



# Disjoint directed and undirected paths and cycles in digraphs

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## ABSTRACT

We show that the following problem is  $\mathcal{NP}$ -complete: Given a digraph  $D$  and distinct vertices  $s, t$  of  $D$ , decide whether the underlying graph of  $D$  contains two internally disjoint  $(s, t)$ -paths  $P$  and  $Q$  such that  $P$  is a directed  $(s, t)$ -path in  $D$ . Using this result we characterize those mixed linkage problems which are polynomially solvable (assuming  $\mathcal{P} \neq \mathcal{NP}$ ). This complements the classical dichotomy by FORTUNE, HOPCROFT, and WYLLIE classifying those directed linkage problems that are polynomially solvable. We furthermore show that, contrary to the case of directed linkages in digraphs, the mixed problem remains  $\mathcal{NP}$ -complete for acyclic digraphs.

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## 1. Introduction

Notation not introduced in this paper is generally consistent with [1].<sup>1</sup> In this paper we do allow (di)graphs and mixed graphs to contain loops and parallel edges/arcs. In their important paper [3] FORTUNE, HOPCROFT, and WYLLIE proved that most directed linkage problems are  $\mathcal{NP}$ -complete. To be more precise let us consider the following linkage problem. We fix a digraph  $P$ .

**Problem 1.1** (*Directed Linkage with Demand Digraph  $P$* ). Given a digraph  $D$  and an injection  $h : V(P) \rightarrow V(D)$ , decide if  $h$  extends to an injection on  $V(P) \cup A(P)$  such that, for every arc  $a = st$  of  $P$ ,  $h(a)$  is an  $(h(s), h(t))$ -path in  $D$  if  $s \neq t$  and a cycle in  $D$  containing  $h(s)$  if  $s = t$ , and, for each  $b \in A(P) - \{a\}$ ,  $V(h(a)) \cap V(h(b)) \subseteq \{h(s), h(t)\}$ .

FORTUNE et al. call this the *Fixed directed subgraph homeomorphism problem*. They have shown that, assuming  $\mathcal{P} \neq \mathcal{NP}$ , the linkage problem with demand digraph  $P$  is polynomially solvable precisely when all arcs of  $P$  have the same head or they all have the same tail. It is an easy consequence of Graph Minors XIII [7] that the undirected analogue is polynomially solvable for any fixed demand graph: It is explicitly solved there for the “linkage case” when the edges of the demand graph are independent and have two endvertices each. For the general case, we may assume that the input graph  $G$  does not contain loops, for if  $P$  contains a loop  $e$  at  $s$  and  $G$  contains a cycle  $C$  of length 1 at  $h(s)$  then we would solve the problem for  $P - e$  and  $G - E(C)$  and extend  $h(e) := C$ , and in the other case the objects of  $G$  to which we are allowed to map the edges of  $P$  cannot contain loops. We introduce a new vertex  $x_{s,f}$  for every pair  $(h(s), f)$ , where  $f$  is an edge incident in  $G$  with  $h(s)$ ,  $s \in V(P)$ , as a subdivision vertex of  $f$  adjacent to  $h(s)$ , and delete all vertices in  $h(V(P))$  from  $G$ ; there are only polynomially many ways (in terms of  $|E(G)|$ ) to map every edge  $e = st$  of  $P$  to a set of subdivision vertices  $\{x_{s,f}, x_{t,g}\}$  such that these sets are pairwise disjoint, and the solutions of the linkage problems corresponding to any such collection of 2-sets are in bijective correspondence with the homeomorphisms from  $P$  to  $G$  extending  $h$ .

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<sup>1</sup> The first edition of the book is available for free download at the URL: <http://www.cs.rhul.ac.uk/books/dbook/main.pdf>.

In this paper we consider a mixed version in the following sense: We study linkages in the underlying graph of a digraph  $D$  with the property that certain paths in these linkages must also be directed paths in  $D$ .

Let us fix a mixed graph  $P = (V', E' \cup A')$  with edges  $E'$  and arcs  $A'$ , and consider a digraph  $D$ . The underlying graph of  $D$  is denoted by  $UG(D)$ . The vertices of  $UG(D)$  are those of  $D$ , and the edges of  $UG(D)$  are obtained from the arc set of  $D$  by suppressing the orientation of each arc.<sup>2</sup> We call an injection  $h$  on  $V' \cup A' \cup E'$  a *mixed homeomorphism from  $P$  to  $D$* , if

(H1) for every vertex  $x$  of  $P$ ,  $h(x)$  is a vertex of  $D$ ,

(H2) for every arc  $a = st$  of  $P$ ,  $h(a)$  is an  $(h(s), h(t))$ -path in  $D$  if  $s \neq t$  and a cycle in  $D$  containing  $h(s)$  if  $s = t$ ,

(H3) for every edge  $e = st$  of  $P$ ,  $h(e)$  is an  $h(s)h(t)$ -path in  $UG(D)$  if  $s \neq t$  and a cycle in  $UG(D)$  containing  $h(s)$  if  $s = t$ , and

(H4) for distinct  $x, y \in A' \cup E'$ ,  $V(h(x)) \cap V(h(y)) \subseteq h(V(x))$ ,

where  $V(x)$  denotes the set of endvertices of an arc or edge from  $P$ . We omit the word “mixed” if the demand graph does not contain arcs or does not contain edges.

In the more specific setting of a digraph  $P$ , [Problem 1.1](#) can thus be reformulated as follows: Given a digraph  $D$  and an injection  $h : V(P) \rightarrow V(D)$ , decide whether  $h$  extends to a homeomorphism from  $P$  to  $D$ . – Here is our generalization.

**Problem 1.2** (*Directed Linkage with Mixed Demand Graph  $P$* ). Given a directed graph  $D$  and an injection  $h : V(P) \rightarrow V(D)$ , decide whether  $h$  extends to a mixed homeomorphism from  $P$  to  $D$ .

Besides being a common generalization of the directed and the undirected linkage problems, [Problem 1.2](#) is partially motivated by a problem on edge-disjoint trees in digraphs (see [Section 5](#)). Since [Problem 1.2](#) is polynomially solvable for an undirected demand graph, one is tempted to conjecture that the mixed case is polynomially solvable at least for those demand graphs where the subgraph formed by the arcs can be handled in polynomial time, i.e. they all have the same head or tail; this is far from being true: We will show that the problem is  $\mathcal{NP}$ -complete whenever the demand graph possesses both arcs and edges ([Section 2](#)). These results carry over to the “edge-disjoint version” of the problem, where instead of (H4) we require images of distinct arcs (edges) only to be arc-disjoint (edge-disjoint) ([Section 3](#)). When restricted to acyclic input digraphs, [Problem 1.1](#) is polynomially solvable for any demand graph [[3](#)], so that one might expect that [Problem 1.2](#) behaves better in that scenario, too; but it does not, as we shall prove, except for the just mentioned case where the demand graph is indeed a digraph or contains directed cycles ([Section 4](#)).

## 2. General mixed demand graphs

**Lemma 2.1.** *Suppose the mixed graph  $P$  is a subgraph of the mixed graph  $Q$  and that the directed linkage problem with mixed demand graph  $P$  is  $\mathcal{NP}$ -complete. Then the directed linkage problem with mixed demand graph  $Q$  is also  $\mathcal{NP}$ -complete.*

**Proof.** Given a digraph  $D$  and an injection  $g : V(P) \rightarrow V(D)$ , we construct, in polynomial time, a digraph  $H$  together with an injection  $h : V(Q) \rightarrow V(H)$  such that  $g$  extends to a mixed homeomorphism from  $P$  to  $D$  if and only if  $h$  extends to a mixed homeomorphism from  $Q$  to  $H$ . Let  $Q - P$  be the mixed graph consisting of all arcs and edges in  $Q$  but not in  $P$  together with all vertices of  $Q$  incident to these edges and arcs. Denote by  $R$  the digraph that we obtain from  $Q - P$  by replacing each edge  $e = uv$  by the path or cycle  $P_e := u \leftarrow v_e \rightarrow w$  where  $v_e$  is a new vertex unique for this edge. We may assume that  $R$  is disjoint from  $D$ . The digraph  $H$  is obtained from  $D$  by adding a copy of  $R$  in such a way that a vertex  $x$  of  $R$  that is also in  $P$  is identified with the vertex  $g(x)$  in  $D$ . Extend  $g$  to a mapping  $h$  from  $V(Q)$  to  $V(H)$  by setting  $h(v) = v$  if  $v \in V(Q) - V(P)$ . For every arc  $a$  in  $Q - P$  we denote by  $P_a$  the corresponding path of length 1 in  $H$ . See [Fig. 1](#).

If  $g$  extends to a mixed homeomorphism  $g$  from  $P$  to  $D$  then we extend the result once more by setting  $g(v) := h(v)$  for  $v \in V(Q) - V(P)$  and  $g(x) := P_x$  for  $x \in A(Q - P) \cup E(Q - P)$ . It is easy to see that this yields an extension of  $h$  which is a mixed homeomorphism from  $Q$  to  $H$ .

For the converse direction, suppose that  $h$  extends to a mixed homeomorphism from  $Q$  to  $H$ ; we denote the extension by  $h$ , too, and choose it in such a way that as many arcs and edges  $x$  of  $Q - P$  as possible are mapped to the respective  $P_x$ . First, if, for some  $a \in A(Q - P)$ , the unique arc from  $P_a$  would be contained in some  $h(x)$ , where  $x \in A(P) \cup E(P)$ , then we know that  $h(a)$  and  $h(x)$  have only objects from  $V(a) \cap V(x)$  in common; hence, if  $x \in A(P)$ , then  $x, a$  must be parallel and  $h(x) = P_a$ , and if  $x \in E(P)$  then  $V(x) = V(a)$  and  $V(h(x)) = V(a)$ . In either case, we may redefine  $h$  by setting  $h(x)$  to the old  $h(a)$  and  $h(a)$  to  $P_a$ , contradicting the choice of  $h$ . Secondly, if, for some  $e \in E(Q - P)$ , the vertex  $v_e$  would be contained in some  $h(x)$ , where  $x \in A(P) \cup E(P)$  then  $x \in E(P)$  and, as  $h(e)$  and  $h(x)$  have only objects from  $V(e) \cap V(x)$  in common, we would know that  $V(e) = V(x)$  and  $V(h(x)) = V(P_e)$ , so that redefining  $h(x)$  to  $h(e)$  and  $h(e)$  to  $P_e$  yields a new mixed homeomorphism, contradicting the choice of  $h$ . Therefore, we may assume that for all  $x \in A(P) \cup E(P)$ ,  $h(x)$  has all its edges in  $E(D)$ , and so  $h|_{V(P) \cup A(P) \cup E(P)}$  is a mixed homeomorphism from  $P$  to  $D$ .  $\square$

The following lemma is trivial to verify.

<sup>2</sup> Note that  $UG(D)$  may have parallel edges.

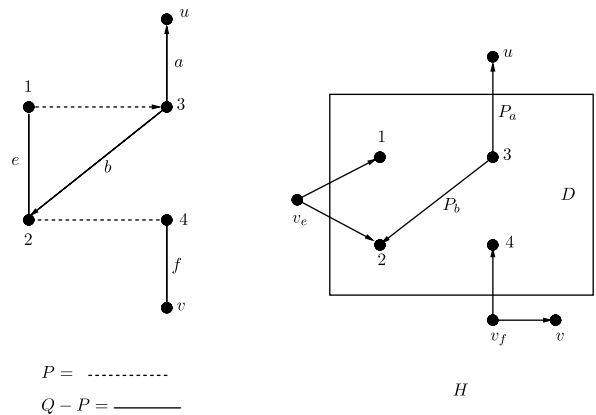


Fig. 1. Reducing the problem for  $P$  to the problem for  $Q$ .

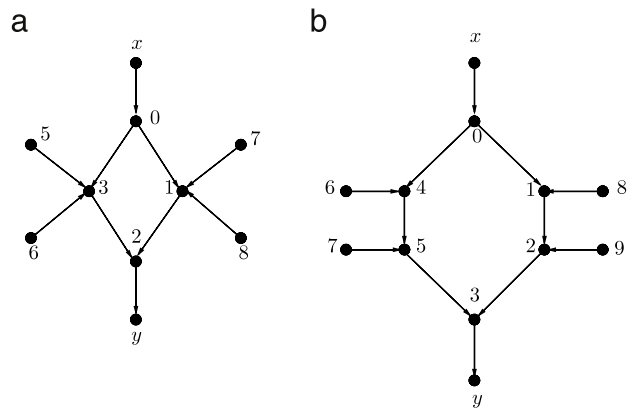


Fig. 2. Switches.

**Lemma 2.2.** Consider the digraph in Fig. 2(a). After fixing  $P$  as one of the two directed  $(x, y)$ -paths exactly one of the paths 536 and 718 will be vertex disjoint from  $P$ . In Fig. 2(b) after fixing  $Q$  as one of the two directed  $(x, y)$ -paths exactly one of the paths 6457 and 8129 will be edge-disjoint from  $Q$  and this path will also be vertex disjoint from  $Q$ .  $\square$

**Lemma 2.3.** If  $P$  consists of an arc  $st$  and an edge  $pq$  then the directed linkage problem with mixed demand graph  $P$  is  $\mathcal{NP}$ -complete.

**Proof.** The reduction is from 3-SAT (see e.g. the definition in [1, Section 18.3]). Let  $\mathcal{F} = C_1 * C_2 * \dots * C_r$  be an instance of 3-SAT with variables  $x_1, x_2, \dots, x_k$ . We construct a digraph  $D$  and an injection  $h : V(P) \rightarrow V(D)$  and show that  $h$  extends to a mixed homeomorphism from  $P$  to  $D$  if and only if  $\mathcal{F}$  is satisfiable.

For each variable  $x_i$ , we let  $H_i$  be the digraph consisting of two internally disjoint  $(z, w)$ -paths of length  $r$  (the number of clauses in  $\mathcal{F}$ ). We associate one of these paths with the literal  $x_i$  and the other with the literal  $\bar{x}_i$ .

See Fig. 4. We form a chain  $H_1 \rightarrow H_2 \rightarrow \dots \rightarrow H_k$  on the subdigraphs corresponding to each variable (see the middle of the figure,  $H_i$  corresponds to the variable  $x_i$ ). With each clause  $C_i$  we associate three switches,  $S_{i,1}, S_{i,2}, S_{i,3}$ , one for each literal it contains. The switches are chosen as in Fig. 2(a). The left paths of these switches (that is, the paths in the left-hand part of Fig. 4) all start at the vertex  $n_i$  and end at  $m_i$ . The right path of each switch is substituted for a (private) arc of  $H_i$  such that the arc is taken from the path which corresponds to  $x_i$  if the literal is  $x_i$  and from the path which corresponds to  $\bar{x}_i$  if the literal is  $\bar{x}_i$ . The substitution is shown for the clause  $C_i = x_1 + \bar{x}_2 + x_5$  in the figure. By the choice of the lengths of the paths in  $H_i$  we can make this substitution so that different arcs in  $H_i$  are substituted by different switches corresponding to several clauses, all of which contain the literal  $x_i$  or  $\bar{x}_i$ . We stack the switches in the order  $S_{1,1}S_{1,2}S_{1,3} \dots S_{r,1}S_{r,2}S_{r,3}$  as shown in the right part of the figure (Part (c) of Fig. 3 shows how to stack several switches). A two-way arc between a clause and some  $H_j$  (shown only for  $C_i$  in Fig. 4) indicates a switch that is substituted for these arcs.<sup>3</sup> We add an arc from  $n_1$  to  $w_k$  in  $H_k$  and choose vertices  $u_1, u_2, v_1, v_2$  as shown (that is,  $u_2$  plays the role of the vertex  $x$  for  $S_{r,3}$  in Fig. 3 and  $v_2$  plays the role of the vertex  $y$  in  $S_{1,1}$ ).

<sup>3</sup> Note that this is the same switch which is shown in the right-hand side of the figure!.

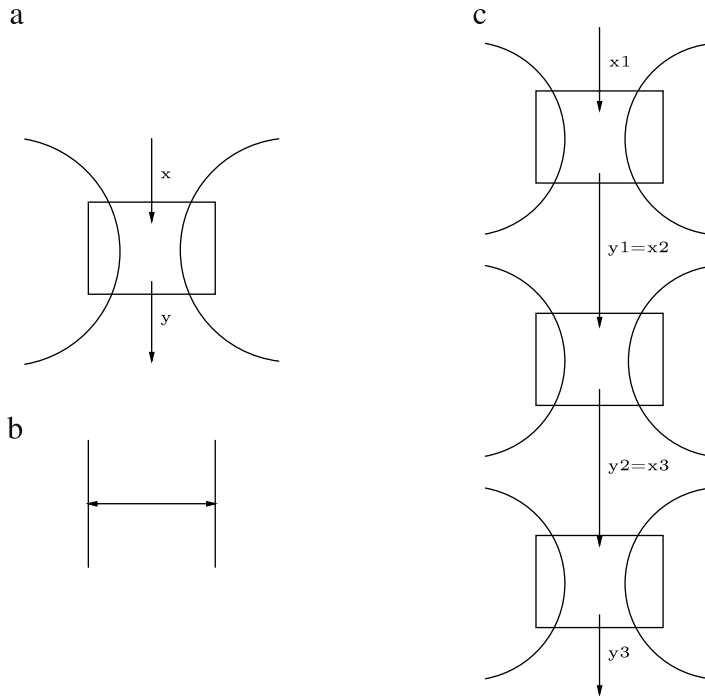


Fig. 3. Part (a) and (b) show schematic pictures of switches. Part (c) shows how to stack 3 switches on top of each other.

It is possible to do all this in such a way that the vertices are disjoint from  $\{s, t, p, q\}$ ; add these four vertices plus the arcs  $\{v_1q, pu_1, su_2, v_2t\}$ . This completes the description of  $D$ , and we take  $h$  as the identity on  $\{s, t, p, q\}$ .

Suppose first that  $h$  extends to a mixed homeomorphism from the mixed graph formed by  $st, pq$  to  $D$ . Then  $h(st)$  is a subpath or a cycle of  $D$  and since it contains neither  $u_1$  nor  $v_1$ ,  $h(st) - \{s, t\}$  is a  $(u_2, v_2)$ -path  $Q$  in  $D$ . Since  $h(st)$  and  $h(pq)$  have only vertices from  $\{s, t, p, q\}$  in common, it follows that  $h(pq) - \{p, q\}$  is a  $u_1v_1$ -path  $P$  in  $UG(D)$ , and that  $P, Q$  are disjoint and disjoint from  $\{s, t, p, q\}$ . It follows from the definition of  $D$  that  $Q$  will use all the arcs that go between two switches (i.e., those arcs that are explicitly shown in the right-hand side of Fig. 4). Hence, by Lemma 2.2, after removing the arcs of  $Q$  the only remaining way to pass through a switch  $S_{i,j}$  is to use either the right path or the left path of  $S_{i,j}$  but not both! By the construction of  $D$ , the path  $P$  must traverse the subdigraphs corresponding to the variables in the order  $H_1, H_2, \dots, H_k$  and each time  $P$  uses precisely one of the two paths in  $H_i$  (recall again that some of the arcs in  $H_i$  in Fig. 4 correspond to the right path of some switch). Let  $T$  be the truth assignment which sets  $x_i := 1$  if  $P$  uses the path corresponding to  $\bar{x}_i$  and let  $x_i := 0$  in the opposite case. We show that this is a satisfying truth assignment for  $\mathcal{F}$ .

It follows from the construction of  $D$  and the remark above on arcs used by  $Q$  that the path  $P$  contains all the vertices  $n_1, \dots, n_{r-1}, m_r$  in that order. Since each of the paths from  $n_j$  to  $m_j$  is part of a switch for every  $j = 1, \dots, r$ , we must use the left path of precisely one of these switches to go from  $n_j$  to  $m_j$ . By Lemma 2.2, every time we use a left path of a switch, the right path cannot also be used. From this we see that for each clause  $C_j, j \in \{1, \dots, r\}$ , it must be the case that at least one of the literals  $y$  (in particular the one whose left path we could use) of  $C_j$  becomes satisfied by our truth assignment. This follows because  $P$  must use the path corresponding to  $\bar{y}$  in the middle. Thus we have shown that  $\mathcal{F}$  is satisfiable.

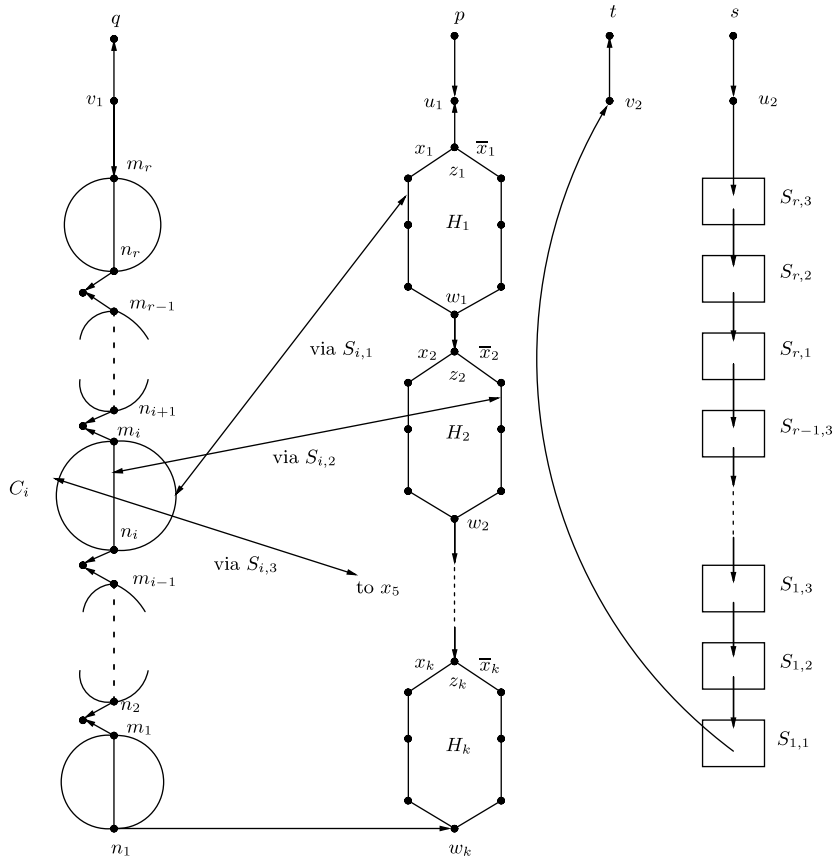
For the converse direction suppose now that  $T$  is a satisfying truth assignment for  $\mathcal{F}$ . Then for every variable  $x_i$  which is true (false) according to  $T$  we will use the subpath corresponding to  $\bar{x}_i$  ( $x_i$ ) in  $H_i$  as a part of a path  $P$  in  $UG(D)$ . For each clause  $C_j$  we can fix one literal which is true and use the left path of the switch that corresponds to that literal as another part of  $P$  (that path cannot be blocked by the way we chose subpaths inside the  $H_i$ 's). By Lemma 2.2, we can extend all these parts to a  $u_1v_1$ -path  $P$  in  $UG(D)$  and find a  $(u_2, v_2)$ -path  $Q$  in  $D$  disjoint from  $P$ . By setting  $h(st) = s \rightarrow Q \rightarrow t$  and  $h(pq) := p \rightarrow P \rightarrow q$  we thus extend  $h$  to a mixed homeomorphism from the mixed graph formed by  $st, pq$  to  $D$ .  $\square$

**Theorem 2.4.** *The directed linkage problem with mixed demand graph  $P$  is polynomially solvable in the following cases:*

- (a)  $P$  has no arcs, or
- (b)  $P$  has no edges and there is some vertex  $s$  in  $V(P)$  that is either the head of all arcs in  $P$  or the tail of all arcs in  $P$ .

*The problem is  $\mathcal{NP}$ -complete for all other mixed demand graphs  $P$ .*

**Proof.** If there are no arcs in  $P$  this follows from [7] (recall our remark in the introduction). In the case (b) when  $P$  has no edges we have the directed subgraph homeomorphism problem and now the claim follows from [3, Theorem 1]. If  $P$  has at least one arc and at least one edge then it follows from Lemmas 2.1 and 2.3 that the directed linkage problem with mixed demand graph  $P$  is  $\mathcal{NP}$ -complete.  $\square$



**Fig. 4.** A schematic picture of the digraph  $D$ . The clause  $C_i$  is given by  $C_i = x_1 + \bar{x}_2 + x_5$ . The two paths in  $H_i$  are drawn as undirected paths because some edges correspond to one of the two undirected paths in a switch. Note that all vertices are distinct, except that we may have  $|\{s, t, p, q\}| \in \{1, 2, 3, 4\}$ .

**Corollary 2.5.** *The following problems are all  $\mathcal{NP}$ -complete for digraphs. Decide whether for a given input digraph  $D$  and vertices  $s \neq t, p \neq q$ , there exist*

- (a) a cycle  $B$  in  $D$  containing  $s$  and a cycle  $C$  in  $UG(D)$  containing  $p$  with  $V(B) \cap V(C) \subseteq \{s\} \cap \{p\}$ ;
- (b) an  $(s, t)$ -path  $P$  in  $D$  and a cycle  $C$  in  $UG(D)$  containing  $p$  with  $V(P) \cap V(C) \subseteq \{s, t\} \cap \{p\}$ ;
- (c) a cycle  $B$  in  $D$  containing  $s$  and a  $pq$ -path  $Q$  in  $UG(D)$  with  $V(B) \cap V(Q) \subseteq \{s\} \cap \{p, q\}$ ;
- (d) an  $(s, t)$ -path  $P$  in  $D$  and a  $pq$ -path  $Q$  in  $UG(D)$  with  $V(P) \cap V(Q) \subseteq \{s, t\} \cap \{p, q\}$ .  $\square$

Note that, by Corollary 2.5(d), it is already  $\mathcal{NP}$ -complete to decide whether the underlying graph of a given digraph  $D$  contains two internally disjoint  $(s, t)$ -paths  $P_1, P_2$  so that  $P_1$  is also a path in  $D$ . This is in contrast to the directed and the undirected versions where the same problem is solvable in linear time (e.g. by flows [1, Section 5.4]).

### 3. Edge-disjoint paths

Let us take a brief look on the edge-disjoint version of Problem 1.2. (We leave the proofs to the reader.) A *weak mixed homeomorphism* from some mixed demand graph  $P$  to a digraph  $D$  is an injection  $h$  from  $V(P) \cup A(P) \cup E(P)$  satisfying (H1), (H2), (H3), and

(H4') for distinct  $x, y \in A(P) \cup E(P), E(h(x)) \cap E(h(y)) = \emptyset$ .

The *Weak directed linkage problem with mixed demand graph  $P$*  asks, given a digraph  $D$  and an injection  $h : V(P) \rightarrow V(D)$ , to decide whether  $h$  extends to a weak mixed homeomorphism from  $P$  to  $D$ .

If  $P$  is an undirected graph, the problem is polynomially solvable by first applying a line graph construction and then a polynomial time algorithm for the linkage problem with an undirected demand graph. If  $P$  is a directed graph we simply apply the so-called *vertex splitting procedure* [1, Section 4.2.4], that is, we replace every vertex  $x$  of  $D$  not in  $h(V(P))$  by two new vertices  $x', x''$  such that every arc of the kind  $ux (xv)$  is replaced by an arc  $ux' (x''v)$  and we add an arc  $x'x''$  from  $x'$  to  $x''$ ; then  $h$  extends to a homeomorphism from  $P$  to this new graph if and only if  $h$  extends to a weak homeomorphism from  $P$  to  $D$ . — There seems to be no convenient reduction for the case of a mixed demand graph which would prove the  $\mathcal{NP}$ -completeness for the remaining mixed cases; however, by replacing the gadget in Fig. 2(a) by the one in Fig. 2(b), it is easy

to check that the proof of [Lemma 2.3](#) works for weak mixed homeomorphisms when we replace the gadget in [Fig. 2\(a\)](#) by the one in [Fig. 2\(b\)](#). Moreover, the construction in [Lemma 2.1](#) can also be employed to prove the corresponding statement for the weak problem, and thus we get

**Theorem 3.1.** *The weak directed linkage problem with mixed demand graph  $P$  is polynomially solvable in the following cases:*

- (a)  $P$  has no arcs, or
- (b)  $P$  has no edges and there is some vertex  $s$  in  $V(P)$  that is either the head of all arcs in  $P$  or the tail of all arcs in  $P$ .

The problem is  $\mathcal{NP}$ -complete for all other mixed demand graphs  $P$ .

#### 4. The case of acyclic digraphs

For the case when the input graph is an acyclic digraph and we ask for internally vertex disjoint paths, FORTUNE, HOPCROFT, and WYLLIE [3] showed, by extending an argument of PERL and SHILOACH [6], that the directed linkage problem is solvable in polynomial time for any demand digraph:

**Theorem 4.1** ([3]). *For every demand digraph  $P$ , there is a polynomial time algorithm to decide if a given directed acyclic digraph  $D$  and an injection  $h : V(P) \rightarrow V(D)$  admits an extension of  $h$  to a homeomorphism from  $P$  to  $D$ .*

**Theorem 4.2.** *The directed linkage problem with mixed demand graph  $P$  is polynomially solvable for acyclic digraphs in the following cases:*

- (a)  $P$  has no arcs, or
- (b)  $P$  has no edges, or
- (c)  $P$  contains a directed cycle.

The problem is  $\mathcal{NP}$ -complete for all other mixed demand graphs.

**Proof.** If  $P$  contains no arcs then the result follows from [Theorem 2.4](#), and if it contains no edges then it follows from [Theorem 4.1](#). The answer to the decision problem is obviously “no” if  $P$  contains a directed cycle, hence we may assume that it does not.

The digraph constructed in the proof of [Lemma 2.3](#) is acyclic, so that at first sight it seems that we could proceed along the same lines as before. However, the proof of [Lemma 2.1](#) does not carry over to the situation when  $Q$  does not have a directed cycle and  $D$  in that proof is acyclic, because the graph  $H$  may contain cycles. This technicality is irrelevant, since if  $Q$  contains at least one arc, at least one edge and no directed cycles, and  $P$  is the subgraph of  $Q$  formed by  $a$  and  $e$ , and  $D$  is obtained from an arbitrary instance of 3-SAT as in the proof of [Lemma 2.3](#), then the digraph  $H$  obtained from such a  $D$  as in the proof of [Lemma 2.1](#) must be acyclic. Hence, in this particular situation, the proof of [Lemma 2.1](#) carries over so that the problem is  $\mathcal{NP}$ -complete in this case.  $\square$

#### 5. Concluding remarks and open problems

The construction in [Lemma 2.3](#) yields an even stronger complexity result.

**Theorem 5.1.** *The directed linkage problem with a mixed demand graph consisting of an arc and an edge is  $\mathcal{NP}$ -complete even when restricted to acyclic digraphs with  $\Delta(D) \leq 3$  and  $\Delta(UG(D)) \leq 4$ .  $\square$*

Hence the question arises whether the directed linkage problem with a mixed demand graph consisting of one arc  $st$  and one edge  $pq$  is still  $\mathcal{NP}$ -complete when restricted to acyclic digraphs  $D$  with  $\Delta(D) \leq 2$  where the degree of every vertex  $x$  in  $UG(D)$  is at most 3.

One might look at versions of [Problem 1.2](#) where only a subset of vertices of the demand graph is fixed; in the extremal case of prescribing no vertex at all we would simply ask for any mixed homeomorphism from the demand graph  $P$  to the input digraph  $D$ . When the demand graph consists of an arc  $st$  and an edge  $pq$  which are not both loops then it is obvious that the problem of deciding whether there is a homeomorphism from  $P$  to  $D$  is polynomially solvable – in each case we could do it by checking the one-arc-deleted subgraphs or their underlying undirected graphs for another arc or a cycle, possibly through a specified vertex. However, we do not know the full complexity status when the demand graph consists of two disjoint loops, one of them being an arc, the other one being an edge.

**Problem 5.2.** Given a digraph  $D$ , decide whether there exists a cycle  $B$  in  $D$  and a cycle  $C$  in  $UG(D)$  such that  $V(B) \cap V(C) = \emptyset$ .

It is  $\mathcal{NP}$ -complete to decide whether for a given vertex  $p$  of an input digraph  $D$  there exists a cycle  $B$  in  $D$  and a cycle  $C$  in  $UG(D)$  containing  $p$  such that  $V(B) \cap V(C) = \emptyset$ ; this follows again from our proof of [Lemma 2.3](#), as every cycle occurring in the digraph  $D$  constructed there *must* pass through  $s = t$  (the same argument shows that it is  $\mathcal{NP}$ -complete to decide whether for two given vertices  $p, q$  of an input digraph  $D$  there exists a cycle  $B$  in  $D$  such that there is a  $pq$ -path in  $UG(D) - V(B)$ ). On the other hand, we know that deciding whether a given digraph or graph contains two disjoint cycles is polynomially

solvable, by the results of [5] (digraphs), and [4] (graphs). Very recently we proved that **Problem 5.2** is polynomially solvable in the case when  $D$  is strongly connected [2]. Based on this result we conjecture that **Problem 5.2** is polynomially solvable.

Also the complexity status of the variant of **Problem 5.2** where we impose  $|V(B) \cap V(C)| = 1$  instead of  $V(B) \cap V(C) = \emptyset$  remains open.

Our original motivation for studying mixed linkage problems comes from the following problem,<sup>4</sup> posed by THOMASSÉ. An *out-branching* with root  $s$  in a digraph  $D$  is a spanning subdigraph of  $D$  without cycles in which each vertex distinct from  $s$  has precisely one arc entering it and  $s$  has no arcs entering it.

**Problem 5.3.** Find a characterization of those pairs  $(D, s)$ , where  $D$  is a digraph and  $s \in V(D)$ , such that  $D$  contains an out-branching with root  $s$  and  $UG(D) - E(B)$  is connected.

The complexity status of this problem is open, too. If we had an out-branching  $B$  as above and any spanning tree  $T$  of  $UG(D)$  then we could solve immediately all weak mixed linkage problems on  $D$  where the demand graph consists of an arc  $st$ ,  $t \neq s$ , and an edge  $e = pq$ ,  $p \neq q$ .

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<sup>4</sup> Listed as Problem 19 on the URL <http://www.cs.elte.hu/egres/> (last visited August 17, 2009).