is organized as follows. In Section 3, we provide some terminology and notation as well as some preliminary results. We show in Section 3 that the orientation completion problem for the class of locally transitive tournaments is NP-complete. We do this by using the (polynomial) equivalence between that problem and the problem of deciding whether a given digraph has a so-called excellent ordering (to be defined in Section 3). Section 4 includes several cases where the orientation problems are shown to be polynomial time solvable. Finally, in Section 5 we make some concluding remarks and state some open problems.

## 2 | TERMINOLOGY AND PRELIMINARY RESULTS

Notation or terminology not introduced here follows [3]. We will use the notation  $[k]$  for the set  $\{1, 2, \dots, k\}$ .

Both graphs and digraphs will be considered in this article. For graphs we assume that they do not contain loops or multiple edges (i.e. they are **simple**) and for digraphs we assume they do not contain loops or two arcs joining the same pair of vertices (i.e. they are **oriented graphs**). Most of the time we will consider the so-called partially oriented graphs that may contain both edges and arcs.

A **partially oriented graph (pog)**  $P = (V, E \cup A)$  is the edge-disjoint union of a graph  $G = (V, E)$  and an oriented graph  $D = (V, A)$  on the same vertex set  $V$ ; thus no two vertices in a partially oriented graph can have two vertices joined by both an edge and an arc. In the case when  $A = \emptyset$ ,  $P$  is just the graph  $G$  and similarly when  $E = \emptyset$ ,  $P$  is the oriented graph  $D$ . When there is either an edge or an arc joining two vertices  $u, v$  in  $P$ , we say that  $u, v$  are **adjacent**, and use  $uv$  to denote the edge between  $u$  and  $v$  and use  $(u, v)$  to denote the arc from  $u$  to  $v$ . If  $(x, y)$  is an arc of  $P$  we say that  $x$  **dominates**  $y$  and call  $y$  an **out-neighbor** of  $x$  and  $x$  an **in-neighbor** of  $y$ . For a vertex  $v \in V(D)$  we denote by  $N^+(v)$  ( $N^-(v)$ ) the set of out-neighbors (in-neighbors) of  $v$ . The **underlying graph** of a pog  $P$ , denoted  $UG(P)$ , is obtained from  $P$  by suppressing the orientation of each arc. A pog  $P$  is **connected** if  $UG(P)$  is a connected graph.

Given a partially oriented graph  $P = (V, E \cup A)$ , **to complete**  $P$  means to obtain an oriented graph by orienting each edge in  $E$  (i.e. replacing each edge by an arc in one of the two possible ways). The central topic of this article is the study of the following problem: Let  $C$  be a fixed class of oriented graphs.

### ORIENTATION COMPLETION PROBLEM FOR $C$

**Instance:** A partially oriented graph  $P$ .

**Question:** Can  $P$  be completed to an oriented graph in  $C$ ?

As an example suppose that  $C$  consists of all transitive tournaments (i.e. tournaments that contain no directed triangle). Then the orientation problem for  $C$  asks whether a partially oriented graph  $P$  can be completed to a transitive tournament. Clearly, if  $UG(P)$  contains a pair of nonadjacent vertices or  $P$  contains a directed cycle, then  $P$  cannot be completed to a transitive tournament. A pair of nonadjacent vertices in  $UG(P)$  or a directed cycle in  $P$  (if one exists) can be found in polynomial time. Otherwise  $P$  can be completed to a tournament in  $C$  as shown below.

**Proposition 1.** *Let  $P = (V, A \cup E)$  be a partially oriented graph. If  $UG(P)$  does not contain a pair of nonadjacent vertices and  $P$  does not contain a directed cycle, then  $P$  can be completed to a transitive tournament*

**Proof.** *Since  $P$  does not contain a directed cycle, its vertices can be ordered  $v_1, v_2, \dots, v_n$  so that each arc  $(v_i, v_j) \in A$  satisfies that  $i < j$ . Now we can now replace each edge  $v_k v_\ell \in E$  with  $k < \ell$  by the arc  $(v_k, v_\ell)$ . Since  $UG(P)$  does not contain a pair of nonadjacent vertices, it is complete and the resulting oriented graph is a transitive tournament completion of  $P$ . ■*

A **local tournament** is an oriented graph  $D$  such that for every vertex  $v$ ,  $N^-(v)$  and  $N^+(v)$  each induces a tournament in  $D$ . If  $N^-(v)$  and  $N^+(v)$  each induces a transitive tournament for every vertex  $v$ , then  $D$  is called **locally transitive**. It follows from these definitions that every tournament is a local tournament and every transitive tournament is a locally transitive local tournament<sup>1</sup>. But neither of the converses is true, as local tournaments or locally transitive local tournaments may not be tournaments. When a local tournament is also a tournament, we call it **complete**. When a locally transitive local tournament is complete, we call it a **locally transitive tournament**.

A graph  $G = (V, E)$  is a **proper circular arc graph** if there is a family of circular arcs  $J_v, v \in V$  on a circle such that no circular arc is contained in another and for any two vertices  $u, v \in V, uv \in E$  if and only if  $J_u \cap J_v \neq \emptyset$ . Such a family of circular arcs is called a **proper circular arc representation** of  $G$ . Given a proper circular arc representation  $J_v, v \in V$  of  $G = (V, E)$ , an orientation of  $G$  can be obtained in such a way that  $(u, v)$  is an arc if and only if  $J_u$  contains the counterclockwise endpoint of  $J_v$ . It is easy to see that this orientation of  $G$  is locally transitive and hence is a local tournament. Thus every proper circular arc graph can be completed to a local tournament. The following theorem due to Skrien [23] assures that the converse is also true for connected graphs.

**Theorem 1.** [23] *Let  $G$  be a connected graph. The following statements are equivalent:*

1.  $G$  can be completed to a local tournament;
2.  $G$  can be completed to a locally transitive local tournament;
3.  $G$  is a proper circular arc graph.

Proper interval graphs form a subclass of proper circular arc graphs. A graph  $G = (V, E)$  is a **proper interval graph** if there is a family of intervals  $I_v, v \in V$  on a line such that no interval is contained in another and for any two vertices  $u, v \in V, uv \in E$  if and only if  $I_u \cap I_v \neq \emptyset$ . Such a family of intervals is called a **proper interval representation** of  $G$ . Given a proper interval representation of  $G$ , one can obtain an acyclic local tournament orientation of  $G$  in a similar way as above for proper circular arc graphs. Conversely, if  $G$  is oriented as an acyclic local tournament then a proper interval graph representation of  $G$  can be obtained from the orientation, cf. [16].

**Theorem 2.** [16] *Let  $G$  be a graph. The following statements are equivalent:*

1.  $G$  can be completed to an acyclic local tournament;
2.  $G$  is a proper interval graph.

A **round ordering** of a digraph  $D$  is a cyclic ordering  $\mathcal{O} = v_1, v_2, \dots, v_n, v_1$  of the vertices of  $D$  such that for each vertex  $v_i$  we have  $N^+(v_i) = \{v_{i+1}, \dots, v_{d^+(v_i)+i}\}$  and  $N^-(v_i) = \{v_{i-d^-(v_i)}, \dots, v_{i-1}\}$  where indices are modulo  $n$ . A digraph that has a round ordering is called **round**. Round digraphs are characterized in [18]. It is easy to see that if an oriented graph has a round ordering then it is locally transitive. The following theorem asserts that the converse is also true.

**Theorem 3.** [1] *A connected oriented graph  $D$  has a round ordering  $\mathcal{O} = v_1, v_2, \dots, v_n, v_1$  of its vertices if and only if  $D$  is a local tournament that is locally transitive. Furthermore, there is a polynomial algorithm for deciding whether a given oriented graph is round and finding a round ordering if one exists.*

Let  $D$  be an oriented graph on the vertices  $v_1, v_2, \dots, v_n$  and let  $D_1, D_2, \dots, D_n$  be vertex disjoint oriented graphs. **To substitute  $D_i$  for  $v_i$**  for each  $i \in [n]$  is to obtain a new oriented graph  $D'$  from  $D$  as follows: First replace each  $v_i$  with the digraph  $D_i$ ,  $i \in [n]$  and then for each arc  $(v_i, v_j)$  in  $D$  we add an arc  $(x, y)$   $D'$  for all  $x \in V(D_i)$  and  $y \in V(D_j)$ . We use  $D[D_1, D_2, \dots, D_n]$  to denote the new digraph  $D'$ . When  $D$  is locally transitive and  $D_i$  is a transitive tournament for each  $i \in [n]$ ,  $D'$  is also locally transitive.

A digraph  $D$  is **regular (k-regular)** if there is a positive integer  $k$  so that every vertex has in- and out-degree  $k$ . In particular, every  $k$ -regular tournament has  $2k + 1$  vertices. A tournament  $T$  is **highly regular** if it is regular and has a round ordering. Moon proved that highly regular tournaments are the basic structure frames for constructing all tournaments that are locally transitive.

**Theorem 4.** [21] *Every locally transitive tournament is obtained from a highly regular tournament  $T$  by substituting a transitive tournament for each vertex of  $T$ .*

Given a locally transitive tournament  $T'$ , we can obtain a highly regular tournament  $T$  as indicated in Theorem 4 by repeatedly identifying two vertices  $x, y$  with  $N^-(x) \setminus \{x, y\} = N^-(y) \setminus \{x, y\}$  and  $N^+(x) \setminus \{x, y\} = N^+(y) \setminus \{x, y\}$  and deleting the loop resulting from the identification, until no two such vertices remain.

Let  $\mathcal{O} = v_1, v_2, \dots, v_n, v_1$  be a cyclic ordering of the vertices of a pog  $P = (V, E \cup A)$ . An arc  $(v_i, v_j) \in A$  **dominates** an arc  $(v_s, v_t) \in A$  with respect to  $\mathcal{O}$  if the vertices of the two arcs appear in the order  $v_i, v_s, v_t, v_j$  in  $\mathcal{O}$ , where we can have  $i = s$  or  $j = t$ . An arc  $(v_i, v_j) \in A$  dominates an edge  $v_p v_q$  if both of the vertices  $v_p, v_q$  occur in the interval  $[v_i, v_j]$  from  $v_i$  to  $v_j$  according to  $\mathcal{O}$ . An arc is **maximal** with respect to  $\mathcal{O}$  if it is not dominated by any other arc. A cyclic ordering of the vertices of a pog  $P$  is **excellent** if  $P$  has no pair of arcs  $(v_i, v_j), (v_s, v_t)$  so that these vertices occur in the order  $v_i, v_t, v_s, v_j$  in the cyclic ordering, where we may have  $i = t$  or  $s = j$ .

**Lemma 1.** *Suppose  $P = (V, E \cup A)$  is pog that has an excellent cyclic ordering  $\mathcal{O} = v_1, \dots, v_n, v_1$  of its vertices. Then  $P$  can be completed to an oriented graph  $D'$  for which the same cyclic ordering  $\mathcal{O}$  is excellent.*

**Proof.** *Let  $P = (V, E \cup A)$  be a pog and let  $\mathcal{O} = v_1, v_2, \dots, v_n, v_1$  be an excellent cyclic ordering of  $D$ . Let  $a_1 = (v_{i_1}, v_{j_1}), a_2 = (v_{i_2}, v_{j_2}), \dots, a_k = (v_{i_k}, v_{j_k})$  be the maximal arcs of  $D$  with respect to  $\mathcal{O}$ . By the assumption of the lemma, for each arc  $a_r$ , every arc  $(v_p, v_q)$  for which both vertices  $v_p, v_q$  occur after in the interval  $[v_i, v_j]$  satisfy that the vertices occur in the order  $v_{i_r}, v_p, v_q, v_{j_r}$ . For each  $r \in [k]$  in increasing order and all indices  $p, q$  with  $v_{i_r}, v_p, v_q, v_{j_r}$  occurring in that order such that  $v_p v_q$  is an edge of  $P$  we orient this edge as the arc  $(v_p, v_q)$ . Let  $D^* = (V, A \cup A^*)$  be the oriented graph consisting of the original arcs and those edges which we have oriented so far. By construction of  $D^*$ ,  $\mathcal{O}$  is an excellent ordering of  $D^*$ . Hence if no edge of  $E$  is still unoriented we are done. It suffices to show that we may orient one of the remaining edges, since then the claim follows by induction on*

the number of unoriented edges. Let  $v_p v_q$  be an edge that was not oriented and orient this as  $(v_p, v_q)$ . We claim that  $\mathcal{O}$  is an excellent ordering of  $D^* \cup \{(v_p, v_q)\}$ . If not then there is an arc  $(v_a, v_b)$  of  $D^*$  such that the vertices occur in the order  $v_p, v_b, v_a, v_q$  but then the edge  $v_p v_q$  is dominated by the arc  $(v_a, v_b)$  and hence by one of the arcs  $a_1, \dots, a_k$ , contradicting that it was not oriented above. ■

**Lemma 2.** *An oriented graph  $D$  has an excellent cyclic ordering  $\mathcal{O}$  if and only if it can be extended to a round local tournament  $D^*$  by adding new arcs. In particular, every excellent ordering of  $D$  is a round ordering of  $D^*$  and conversely.*

**Proof.** *Suppose first that  $D$  can be extended to a round local tournament  $D^*$ . According to Theorem 3 there is a round ordering  $\mathcal{O} = v_1, v_2, \dots, v_n, v_1$  of  $V(D^*) = V(D)$ . We claim that this ordering is also excellent. If not, then there are arcs  $(v_i, v_j)$  and  $(v_s, v_t)$  so that the vertices occur in the order  $v_i, v_t, v_s, v_j$  according to  $\mathcal{O}$ . Since  $\mathcal{O}$  is a round ordering, we have that  $(v_i, v_t)$  and  $(v_t, v_j)$  are arcs of  $D^*$  but then the neighbors of  $v_t$  do not occur correctly according to  $\mathcal{O}$ , contradiction. So  $\mathcal{O}$  is an excellent ordering of  $D^*$  and hence also of the subdigraph  $D$ . To prove the only if part let  $\mathcal{O} = v_1, v_2, \dots, v_n, v_1$  be an excellent cyclic ordering of the oriented graph  $D$ . It suffices to observe that for every maximal arc  $(v_i, v_j)$  with respect to  $\mathcal{O}$  and any pair of nonadjacent vertices  $v_a, v_b$  in the interval  $[v_i, v_j]$  with  $v_a$  before  $v_b$  we may add the arc  $(v_a, v_b)$  and still have an excellent ordering of the resulting oriented graph. Now the claim follows by induction on the number of such nonadjacent pairs. ■*

For a given oriented graph  $D$  we denote by  $D^c$  the partially oriented complete graph obtained from  $D$  by adding an edge between each pair of nonadjacent vertices.

**Lemma 3.** *If  $D$  is a round oriented graph, then  $D^c$  can be completed to a locally transitive tournament.*

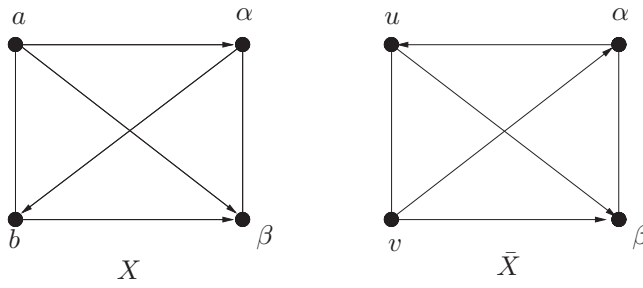
**Proof.** *We prove the statement by induction on the number of vertices in  $D$  that are not adjacent to all other vertices. By Theorem 3, the base case where there is no such vertex is true. So assume that all round oriented graphs on  $n$  vertices with at most  $k$  vertices as above can be completed to a locally transitive tournament and let  $D$  be a round digraph with  $k + 1$  vertices each of which has a nonneighbor. Let  $\mathcal{O} = v_1, v_2, \dots, v_n, v_1$  be a round ordering of  $D$ . W.l.o.g. the vertex  $v_1$  has a nonneighbor, so we have that  $v_{d^+(v_1)+2} \neq v_{n-d^-(v_1)}$ . We claim that there is no arc  $(v_p, v_q)$  with  $1 \leq q < p < n - d^-(v_1)$ . Suppose such an arc does exist. Then we have  $p > d^+(v_1) + 1$  because  $\mathcal{O}$  is excellent and we have  $q > 1$  since  $v_p$  is not adjacent to  $v_1$ . But this contradicts the fact that the vertex  $v_p$  sees its out-neighborhood as an interval just after itself according to  $\mathcal{O}$  because  $v_1$  is not-adjacent to  $v_p$ . Thus if we add all the arcs  $(v_1, v_{d^+(v_1)+2}), \dots, (v_1, v_{n-d^-(v_1)-1})$  to  $D$  the order  $\mathcal{O}$  is an excellent ordering of the resulting digraph  $D'$ . By Lemmas 1 and 2 this implies that  $D'$  can be extended to a round local tournament  $D''$  by adding new arcs. Now the claim follows by induction since  $D'$  has less vertices with nonneighbors than  $D$  does. ■*

### 3 | HARDNESS RESULTS ON EXCELLENT ORDERINGS AND COMPLETIONS TO LOCALLY TRANSITIVE TOURNAMENTS

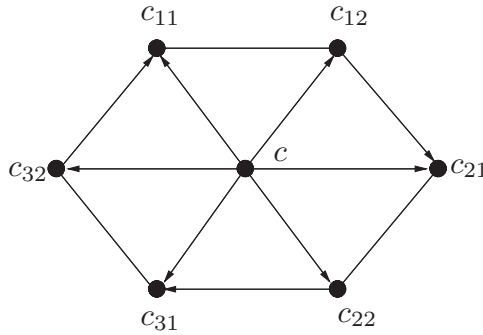
We shall use the following consequence of Lemmas 1, 2, and 3 as well as Theorem 3.

**Lemma 4.** *An oriented graph  $D$  has an excellent ordering if and only if the pog  $D^c$  has a completion to a tournament  $T$  that is locally transitive. Furthermore, given an excellent ordering of  $D$  we can construct  $T$  in polynomial time and conversely, given  $T$  we can obtain an excellent ordering of  $D$  in polynomial time.*

The following is easy to check.



**FIGURE 1** Two different labelings of the same partially oriented complete graph on four vertices. For later convenience we name these  $X, \bar{X}$ .



**FIGURE 2** A partially oriented wheel  $W$

**Proposition 2.** Each of the two labelings  $X, \bar{X}$  of the same partially oriented complete graph in Figure 1 have exactly two completions to a locally transitive tournament. For  $X$  these are obtained by orienting the two edges  $ab, \alpha\beta$  as either  $(b, a), (\beta, \alpha)$  or  $(a, b), (\alpha, \beta)$ . For  $\bar{X}$  they are obtained by orienting the two edges  $uv, \alpha\beta$  as either  $(v, u), (\alpha, \beta)$  or  $(u, v), (\beta, \alpha)$ .

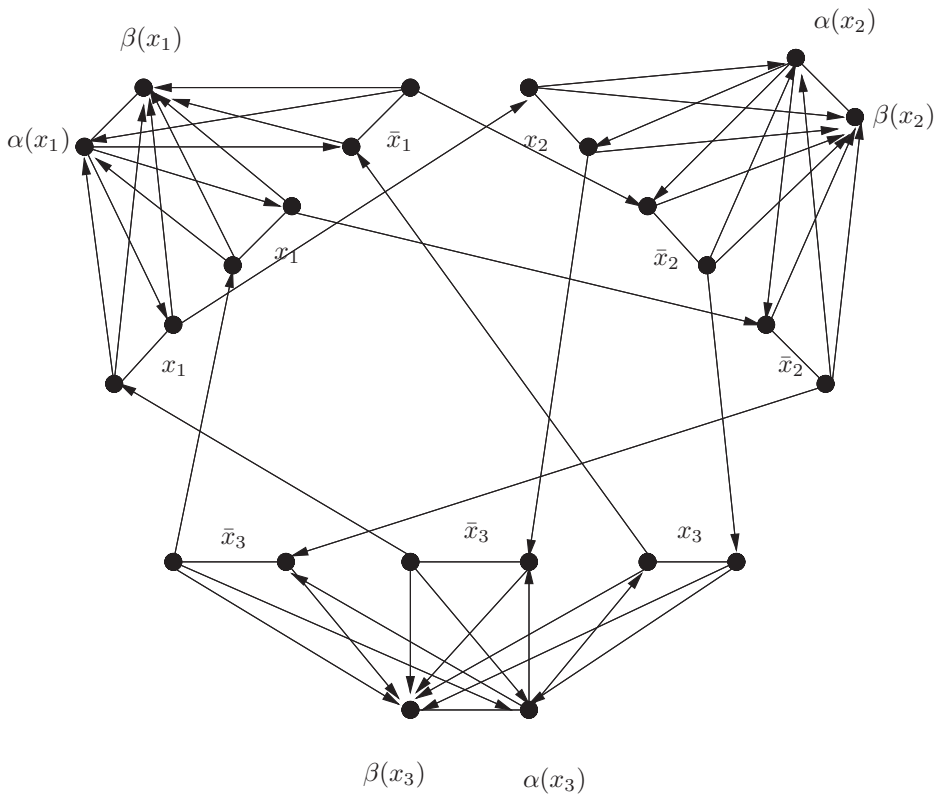
**Lemma 5.** Consider the partially oriented 6-wheel  $W$  in Figure 2. Let  $D$  be an orientation completion of  $W$ . Then  $D$  does not have an excellent ordering if and only if the three edges  $c_{11}c_{12}, c_{21}c_{22}, c_{31}c_{32}$  are oriented as  $(c_{11}, c_{12}), (c_{21}, c_{22}), (c_{31}, c_{32})$ .

**Proof.** If the three edges  $c_{11}c_{12}, c_{21}c_{22}, c_{31}c_{32}$  are oriented as  $(c_{11}, c_{12}), (c_{21}, c_{22}), (c_{31}, c_{32})$  then the vertex  $c$  has a directed 6-cycle in its out-neighborhood and hence  $D^c$  has no completion to a locally transitive tournament. By Lemma 2,  $D$  has no excellent ordering. On the other hand, if  $D$  contains at least one of arcs  $(c_{12}, c_{11}), (c_{22}, c_{21}), (c_{32}, c_{31})$ , then  $D$  is acyclic. By Proposition 1,  $D^c$  can be completed to a transitive tournament and hence by Lemma 4,  $D$  has an excellent ordering. ■

**Theorem 5.** The following polynomially equivalent problems are NP-complete.

- Deciding whether an oriented graph has an excellent ordering.
- Deciding whether a given partially oriented complete graph can be completed to a locally transitive tournament.

**Proof.** We describe polynomial reductions from 3-SAT to these problems.



**FIGURE 3** Part of the digraph  $H'(\mathcal{F})$  when  $\mathcal{F} = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3)$ . For better readability the vertices  $c_1, c_2, c_3$  are not shown.

Let  $\mathcal{F}$  be an instance of 3-SAT with variables  $x_1, x_2, \dots, x_n$  and clauses  $C_1, C_2, \dots, C_m$ , where each clause is of the form  $(\ell_1 \vee \ell_2 \vee \ell_3)$  and each  $\ell_i$  is either one of the variables  $x_j$  or the negation  $\bar{x}_j$  of such a variable.

Let  $p_i$  ( $q_i$ ) be the number of times variable  $x_i$  ( $\bar{x}_i$ ) occurs as a literal in  $\mathcal{F}$ . The enumeration of the clauses  $C_1, \dots, C_m$  induces an ordering on the occurrences of the same literal in the formula. Guided by this ordering we now construct a partially oriented graph  $H' = H'(\mathcal{F})$  as follows:

Let  $X, \bar{X}$  be as in Figure 1. For each variable  $x_i$  we form the partially oriented graph  $X_i$  from  $p_i$  copies of  $X$  and  $q_i$  copies of  $\bar{X}$  (these  $p_i + q_i$  graphs are vertex disjoint) by identifying all the  $\alpha$  vertices and all the  $\beta$  vertices and denote these identified vertices by  $\alpha(x_i), \beta(x_i)$ , respectively. Denote the  $p_i$  copies of  $a, b$  by  $a_{i,1}, \dots, a_{i,p_i}, b_{i,1}, \dots, b_{i,p_i}$  and the  $q_i$  copies of  $u, v$  by  $u_{i,1}, \dots, u_{i,q_i}, v_{i,1}, \dots, v_{i,q_i}$ .

Take  $m$  disjoint copies  $W_1, W_2, \dots, W_m$  of the partially oriented 6-wheel from Figure 2 where the vertices of  $W_i$  are denoted  $c_i, c_{11}^i, c_{12}^i, c_{21}^i, c_{22}^i, c_{31}^i, c_{32}^i$ . Make the following association between literals of  $\mathcal{F}$  and the  $W_i$ 's: If  $C_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$  we associate the vertices  $c_{j_1}^i, c_{j_2}^i$  with the literal  $\ell_{i,j}$  of  $C_i, j \in [3]$ .

Now we make the following vertex identifications. For each clause  $C_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$  we identify the vertices  $c_{11}^i, c_{12}^i, c_{21}^i, c_{22}^i, c_{31}^i, c_{32}^i$  with vertices from the union of the graphs  $X_1, \dots, X_n$  as follows: If  $\ell_{i,j} = x_r$  and this is the  $h$ 'th occurrence of variable  $x_r$  according to the induced ordering of that literal, then identify  $c_{j_1}^i$  with  $a_{r,h}$  and  $c_{j_2}^i$  with  $b_{r,h}$ . If  $\ell_{i,j} = \bar{x}_r$  and this is the  $t$ 'th occurrence of  $\bar{x}_r$ ,

according to the induced ordering of that literal, then identify  $c_{j_1}^i$  with  $u_{r,t}$  and  $c_{j_2}^i$  with  $v_{r,t}$ . Note that even after these identifications each of the subdigraphs  $W_1, \dots, W_m$  are still vertex disjoint.

Clearly we can construct  $H'$  in polynomial time from  $\mathcal{F}$ . Denote by  $H$  the oriented graph obtained from  $H'$  by deleting all (unoriented) edges. It is easy to check that the in- and out-neighborhoods of each vertex in  $H$  is acyclic.

By Lemma 4, the two problems in the statement of the theorem are polynomially equivalent, so it suffices to show that  $H$  has an excellent ordering if and only if  $\mathcal{F}$  is satisfiable.

First suppose that  $H$  has an excellent ordering. By Lemma 4 this means that the partially oriented complete graph  $H^c$  has a completion as a locally transitive tournament  $T$ . We claim that the following is a satisfying truth assignment: If the edge  $\alpha(x_i)\beta(x_i)$  is oriented as  $(\alpha(x_i), \beta(x_i))$  in  $T$ , then let  $x_i = \text{False}$  and if it is oriented as  $(\beta(x_i), \alpha(x_i))$  then let  $x_i = \text{True}$ . First observe that, by Proposition 2, this implies that for each  $i \in [n]$  the variable  $x_i$  is false if and only if each of the edges  $a_{i,j}b_{i,j}$ ,  $j \in [p_i]$  are oriented as  $(a_{i,j}, b_{i,j})$  and each of the edges  $u_{i,r}v_{i,r}$ ,  $r \in [q_i]$  are oriented as  $(v_{i,r}, u_{i,r})$ .

We now use this to show that each of the clauses of  $\mathcal{F}$  are satisfied by our truth assignment. As  $T$  is locally transitive, for each of the induced subdigraphs  $T(W_j)$ ,  $j \in [m]$  the out-neighborhood of  $c_j$  is acyclic that implies that at least one of three arcs of  $H$  that correspond to the literals of  $\mathcal{F}$  is oriented as  $(c_{j_2}, c_{j_1})$ . If this arc corresponds to the literal  $x_s$  then, by the identification rule above this is an arc of the form  $(b_{s,t}, a_{s,t})$  so the variable  $x_s$  is true and  $C_j$  is satisfied. If the arc corresponds to the literal  $\bar{x}_s$  then the identification rule implies that this is an arc of the form  $(v_{s,t}, u_{s,t})$ , implying that  $\bar{x}_s$  is true so again  $C_j$  is satisfied. Thus we have shown that  $\mathcal{F}$  is satisfiable if  $H^c$  has a locally transitive completion ( $H$  has an excellent ordering).

Now suppose that  $t : \{x_1, \dots, x_n\} \rightarrow \{\text{True}, \text{False}\}$  is a satisfying truth assignment for  $\mathcal{F}$ . We shall use this truth assignment to construct an excellent ordering of the pog  $H'$ . Recall that this is also an excellent ordering of the directed part  $H$  of  $H'$ .

We first orient the edges  $\alpha(x_1)\beta(x_1), \dots, \alpha(x_n)\beta(x_n)$  as follows: If  $x_i = \text{True}$  then orient  $\alpha(x_i)\beta(x_i)$  as  $(\beta(x_i), \alpha(x_i))$  and otherwise orient as  $(\alpha(x_i), \beta(x_i))$ . Denote by  $\hat{H}$  the resulting pog. It follows from Proposition 2, the way we made identifications between vertices of the  $W_j$ 's and variable vertices and the fact that  $t$  is a satisfying truth assignment that we can now orient all the remaining edges of  $\hat{H}$  (recall that those correspond to the literals) uniquely so that the resulting full orientation  $\vec{H}$  of  $H'$  satisfies that the in- and out-neighborhood of each vertex is still acyclic.

We now construct an excellent ordering for  $\vec{H}$ . Denote by  $A(x_i)$  ( $B(x_i)$ ),  $i \in [n]$  the set of out-neighbors (in-neighbors) of  $\alpha(x_i)$  in  $\vec{H}$ . Note that if  $t(x_i) = \text{False}$ , then we must have  $A(x_i) = \{b_{i,1}, \dots, b_{i,p_i}, u_{i,1}, \dots, u_{i,q_i}, \beta(x_i)\}$ ,  $B(x_i) = \{a_{i,1}, \dots, a_{i,p_i}, v_{i,1}, \dots, v_{i,q_i}\}$  and there is no oriented arc from  $A(x_i)$  to  $B(x_i)$ . Similarly, if  $t(x_i) = \text{True}$ , then  $A(x_i) = \{b_{i,1}, \dots, b_{i,p_i}, u_{i,1}, \dots, u_{i,q_i}\}$ ,  $B(x_i) = \{a_{i,1}, \dots, a_{i,p_i}, v_{i,1}, \dots, v_{i,q_i}, \beta(x_i)\}$  and there is no oriented arc from  $B(x_i)$  to  $A(x_i)$ .

Furthermore observe that  $\beta(x_i)$  has no out-neighbor when  $t(x_i) = \text{False}$  and precisely one out-neighbor, namely  $\alpha(x_i)$ , when  $t(x_i) = \text{True}$ . Let  $1 \leq i_1 < i_2 < \dots < i_k \leq n$  and  $1 < j_1 < j_2 < \dots < j_g \leq n$  denote the indices of the true, respectively the false variables. Consider the following cyclic ordering  $\mathcal{O}$  of  $V(\vec{H})$ :

$$\alpha(x_{i_1}), \alpha(x_{i_2}), \dots, \alpha(x_{i_k}), c_1, c_2, \dots, c_m, A(x_{i_1}), \dots, A(x_{i_k}), B(x_{j_1}), \dots, B(x_{j_g}), \alpha(x_{j_1}), \dots, \alpha(x_{j_g}), A(x_{j_1}), \dots, A(x_{j_g}), B(x_{i_1}), \dots, B(x_{i_k}), \alpha(x_{i_1}),$$

where the ordering inside each  $A(x_i), B(x_i)$  is as according to the way we listed those sets above.

We shall prove that the ordering  $\mathcal{O}$  is excellent. Suppose for contradiction that there is a pair of arcs  $(v_i, v_j)$  and  $(v_s, v_t)$  with the vertices occurring in the order  $v_i, v_t, v_s, v_j$  according to  $\mathcal{O}$ .

- We cannot have  $v_i = \alpha(x_{i_f})$  for some  $f \in [k]$  because there is no backward arc in the interval of  $\mathcal{O}$  from  $\alpha(x_{i_f})$  to (the end of)  $A(x_f)$  ( $\alpha(x_{i_f})$  is only adjacent to vertices in  $A(x_{i_f})$ ). Similarly, we cannot have  $v_i$  in the interval  $[\alpha(x_{j_1}), \alpha(x_{j_g})]$ .
- We cannot have  $v_i = c_p$  for some  $p \in [m]$  because the only arcs incident to  $c_p$  are from  $c_p$  to the six vertices that correspond to its three literals and we ordered the  $A$  and  $B$  sets and  $\alpha(x_{j_1}), \dots, \alpha(x_{j_g})$  in such a way that any arc between them goes forward in the ordering. In particular there is no backward arc with respect to the ordering in the interval

$$A(x_{i_1}), \dots, A(x_{i_k}), B(x_{j_1}), \dots, B(x_{j_g}), \alpha(x_{j_1}), \dots, \alpha(x_{j_g}), A(x_{j_1}), \dots, A(x_{j_g}), B(x_{i_1}), \dots, B(x_{i_k})$$

- We cannot have  $v_i$  in the interval  $A(x_{i_1}), \dots, A(x_{i_k})$  since all out-neighbors of those vertices are in the interval  $B(x_{i_1}), \dots, B(x_{i_k})$  and then the remark above implies the claim. Similarly, we cannot have  $v_i$  in the interval  $A(x_{j_1}), \dots, A(x_{j_g})$ .
- We cannot have  $v_i$  in the interval  $B(x_{j_1}), \dots, B(x_{j_g})$  because there are no backward arcs in the interval  $B(x_{j_1}), \dots, B(x_{j_g}), \alpha(x_{j_1}), \dots, \alpha(x_{j_g}), A(x_{j_1}), \dots, A(x_{j_g})$  and this contains all out-neighbors of such a  $v_i$ .
- Finally we cannot have  $v_i$  in the interval  $B(x_{i_1}), \dots, B(x_{i_k})$  because all arcs out of a vertex in this interval remains inside the interval  $B(x_{i_1}), \dots, B(x_{i_k}), \alpha(x_{i_1}), \alpha(x_{i_2}), \dots, \alpha(x_{i_k})$  and there is no backward arc here.

Thus we have shown that  $\mathcal{O}$  is excellent and hence, by Lemma 4, the partially oriented complete graph  $H^c$  has a completion to a locally transitive tournament. ■

## 4 | POLYNOMIAL CASES

Let  $G = (V, E)$  be a graph. The **auxiliary graph**  $G^+$  of  $G$  is defined as follows. The vertex set of  $G^+$  consists of all ordered pairs  $(u, v)$  for all  $uv \in E$  (note that every edge of  $G$  gives rise to two vertices of  $G^+$ ). Two vertices  $(u, v)$  and  $(u', v')$  of  $G^+$  are adjacent if and only if one of the following conditions holds:

- $u = u'$  and  $vv' \notin E$ ;
- $uu' \notin E$  and  $v = v'$ ;
- $u = v'$  and  $v = u'$ .

**Lemma 6.** [16] A graph  $G$  is local tournament orientable if and only if  $G^+$  is bipartite. Moreover, when  $G^+$  is bipartite, for any two vertices  $(u, v), (u', v')$  of odd distance in  $G^+$ , a local tournament of  $G$  must contain exactly one of them as an arc. In particular, the arcs of every local tournament orientation of  $G$  correspond to a color class of  $G^+$ .

**Theorem 6.** The orientation completion problem is polynomial time solvable for the class of local tournaments.

**Proof.** Let  $P = (V, A \cup E)$  be a partially oriented graph and let  $G = UG(P)$ . The arc set  $A$  corresponds to a subset  $S$  of the vertex set of  $G^+$ . According to Lemma 6  $P$  can be completed to a local tournament if and only if  $G^+$  is bipartite and  $S$  is contained in a color class of  $G^+$ . Checking whether  $G^+$  is bipartite and in the case when  $G^+$  is bipartite whether  $S$  is contained in a color class of  $G^+$  can be done in polynomial time. ■

An oriented graph  $D = (V, A)$  is called **quasi-transitive** if for any three distinct vertices  $x, y, z$ ,  $(x, y) \in A$  and  $(y, z) \in A$  together imply there is an arc between  $x$  and  $z$  in either direction, cf. [4]. In a similar way we can define an auxiliary graph for each graph  $G$  that can be used to determine whether  $G$  can be completed to a quasi-transitive oriented graph, cf. [16]. This also implies that the orientation completion problem for the class of quasi-transitive oriented graphs is solvable in polynomial time.

In contrast to Theorem 5 that shows that the orientation problem for locally transitive tournaments is NP-complete, the orientation problem for transitive oriented graphs is polynomial time solvable.

**Theorem 7.** [19] *Deciding whether a given partially oriented complete graph can be completed to a transitive oriented graph is polynomial time solvable.*

We next consider the orientation completion problem for acyclic local tournaments. By Theorem 2, a partially oriented graph  $P$  can be completed to an acyclic local tournament if and only if  $UG(P)$  is a proper interval graph. Since every (proper) interval graph is chordal, it has a perfect elimination ordering, which is a vertex ordering  $<$  such that if  $x < y < z$  and  $xy, xz$  are edges then  $yz$  is an edge.

There is a simple algorithm that determines whether a graph is a proper interval graph and completes it to an acyclic local tournament (orientation) if it is, cf. [16]. Let  $G$  be a graph and let  $<$  be a vertex ordering of  $G$ . For two ordered pairs  $(u, v), (u', v')$  of vertices of  $G$ , we say that  $(u, v)$  is **lexicographically smaller than**  $(u', v')$  **with respect to**  $<$  if either  $u < u'$  or  $u = u'$  and  $v < v'$ . The following algorithm is taken from [16].

**Lexicographic 2-Coloring:** Let  $G = (V, E)$  be a graph that is chordal and whose auxiliary graph  $G^+$  is bipartite.

1. Find a perfect elimination ordering  $<$  of  $G$ .
2. While there exist uncolored vertices in  $G^+$ : color lexicographically the smallest uncolored vertex  $(u, v)$  red and extend it to a red/blue coloring of the connected component of  $G^+$  containing  $(u, v)$ .

**Lemma 7.** [16] *Let  $G = (V, E)$  be a graph that is chordal and whose auxiliary graph  $G^+$  is bipartite. Let  $R$  be the set of all red vertices of  $G^+$  obtained by the Lexicographic 2-Coloring algorithm above. Then  $G$  is a proper interval graph if and only if  $(V, R)$  is an acyclic local tournament orientation of  $G$ .*

Rose, Tarjan and Lueker [22] developed a linear time algorithm for finding a perfect elimination ordering in a chordal graph. The algorithm is called the **Lexicographic Breadth First Search** (LBFS) that is a refinement of the classical **Breadth First Search** for graphs. Beginning with an arbitrary vertex of the graph, LBFS always labels the next vertex to be one whose neighborhood among the labeled vertices is the lexicographically largest. Hence if  $v_n, v_{n-1}, \dots, v_{i+1}$  are labeled, then the next labeled vertex  $v_i$  is one for which  $\{j : j > i \text{ and } v_i v_j \text{ is an edge}\}$  is the lexicographically largest among all unlabeled vertices. (For two distinct subsets  $S, T \subseteq \{v_n, v_{n-1}, \dots, v_{i-1}\}$ ,  $S$  is **lexicographically larger** than  $T$  if the vertex in  $S \triangle T$  with the largest subscript belongs to  $S$ .)

Suppose that  $G$  is local tournament orientable. By Lemma 6, if  $D$  is a local tournament orientation of  $G$ , then  $D$

- does not contain two arcs whose corresponding vertices are of odd distance in  $G^+$ , and
- does not contain exactly one of any two arcs whose corresponding vertices are of even distance in  $G^+$ .

A partial orientation of  $G$  is called **consentaneous** if it satisfies the two properties listed above.

**Theorem 8.** *Let  $P = (V, A \cup E)$  be a partially oriented graph. Suppose that  $UG(P)$  is a proper interval graph and  $P$  is consentaneous. Then  $P$  can be completed to an acyclic local tournament if and only if  $P$  does not contain a directed cycle.*

**Proof.** *If  $P$  contains a directed cycle then it cannot be completed to an acyclic oriented graph and hence not to an acyclic local tournament. For the other direction, we first show that  $P$  admits a perfect elimination ordering  $v_1, v_2, \dots, v_n$  such that all arcs are forward, that is, if  $(v_i, v_j)$  is an arc then  $i < j$ . To obtain such an ordering we apply a modified LBFS beginning with a vertex of outdegree 0, with preferences (in the case of ties) given to vertices having no out-neighbors among unlabeled vertices.*

*Let  $v_1, v_2, \dots, v_n$  be an ordering obtained by the modified LBFS. According to [22], it is a perfect elimination ordering. Suppose that the ordering contains backward arc. Let  $(v_i, v_j) \in A$  be a backward arc having the largest subscript  $i$ . Since  $(v_i, v_j)$  is backward, we have  $i > j$ . The choice of  $v_n$  implies  $n > i$ . Since  $i > j$ , at the time of labeling  $v_i$  the vertex  $v_j$  is an unlabeled out-neighbor of  $v_i$ . The LBFS rule ensures that  $v_i$  is a vertex having the lexicographically largest neighborhood among the vertices  $v_n, \dots, v_{i+1}$ . If the neighborhood of  $v_i$  (among the labeled vertices) is lexicographically larger than the neighborhood of  $v_j$ , some vertex  $v_\ell$  with  $\ell > i$  adjacent to  $v_i$  but not to  $v_j$  in  $P$ . The assumption that  $P$  is consentaneous implies  $(v_\ell, v_i)$  is an arc that is backward with respect to the ordering. This contradicts the choice of  $(v_i, v_j)$ . Hence  $v_i$  and  $v_j$  must have the same neighborhood among the labeled vertices. But then the rule prefers  $v_j$  to  $v_i$  for the next labeled vertex, unless  $v_j$  has an out-neighbor  $v_k$  among unlabeled vertices. Again, the choice of  $(v_i, v_j)$  and the assumption that  $P$  is consentaneous together imply that  $v_k, v_j$  have the same neighborhood among the labeled vertices. Hence  $v_k, v_i$  have the same neighborhood among the labeled vertices. The rule prefers  $v_k$  to  $v_i$  for the next labelled vertex, unless  $v_k$  has an out-neighbor among unlabeled vertices. Continuing this way, we obtain a directed cycle, which contradicts the assumption. Hence  $v_1, v_2, \dots, v_n$  is a perfect elimination ordering of  $P$  that contains no backward arcs.*

*Now we apply the Lexicographic 2-Coloring algorithm using the perfect elimination ordering to obtain a red/blue coloring of the vertices of  $UG(P)^+$ . Let  $R$  be the set of red vertices of  $UG(P)^+$  produced by the algorithm. Since the perfect elimination ordering has no backward arc from  $A$ ,  $A \subseteq R$ . Hence by Lemma 7,  $(V, R)$  is an acyclic local tournament that is an orientation completion of  $P$ . ■*

**Corollary 1.** *The orientation completion problem for the class of acyclic local tournaments is solvable in polynomial time.*

**Proof.** *Suppose that a partially oriented graph  $P = (V, A \cup E)$  is given. Denote  $G = UG(P)$ . If  $G^+$  is not bipartite, then the answer is "no." Assume that  $G^+$  is bipartite. Note that each arc in  $A$  corresponds to a vertex in  $G^+$ . If some two arcs in  $A$  correspond to two vertices in the same component of  $G^+$  but in different color classes, then  $P$  cannot be completed to a local tournament and therefore cannot be completed to an acyclic local tournament so the answer is "no." Otherwise, by letting  $A'$  consist of all color classes containing at least one vertex from  $A$ , we obtain the minimal consentaneous partial oriented graph  $P' = (V, A' \cup E')$  with  $A' \supseteq A$ . If  $P'$  contains a directed cycle, then the answer is again "no" by Lemma 6 and Theorem 8. Otherwise,  $P'$  contains no directed cycle and we can complete  $P'$  to an acyclic local tournament orientation of  $P'$  according to Theorem 8. This acyclic local tournament is also a completion of  $P$ . All these steps can be done in polynomial time. ■*

**Corollary 2.** [20] *The problem of extending partial proper interval representations of proper interval graphs is solvable in polynomial time.*

**Proof.** *We show how to reduce the problem of extending partial proper interval representations of proper interval graphs to the orientation completion problem for the class of acyclic local tournaments that is polynomial time solvable according to Corollary 1. Suppose that  $G$  is a proper interval graph*

and  $H$  is an induced subgraph of  $G$ . Given a proper interval representation  $I_v, v \in V(H)$  of  $H$  (i.e. a partial proper interval representation of  $G$ ), we obtain an orientation of  $H$  in such a way that  $(u, v)$  is an arc if and only if  $I_u$  contains the left endpoint of  $I_v$ . The oriented edges together with the remaining edges in  $G$  yield a partial orientation of  $G$ . This partial orientation of  $G$  can be completed to an acyclic local tournament if and only if the partial representation of  $H$  can be extended to a proper interval representation of  $G$ . ■

We now return to the orientation completion problem for local transitive tournaments. This problem is NP-complete in general and remains so even for complete graphs as shown in Section 3. We will show that the problem becomes polynomial time solvable if the input partially oriented graphs are all friendly (see definition below).

Let  $P = (V, A \cup E)$  be a partially oriented graph and let  $G = UG(P)$ . A triple of vertices  $x, y, z$  in  $P$  is called **bad** if

- $xyzx$  is a triangle in  $G$ ,
- the three edges in  $xyzx$  correspond to vertices from three different connected components in  $G^+$ , and
- exactly two edges in  $xyzx$  are oriented in  $P$ .

If  $P$  is consentaneous and has no bad triple then it is called **friendly**. When  $G$  is a complete graph, the three edges of each fixed triangle correspond to vertices from three connected components of  $G^+$ . Thus if a partially oriented complete graph is friendly then the arcs induce vertex disjoint tournaments.

**Lemma 8.** *Let  $P = (V, A \cup E)$  be a partially oriented complete graph. Suppose that the arc set  $A$  induces two vertex disjoint tournaments  $T'$  and  $T''$  with  $V(T') \cup V(T'') = V$  that are both locally transitive. Then  $P$  can be completed to a tournament that is also locally transitive.*

**Proof.** Since  $T'$  is locally transitive, by Theorem 4  $T'$  is obtained from a highly regular tournament  $H'$  with round ordering  $u_0, u_1, \dots, u_{2a}$  by substituting transitive tournament  $T_i$  for  $u_i$  for each  $i = 0, 1, \dots, 2a$ , that is,  $T' = H'[X_0, X_1, \dots, X_{2a}]$ . Similarly we have  $T'' = H''[Y_0, Y_1, \dots, Y_{2b}]$  where  $H''$  is a highly regular tournament with round ordering  $v_0, v_1, \dots, v_{2b}$  and each  $Y_i$  is a transitive tournament replacing  $v_i$ . Without loss of generality assume  $a \geq b$ . Let  $T = H'[(X_0 \cup Y_0), \dots, (X_b \cup Y_b), X_{b+1}, \dots, X_a, (X_{a+1} \cup Y_{b+1}), \dots, (X_{a+b} \cup Y_{2b}), X_{a+b+1}, \dots, X_{2a}]$ . It is easy to verify that the tournament  $T$  is an orientation completion of  $P$  and is locally transitive. ■

The following theorem characterizes friendly partially oriented complete graphs that can be completed to a tournament that is locally transitive.

**Theorem 9.** *Let  $P = (V, A \cup E)$  be a friendly partially oriented complete graph. Then  $P$  can be completed to a tournament that is locally transitive if and only if no directed triangle is contained in the in-neighborhood or the out-neighborhood of any vertex in  $P$ .*

**Proof.** We only prove the sufficiency as the necessity is obvious. Suppose that  $P$  has no directed triangle contained in the in-neighborhood or the out-neighborhood of any vertex. Since it is friendly,  $V$  can be partitioned into vertex disjoint tournaments such that no arc is between any two of them. Since no directed cycle is contained in the in-neighborhood or the out-neighborhood of any vertex, each tournament is locally transitive. By Lemma 8, any two such tournaments can be completed to a tournament that is locally transitive and therefore  $P$  can be completed to a tournament that is locally transitive. ■

By Theorem 9, to determine whether a friendly partially oriented complete graph can be completed to a tournament that is locally transitive, one only needs to check if it has a directed triangle contained in the in-neighborhood or the out-neighborhood of a vertex. In the case when no such a triangle exists, following the proof of Theorem 9 we can complete the given partially oriented graph to a locally transitive tournament. All these can be done in polynomial time. Hence we have the following:

**Corollary 3.** *The problem of deciding whether a friendly partially oriented complete graph can be completed to a tournament that is locally transitive and constructing such an orientation completion (if one exists) is solvable in polynomial time.*

Let  $P = (V, A \cup E)$  be a partially oriented graph and  $G = UG(P)$ . Two vertices in  $P$  are **similar** if they have the same closed neighborhood in  $G$ . An edge  $xy$  (oriented or not) in  $G$  is called **balanced** if  $x$  and  $y$  are similar and is called **unbalanced** otherwise. A **cell** of  $P$  is a maximal subgraph of pairwise similar vertices. Note that each cell is a complete subgraph and two cells are either completely adjacent or completely nonadjacent. A vertex of  $P$  is **universal** if it is adjacent to all other vertices. Clearly universal vertices are similar to each other. The cell on the universal vertices of  $G$  will be called **universal** and any other cell will be called **nonuniversal**. Each edge  $xy$  in a cell corresponds to a connected component of  $G^+$  on the two vertices  $(x, y), (y, x)$ . We shall call such a component of  $G^+$  a **thin** component and others **thick** components.

We recall some results from [17] that are described in the next two theorems and will be useful in the discussion.

**Theorem 10.** [17] *Let  $D$  be a connected local tournament. Then the following statements hold:*

- *If  $B$  is a nonuniversal cell then  $B$  induces a transitive tournament.*
- *If  $B$  is a nonuniversal cell and  $v \notin B$  is adjacent to the vertices in  $B$ , then  $v$  either completely dominates  $B$  or is completely dominated by  $B$ .*
- *If  $B$  and  $B'$  are two adjacent nonuniversal cells, then either  $B$  completely dominates  $B'$  or is completely dominated by  $B'$ .*

**Theorem 11.** [17] *Let  $G$  be a connected graph that is local tournament orientable and let  $C_1, C_2, \dots, C_k$  be the connected components of the complement  $\overline{G}$  of  $G$ . Then the following statements hold.*

- *Suppose that  $\overline{G}$  is not bipartite. Then  $\overline{G}$  has exactly one connected component (i.e.  $k = 1$ ) and  $G^+$  has exactly one thick component. Each cell of  $G$  is nonuniversal. A local tournament orientation of  $G$  is locally transitive if and only if each cell is a transitive tournament.*
- *Suppose that  $\overline{G}$  is bipartite. Then the vertices in each fixed thick component of  $G^+$  correspond to either all unbalanced edges of  $G$  within a fixed  $C_i$  (which are nonedges of  $C_i$ ) or all edges between two fixed  $C_i$  and  $C_j$  ( $i \neq j$ ). In the case when  $k = 1$  and  $G$  has at least two vertices, each cell is nonuniversal,  $G^+$  has exactly one thick component, and moreover a local tournament orientation of  $G$  is locally transitive if and only if each cell is a transitive tournament.*

**Theorem 12.** *Let  $G$  be a connected graph that is local tournament orientable. Then a friendly partial orientation  $P = (V, A \cup E)$  of  $G$  can be completed to a locally transitive local tournament if and only if  $P$  has no directed cycle contained in*

- *a nonuniversal cell, or*
- *the in-neighborhood or the out-neighborhood of a vertex.*

**Proof.** The necessity follows from Theorem 10 and the fact that a locally transitive local tournament has no directed cycle contained in the in-neighborhood or the out-neighborhood of any vertex.

For the sufficiency suppose that  $P$  has no directed cycle contained in a non-universal cell, or in the in-neighborhood or the out-neighborhood of a vertex. In view of Theorem 9, we may assume that  $G$  is not complete. This implies that  $\bar{G}$  has at least one nontrivial component and  $G^+$  has at least one thick component.

We first explain how to complete the orientation of each nonuniversal cell in  $P$ . By assumption no directed cycle is contained in any nonuniversal cell. Thus the orientation of each cell can be completed to a transitive tournament. If  $G^+$  has only one thick component then the resulting orientation of  $G$  can be further completed to a local tournament orientation of  $G$  that is locally transitive by Theorem 11. In particular, when  $\bar{G}$  is not bipartite or  $\bar{G}$  is bipartite and has exactly one connected component, the partial orientation of  $\bar{G}$  can be completed to a local tournament that is locally transitive. So we may assume that  $\bar{G}$  is bipartite and has at least two components. We may assume furthermore that each nonuniversal cell is a transitive tournament.

Let  $C_1, C_2, \dots, C_q$  be the connected components of  $\bar{G}$  where each  $C_i$  with  $1 \leq i \leq p$  is nontrivial and the rest are trivial (i.e., each consisting of a universal vertex of  $G$ ). Consider a fixed  $C_i$ . According to Theorem 11, all unbalanced edges of  $G$  within  $C_i$  correspond to the vertices of a component of  $G^+$ . Since the partial orientation is friendly and hence consentaneous by definition, either all unbalanced edges of  $G$  within  $C_i$  are oriented or none of them is. In the latter case we orient all unbalanced edges of  $G$  within  $C_i$  using any one of the two color classes of the component of  $G^+$  corresponding the edges. Thus all edges of  $G$  within  $C_i$  are now oriented. The only edges in  $G$  that are not oriented (if any) are between the components  $C_1, C_2, \dots, C_q$ .

Arbitrarily choose a vertex  $s_i \in C_i$  for each  $i = 1, 2, \dots, q$  and consider the (complete) subgraph  $K$  of  $G$  induced by  $s_1, \dots, s_q$ . Note that in each triangle of  $K$  the three edges correspond to vertices from three components of  $G^+$ . Since  $P$  is friendly, there cannot be exactly two edges in each triangle of  $K$  are oriented in  $P$ . This means that the subgraph of  $P$  induced by  $s_1, s_2, \dots, s_q$  is a friendly partial orientation of  $K$ . By Theorem 9, it can be completed to a full orientation of  $K$  that is locally transitive. Denote this tournament by  $R$ . We explain how  $R$  can guide us to orient the remaining unoriented edges (between the components).

Let  $(S_i, T_i)$  be the bipartition of  $C_i$  for each  $i = 1, 2, \dots, p$ . Without loss of generality assume  $s_i \in S_i$  for each  $i = 1, 2, \dots, p$ . Note that each  $C_i$  with  $p+1 \leq i \leq q$  consists of the single vertex  $s_i$ . Suppose that  $(s_i, s_j)$  is an arc in  $R$ . In the case when  $(s_i, s_j)$  is in  $P$ , all edges between  $C_i$  and  $C_j$  are oriented since  $P$  is consentaneous. So assume  $(s_i, s_j)$  is not in  $P$ . If  $1 \leq i, j \leq p$ , then orient the edges between  $C_i$  and  $C_j$  in such a way that  $S_i \rightarrow S_j \rightarrow T_i \rightarrow T_j \rightarrow S_i$ ; if  $1 \leq i \leq p$  and  $p+1 \leq j \leq q$ , then  $S_i \rightarrow s_i \rightarrow T_i$ ; if  $1 \leq j \leq p$  and  $p+1 \leq i \leq q$ , then  $T_j \rightarrow s_i \rightarrow S_j$ ; if  $p+1 \leq i, j \leq q$ , then  $s_i \rightarrow s_j$ . We remark that in the case when  $(s_i, s_j)$  is in  $P$ , the edges between  $C_i$  and  $C_j$  are oriented in the same way as defined above.

Let  $D$  denote the full orientation of  $G$ . We will show that  $D$  is locally transitive. According to Theorem 11 the subdigraph  $D_i$  of  $D$  induced by  $V(C_i)$  is locally transitive for each  $i$ . Hence by Theorem 3  $D_i$  has a round ordering  $u_1, u_2, \dots, u_t, u_1$ . We claim that  $S_i$  induces a transitive tournament in  $D_i$ . Suppose to the contrary that the tournament induced by  $S_i$  contains a directed triangle. Without loss of generality assume  $u_1 u_a u_b u_1$  is such a triangle. Recall that  $C_i$  is a connected component in  $\bar{G}$ . In particular, each vertex of  $D_i$  has a non-neighbor in  $D_i$ . Let  $X = N_{D_i}^+(u_b) \cap N_{D_i}^-(u_a)$  and  $Y = \{u_{a+1}, u_{a+2}, \dots, u_{b-1}\}$ . Note that the vertices of  $X$  appear consecutively in the round ordering, that is,  $X = \{u_c, u_{c+1}, \dots, u_1, \dots, u_d\}$  for some  $c, d$ . Let  $W = \{u_{b+1}, u_{b+2}, \dots, u_{c-1}\}$  and  $Z = \{u_{d+1}, u_{d+2}, \dots, u_{a-1}\}$ . The vertices in  $W \cup X$  are pairwise adjacent as they are out-neighbors of  $u_b$ . Similarly vertices in  $Z \cup X$  are pairwise adjacent as they are in-neighbors of  $u_a$ . Therefore the non-neighbors of each vertex in  $X$  can only be in  $Y$ . In particular the nonneighbors of each vertex in  $X \cap S_i$

are in  $Y \cap T_i$  since any two vertices in  $S_i$  are adjacent in  $G$ . Since  $u_a \in S_i$ , it is adjacent to every vertex in  $W \cap S_i$ . Every vertex in  $W \cap S_i$  is adjacent to all vertices in  $Y \cap T_i$  as they are out-neighbors of  $u_a$ . Similarly, every vertex in  $Z \cap S_i$  is adjacent to all vertices in  $Y \cap T_i$  as they are in-neighbors of  $u_b$ . Hence the nonneighbors of each vertex in  $Y \cap T_i$  can only be in  $X \cap S_i$ . It follows that any path in  $\overline{G}$  joining  $u_1$  can only have vertices in  $(X \cap S_i) \cup (Y \cap T_i)$ . Since  $u_a \notin (X \cap S_i) \cup (Y \cap T_i)$ , there is no path in  $\overline{G}$  joining  $u_1$  and  $u_a$ , which contradicts the fact that they are in the same connected component of  $\overline{G}$ . Therefore  $S_i$  must induce a transitive tournament in  $D$ . By symmetry,  $T_i$  also induces a transitive tournament.

Suppose to the contrary that  $D$  has a directed triangle  $xyz$  that is contained in the out-neighborhood of  $w$ . The proof is similar if  $D$  has a triangle that is contained in the in-neighborhood of a vertex. We claim each  $D_i$  (the subdigraph induced by  $V(C_i)$ ) contains at most one vertex from  $\{w, x, y, z\}$ . Suppose  $D_i$  contains two vertices (say  $x, y$ ) from the triangle. If  $x, y \in S_i$ , then  $z \in T_i$  because every vertex not in  $D_i$  either dominates both  $x, y$  or is dominated by both of them and no vertex in  $S_i$  forms a directed triangle with  $x, y$ . Since  $w$  dominates each of  $x, y, z$ , it can only be in  $D_i$ . Thus all four vertices  $w, x, y, z$  are in  $D_i$ , a contradiction to the fact that  $D_i$  is locally transitive. A similar argument shows that  $T_i$  cannot contain both  $x, y$ . By symmetry we may assume that  $x \in S_i$  and  $y \in T_i$ . The above proof implies  $z$  cannot be in  $D_i$  (as otherwise either  $z, x$  would be both in  $S_i$  or  $z, y$  would be both in  $T_i$ ). Since  $w$  dominates both  $x, y$  and no vertex not in  $D_i$  can have this property,  $w$  is also in  $D_i$ . Since  $z \notin D_i$  is dominated by both  $w, y$ , the vertex  $w$  must be in  $T_i$ . Since  $x$  is an in-neighbor of  $y$ , it is adjacent to all in-neighbors of  $y$  in  $T_i$ . Since  $w$  dominates  $y$  and  $T_i$  induces a transitive tournament in  $D$ , each out-neighbor of  $y$  is an out-neighbor of  $w$ . Since  $x$  is an out-neighbor of  $w$ , it is adjacent to all out-neighbors of  $w$  and in particular to all out-neighbors of  $y$ . Hence  $x$  is adjacent to all other vertices in  $D_i$ . This contradicts the fact that  $C_i$  is a connected component in  $\overline{G}$ . If  $D_i$  contains  $w$  and one of  $x, y, z$  (say  $x$ ). Then  $w, x$  cannot be both in  $S_i$  or both in  $T_i$  as  $z \notin D_i$  dominates  $x$  and is dominated by  $w$ . On the other hand, since  $w$  and  $x$  both dominate  $y \notin D_i$ , they must be either both in  $S_i$  or both in  $T_i$ . So  $D_i$  cannot contain both  $w$  and  $x$ . Therefore each  $D_i$  contains at most one vertex from  $\{w, x, y, z\}$ . Assume that  $w \in D_\alpha, x \in D_\beta, y \in D_\gamma$ , and  $z \in D_\ell$ . For each  $i \in \{\alpha, \beta, \gamma, \ell\}$ , let  $v_i$  be the only vertex in  $V(D_i) \cap \{w, x, y, z\}$  (which may or may not be the vertex  $s_i$  (defined above) that is also contained in  $D_i$ ). Denote by  $F$  the subdigraph induced by  $\{w, x, y, z\}$  ( $= \{v_\alpha, v_\beta, v_\gamma, v_\ell\}$ ). Suppose that  $(s_i, s_j)$  is an arc in  $D$ . Then  $(v_i, v_j)$  is an arc if either  $v_i \in S_i$  and  $v_j \in S_j$  or  $v_i \in T_i$  and  $v_j \in T_j$ ; otherwise  $(v_j, v_i)$  is an arc in  $D$ . It follows that the subdigraph induced by  $\{s_\alpha, v_\beta, v_\gamma, v_\ell\}$  is either the same as  $F$  or is the same as the one obtained from  $F$  by reversing the arcs between  $v_\alpha$  and the other three vertices. In any case it is easy to verify that the subdigraph induced by  $\{s_\alpha, v_\beta, v_\gamma, v_\ell\}$  has a directed triangle contained in the in-neighborhood of the fourth vertex or a directed triangle contained in the out-neighborhood of the fourth vertex. It follows that the subdigraph induced by  $s_\alpha, s_\beta, s_\gamma, s_\ell$  has a directed triangle contained in the in-neighborhood of the fourth vertex or a directed triangle contained in the out-neighborhood of the fourth vertex. This is a contradiction to the fact that the subdigraph induced by  $s_\alpha, s_\beta, s_\gamma, s_\ell$  is locally transitive. Therefore  $D$  is a locally transitive local tournament that is an orientation completion of the given partial orientation  $P$  of  $G$ . ■

**Corollary 4.** *The problem of deciding whether a friendly partially oriented graph can be completed to a locally transitive local tournament and obtaining such an orientation completion (if one exists) is solvable in polynomial time*

**Proof.** By Theorem 12, it suffices to check whether the input friendly partially oriented graph has a directed cycle in a nonuniversal cell or a directed cycle contained in the in-neighborhood or the out-neighborhood of a vertex. This can be done in polynomial time. When no such directed cycle is

contained in the input partially oriented graph, the constructive proof of Theorem 12 explains how to complete it to a locally transitive local tournament. ■

**Theorem 13.** *The problem of extending partial proper circular arc representations of proper circular arc graphs is solvable in polynomial time.*

**Proof.** We show how to reduce the problem of extending partial representations of proper circular arc graphs to the problem of deciding whether a friendly partially oriented graph can be completed to a locally transitive local tournament. Since the latter problem is solvable in polynomial time by Corollary 4 so is the former one. Let  $G = (V, E)$  be a proper circular arc graph and  $H$  be an induced subgraph of  $G$ . Suppose that  $J_v, v \in V(H)$  is a proper circular arc representation of  $H$  (i.e. a partial proper circular arc representation of  $G$ ). We obtain an orientation of  $H$  in such a way that  $(u, v)$  is an arc if and only if  $J_u$  contains the counterclockwise endpoint of  $J_v$ . Let  $C_1, C_2, \dots, C_r$  be the connected components of  $\overline{G}$ , each of which contains at least one vertex from  $H$ . Let  $H'$  be the subgraph of  $G$  induced by  $V(C_1) \cup V(C_2) \cup \dots \cup V(C_r)$ . We extend the orientation from  $H$  to  $H'$  as follows: We first orient all unoriented edges in the cells of  $H'$  so that each becomes transitive tournament. For each  $i = 1, 2, \dots, r$ , if any unbalanced edge within  $C_i$  is an oriented edge, then we extend the orientation of  $H$  to all unbalanced edges of  $G$  within  $C_i$  using a color class of the corresponding thick component of  $G^+$  (see Theorem 11). For each pair  $i, j$ , there must be at least one oriented edge between  $C_i$  and  $C_j$ , we extend the orientation of  $H$  to all edges of  $G$  between  $C_i, C_j$  using a color class of the corresponding thick component of  $G^+$  (see Theorem 11). Thus we obtain a partial orientation of  $H'$ , which yields a partial orientation  $P = (V, A \cup E')$  of  $G$  where  $A$  consists of all arcs in  $H'$ . The definition of  $H'$  and the orientation of  $H'$  ensures that  $P$  is consentaneous. It also implies that in any triangle of  $G$ , if the three edges in the triangle correspond to vertices from three different connected components in  $G^+$ , then they are all oriented in  $H'$  and hence in  $P$ . Therefore  $P$  is friendly. It remains to show that the proper circular arc representation of  $H$  can be extended to a proper circular arc representation of  $G$  if and only if  $P$  can be completed to a locally transitive local tournament. Suppose that  $G$  has a proper circular arc representation that extends the proper circular arc representation of  $H$ . Then we can obtain an orientation of  $G$  that is a locally transitive local tournament using the representation of  $G$  (in a similar way as above for  $H$ ). By Theorem 11 and the definition of  $P$ , the orientation of  $G$  is an orientation completion of  $P$ . Conversely, suppose that  $P$  is completed to a locally transitive local tournament. Then it is possible to extend the proper circular arc representation of  $H$  to a proper circular arc representation of  $G$  (see Theorem 3.1 in [16]). ■

## 5 | REMARKS AND OPEN PROBLEMS

A digraph  $D = (V, A)$  is  $k$ -arc-strong for some  $k > 0$  if it remains strongly connected after the deletion of any subset  $A' \subseteq A$  of at most  $k - 1$  arcs. A digraph  $D = (V, A)$  is  $k$ -strong if  $|V| \geq k + 1$  and  $D - X$  is strong for every subset  $X \subset V$  of size at most  $k - 1$ .

As we mentioned in the introduction, the orientation completion problem is polynomially solvable for the class of strong digraphs. It is natural to ask about the complexity of the problem for the class of  $k$ -arc-strong, respectively the class of  $k$ -strong digraphs.

**Theorem 14.** *The orientation completion problem is polynomially solvable for the class of  $k$ -arc-strong digraphs.*

**Proof.** (Sketch) It is well-known, see e.g. [3, Section 11.8] that there is a polynomial algorithm based on submodular flows for deciding whether a given undirected graph  $G$  has a  $k$ -arc-strong orientation.

The way this works is that first an arbitrary orientation  $D$  is assigned to the edges of  $G$  and then using a submodular flow algorithm, we can determine whether we can reorient some arcs so that the result is  $k$ -arc-strong (which by Nash-Williams orientation theorem (see e.g. [3, Theorem 11.5.3]) is the case precisely when  $G$  is  $2k$ -edge-connected). Furthermore we can find a set of arcs whose reversal results in a  $k$ -arc-strong digraph if such a set exists. This approach does not quite work for the orientation completion problem since the starting orientation that we chose has to agree with the directed arcs of the input  $pog$   $P$  and none of these may be reversed. The solution is to use a polynomial minimum cost submodular flow algorithm (see e.g. [11,12]) in which the two different orientations of an edge have (possibly different) costs assigned to them and the goal is to find a minimum cost submodular flow, corresponding to a minimum cost set of arcs of  $D$  whose reversal leads to a  $k$ -arc-strong orientation of  $G$ . Now we just have to assign cost one to any arc that is opposite to one of the arcs of  $P$  and zero to all other orientations (including both orientations of the edges of  $P$ ), implying that  $P$  has a  $k$ -arc-strong completion if and only if the min-cost feasible submodular flow has cost zero. ■

**Theorem 15.** For any natural number  $k \geq 3$  the orientation completion problem is NP-complete for the class of  $k$ -strong digraphs.

**Proof.** It was shown in [8] that for every natural number  $k \geq 3$  it is NP-complete to decide whether an undirected graph has a  $k$ -strong orientation. Hence also the more general orientation completion problem is NP-complete for the class of  $k$ -strong digraphs. ■

Thomassen [25] proved that a graph  $G$  has a 2-strong orientation if and only if  $G$  is 4-edge-connected and  $G - v$  is 2-edge-connected for every vertex  $v$ . This implies that the orientation problem for the class of 2-strong digraphs is polynomial. It is also easy to check whether a given digraph is 2-strong. However, as far as we know, the complexity of deciding whether the edges of a mixed graph<sup>2</sup> can be oriented so that the resulting digraph is 2-strong is open.

**Problem 1.** What is the complexity of the orientation completion problem for the class of 2-strong digraphs?

A digraph is an **in-tournament** if the set of in-neighbors of every vertex induces a tournament.

**Theorem 16.** The orientation completion problem is polynomial for the class of in-tournaments.

**Proof.** In [3, Section 11.1.4] it is shown how to reduce the problem of deciding whether a graph can be oriented as an in-tournament to an instance of 2-SAT. It is not difficult to see that we may extend that reduction to work when the input is a  $pog$  instead of a graph. We leave the details to the interested reader. ■

**Proposition 3.** [5] A graph is chordal if and only if it has an orientation as an acyclic in-tournament.

**Problem 2.** What is the complexity of the orientation completion problem for the class of acyclic in-tournaments?

We may also ask about the complexity of other properties of the target graph such as having a directed cycle factor, that is, a spanning collection of vertex-disjoint directed cycles. The following result shows that for this class the orientation completion problem is hard.

**Theorem 17.** It is NP-complete to decide whether  $pog$   $P$  has a completion  $D$  with a directed cycle factor.

**Proof.** It was shown in [2] that is NP-complete to decide whether a bipartite digraph  $B$  has a directed cycle-factor  $C_1, C_2, \dots, C_k$  so that no  $C_i$  has length 2. Let  $B$  be given and form the  $pog$   $P$  from  $B$  by

replacing the two arcs of each directed 2-cycle by an edge. It is easy to see that  $P$  has a completion with a directed cycle factor if and only if  $B$  has a cycle factor with no directed 2-cycle, implying the theorem. ■

Let  $\pi = \{(s_1, t_1), \dots, (s_k, t_k)\}$  be a set of  $k$  pairs of distinct vertices in a (di)graph  $H$ . A  $\pi$ -linkage in  $H$  is a collection of  $k$  disjoint paths  $R_1, \dots, R_k$  so that  $R_i$  starts in  $s_i$  and ends in  $t_i$ . For a given class  $C$  of digraphs, the  $C$ - $\pi$ -linkage completion problem is as follows: given a pog  $P = (V, E \cup A)$  and a set  $\pi$  of  $k$  terminal pairs in  $V$ ; it is possible to complete the orientation of  $P$  so that the resulting oriented graph is in  $C$  and has a  $\pi$ -linkage?

For general digraphs the  $\pi$ -linkage problem, and hence also the completion version, is NP-complete already when  $k = 2$  and even if the digraph is highly connected [10,24]. Chudnovsky et al. [7] proved that the  $\pi$ -linkage problem is polynomial for semicomplete digraphs (that is digraphs whose underlying graph is complete). This implies that the **tournament- $\pi$ -linkage completion** problem is polynomial because such a completion is possible if and only if the digraph  $\overleftrightarrow{P}$  that we obtain from  $P$  by replacing each undirected edge by a directed 2-cycle is semicomplete and has a  $\pi$ -linkage (no two paths in a linkage intersect).

**Problem 3.** *What is the complexity of the local-tournament- $\pi$ -linkage completion problem when  $k \geq 2$  is fixed?*

The following generalizes round orderings. A cyclic ordering  $\mathcal{O} = v_1, v_2, \dots, v_n, v_1$  is **nice** if there is no triple  $v_k, v_i, v_j$  where  $(v_i, v_k), (v_j, v_i)$  are arcs and the vertices occur in the given order according to  $\mathcal{O}$ . It is easy to check that every excellent cyclic ordering is also nice but the converse need not hold.

**Problem 4.** *What is the complexity of deciding whether an oriented graph  $D$  has a nice cyclic ordering?*

It was shown in [13] that when the input is just a set of cyclic triples (not coming from in-and out-neighbors of a digraph) and the goal is to find a cyclic ordering that agrees with all the triples, the problem is NP-complete.

**Problem 5.** *Does every oriented graph with a nice cyclic ordering also have an excellent cyclic ordering?*

If true, then Theorem 5 would imply that Problem 4 is NP-complete.

## ENDNOTES

<sup>1</sup> Locally transitive local tournaments are previously called **local transitive tournaments**, cf. [17]

<sup>2</sup> Mixed graphs are more general than partially oriented graphs, in that they may have two vertices joined by both an arc and an edge and sometimes also parallel arcs and edges are allowed.

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