Problems on digraphs

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Abstract

The purpose of this file, which will continue to be updated, is to collect problems and observations that can help the students and colleagues participating in the collaboration between Jørgen Bang-Jensen and the group of Jin Yan at Shandong University to find interesting problems to work on.

1 Preliminaries

All the notation and results not mentioned here, as well as many of the results and problems below can be found in [5, 6].

2 Highly connected spanning oriented subdigraphs of digraphs

Conjecture 2.1 (Jackson and Thomassen)**.** Every $2k$-strong digraph contains a spanning oriented k-strong subdigraph. That is, we can delete one arc from every 2-cycle so that the resulting spanning oriented subdigraph is k-strong.

True for $k = 1$ by the theorem of Boesch and Tindell [20] on strong orientations of mixed graphs. For semicomplete we have the following stronger conjecture.

Conjecture 2.2 (Bang-Jensen and Jordán)**.** Every $(2k-1)$-strong semicomplete digraph with at least $2k+1$ vertices contains a spanning k-strong subtournament.

- This would be best possible as there are $(2k-2)$-strong semicomplete digraphs with no spanning k-strong subtournament
- True for $k = 2$ (Bang-Jensen and Jordán [13]).
- True for $k = 3$ [31](Wang, Qi and Yan)
- Best known bound for general k is that every $(3k-2)$-strong semicomplete digraph has a spanning k-strong tournament (Guo [25])

Recall the following very useful result of Bang-Jensen, Gutin and Yeo

Theorem 2.3. *Let $D = S[H_1, H_2, \ldots, H_s]$ be a composition of a strong digraph S on at least 3 vertices and arbitrary digraphs $H_1, H_2, \ldots, H_s$, each with at least one vertex. Let $D_0 = S[I_1, I_2, \ldots, I_s]$ be obtained from D by deleting every arc which lies inside some H_i. Then D_0 is k-strong if and only if D is k-strong.*

Theorem 2.4 (Bang-Jensen and Huang)**.** *[10] Let D be a quasi-transitive digraph.*

(a) If D is not strong, then there exists a transitive oriented graph T such that $D = T[H_1, H_2, \ldots, H_t]$, where each H_i is a strongly connected quasi-transitive digraph.

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(b) If D is strong, then there exists a strong semicomplete digraph S such that $D = S[Q_1, Q_2, \dots, Q_s]$ where each Q_i is either a single vertex or a non-strong quasi-transitive digraph. Furthermore, if S has a directed 2-cycle $s_i s_j s_i$, then the digraphs Q_i and Q_j each have just one vertex. (see digraphs book edition Lemma 4.8.4)

Armed with these results we can no prove the following:
Let $\mathcal{H}$ be the class of compositions of the form $D = S[H_1, H_2, \dots, H_s]$ where S is a strong semicomplete digraph with at least 3 vertices and with the property that if $s_i s_j s_i$ is a 2-cycle in S, then each of H_i and H_j have size one.

Let $f(k)$ denote the smallest function such that every $f(k)$-strong semicomplete digraph contains a k-strong tournament.
By a result of Guo [25]we know that $f(k) \leq 3k - 2$

Theorem 2.5. *Every $f(k)$-strong digraph from $\mathcal{H}$ contains a spanning oriented subdigraph which is k-strong.*

Proof:

- Let $D = S[H_1, H_2, \dots, H_s]$ be a digraph from $\mathcal{H}$ which is $f(k)$-strong.
- By Theorem 2.3 the digraph $D_0 = S[I_1, I_2, \dots, I_s]$ which is obtained by deleting all eachs inside each H_i is also $f(k)$-strong.
- Now let $D^* = S[TT_{k_1}, TT_{k_2}, \dots, TT_{k_s}]$ be obtained from D_0 by adding the arcs of a transitive tournament TT_{k_i} on $k_i = |V(H_i)|$ vertices to the vertices of I_i.
- Then D^* is $f(k)$-strong so by the definition of $f(k)$, D^* contains a spanning k-strong tournament $T^* = T[TT_{k_1}, TT_{k_2}, \dots, TT_{k_s}]$ where T is a spanning subtournament of S.
- By Theorem 2.3 the extended tournament $T_0 = T[[I_1, I_2, \dots, I_s]$ is also k-strong and it is a spanning oriented subdigraph of D.

□

Note how we used the assumption that is $s_i s_j s_i$ is a 2-cycle of S, then H_i and H_j have size 1. This means that T will contain precisely one of the arcs $s_i s_j$ and $s_j s_i$. If one or both of H_i, H_j had been larger then we cannot (a priori) guarantee that T contains either all arcs from $V(H_i)$ to $V(H_j)$ or all arcs in the other direction.

Corollary 2.6. *The following are consequences of the above theorem.*

- *Every $(3k-2)$-strong quasi-transitive digraph contains a spanning oriented quasi-transitive digraph which is k-strong.*
- *Every 3-strong quasi-transitive digraph contains a spanning oriented 2-strong quasi-transitive digraph.*
- *Every 5-strong quasi-transitive digraph contains a spanning oriented 3-strong quasi-transitive digraph.*

Problem 2.7. Can we extend the idea above to general extended semicomplete digraphs, that is, S may contain a 2-cycle $s_i s_j s_i$ where at least one of both of H_i, H_j contain more than one vertex?

Problem 2.8. Find a (small) function $f(k)$ such that every $f(k)$-strong (semicomplete) split digraph D contains a spanning oriented subdigraph which is a k-strong spliot digraph.

Perhaps we can use the same ideas as Kühn et al. used to prove the Kelly conjecture for very large regular tournaments to attack the following problem.

Problem 2.9. Does there exist an integer n_0 such that every $2k$-strong semicomplete digraph on $n \geq n_0$ vertices contains a spanning k-strong tournament?

If this works, then we may try more general digraphs with minimum semi-degree at close to n.

3 Hamiltonian cycles

Theorem 3.1 (Bang-Jensen and Thomassen). *[17] The following problem is NP-complete: Given a tournament T and k arcs $u_1v_1, \dots, u_kv_k$ of T; Does T have a hamiltonian cycle which uses all of the arcs $u_1v_1, \dots, u_kv_k$?*

Corollary 3.2. *The hamiltonian cycle problem and the hamiltonian path problem is NP-complete for split digraphs.*

Proof: Let $T = (V, A), u_1v_1, \dots, u_kv_k$ be an instance of the problem above and form a split digraph D by adding k new vertices $z_1, z_2, \dots, z_k$ and the arcs u_iz_i, z_iv_i for $i \in [k]$. Then D has a hamiltionian cycle if and only if T has a hamiltonian cycle which uses all of the arcs $u_1v_1, \dots, u_kv_k$. To see that also the hamiltonian path problem is NP-complete, just replace z_1 by two vertices z_1, z'_1 and the arcs u_1z_1, z_1v_1 by $u_1z'_1, z_1v_1$. Now the new split digraph has a hamiltonian path if and only if it has such a path from z_1 to z'_1 and this is if and only if T has a hamiltonian cycle through the arcs $u_1v_1, \dots, u_kv_k$. $\square$

Theorem 3.3 (Bang-Jensen, Gutin, Yeo). *[7] The hamiltonian cycle problem is polynomial for semicomplete multipartite digraphs*

As every semicomplete split digraph is a semicomplete multipartite multipartite digraph (in which only one partite set is non-trivial) we get the following corollarty.

Corollary 3.4. *The hamiltonian cycle problem is polynomial for semicomple split digraphs*

A cycle subdigraph $\mathcal{F} = C_1 \cup \ldots \cup C_k$ of a digraph is **irreducible** if D has no other cycle subdigraph $\mathcal{F}'$ which covers all the vertices of $\mathcal{F}$ (and possibly more) but has fewer than k cycles.

In [32] Yeo showed that in a strong semicomplete multipartite digraph the irreducible cycle factors have a lot of structure. In particular, if $\mathcal{F} = C_1 \cup \ldots \cup C_k$ is an irreducible cycle factor and uv is an arc from $V(C_j)$ to $V(C_i)$ for some $1 \leq i < j \leq k$ then the successor u^+ of u on C_j and the predecessor v^- of v on C_i belong to the same partite set. If D is a semicomplete split digraph there is one one choice for this partite set so perhaps we can use this to obtain a nice characterization of hamiltonian semicomplete split digraphs.

Note that being strong and having a cycle factor is **not** sufficient to guarantee that a semicomplete split digraph is hamiltionian. To see this consider the digraph with vertices a, b, c, d and the arcs $ab, ac, ba, bc, cd, dc, db$.

Question 3.5. Can we construct larger examples and also with larger connectivity? For general semicomplete multipartite digraphs it is known that even very large strong connectivity and a cycle factor is not sufficient to guarantee a hamiltonian cycle (see Figure 5.6 in [5]).

4 Minimum strong spanning subdigraphs

MINIMUM STRONG SPANNING SUBDIGRAPH PROBLEM (MSSS PROBLEM)
Input: A strong digraph $D = (V, A)$
Output: A strong spanning subdigraph $D' = (V, A')$ of D with $|A'|$ minimium amond all strong spanning subdigraphs of D.

Clearly the MSSS problem is NP-complete for general digraphs as $|A'| = |V|$ if and only if D has a hamiltonian cycle. So the MSSS problem can only be polynomial for a class of digraphs if the hamiltonian cycle problem is also polynomial for that class.

Remark 4.1. Every strong digraph $D = (V, A)$ on n vertices contains a spanning strong subdigraph with at most $2(n - 1)$ arcs: just take the union of an out-branching and an in-branching rooted at the same vertes s of V. This gives a strong spanning subdigraph of D as every vertex v can reach s via the in-branching and be reached from s via the out-branching.

Problem 4.2. Is there a polynomial algorithm for the MSSS problem for semicomplete split digraphs?

For extended semicomplete digraphs, quasi-transitive digraphs and for semicomplete bipartite digraphs there are nice characterizations of the size of a minimum strong spanning subdigraph and algorithms to find such digraphs [18] (I may add more references later).

Problem 4.3. Can we characterize the size of a minimum strong spanning subdigraph of a semicomplete split digraph?

5 Hamiltonian connectivity and long paths

A digraph $D = (V, A)$ is **hamiltonian-connected** if it has an (x, y)-hamiltonian path for everty choice of distinct vertices x, y.

- Every 4-strong semicomplete digraph is hamiltonian-connected [29]
- Every 4-strong locally semicomplete digraph is hamiltonian-connected [24]
- There is no k so that every k-strong semicomplete split digraph is hamiltonian connected: Let the semicomplete split digraph $D = (V_1 \cup V_2, A)$ consist of an independent set V_1 of size $k + 2$, a set of vertices V_2 of size k which induces a complete digraph and let every vertex of V_1 form a 2-cycle with every vertex of V_2. Then D is k-strong but it has no hamiltonian path.

Question 5.1. Is there some connectivity condition along with an extra assumption which excludes the example above and which will guarantee that a semicomplete split digraph is hamiltonian-connected?

It is clearly necessary that D has a hamiltonian path so one condition is that $|V_1| \leq |V_2| + 1$. Note that, since semicomplete split digraphs are semicomplete multipartite digraphs, a semicomplete split digraph has a hamiltonian path if and only if it has a 1-path-cycle factor.

Theorem 5.2. *[14] There exists a polynomial algorithm for deciding the existence of a hamiltonian path from x to y in a semicomplete digraph.*

Problem 5.3. Is there a polynomial algorithm for finding a longest (x, y)-path in a semicomplete digraph?

Even the longest path with end vertices x, y is open.

Problem 5.4. Is there a polynomial algorithm for deciding the existence of an (x, y)-hamiltionian path in locally semicomplete digraphs?

Problem 5.5. Is there a polynomial algorithm for deciding the existence of an (x, y)-hamiltionian path in quasi-transitive digraphs?

Remark 5.6. Note that no degree of strong connectivity will guarantee that a quasi-transitive digraph is hamiltonian-connected. This follows from the fact that there are k-strong non-hamiltonian quasi-transitive digraphs for every positive integer k.

Theorem 5.7. *[26] An extended semicomplete digraph $D = S[I_1, I_2, \ldots, I_s]$ is hamiltonian if and only if it is strong and has a cycle factor.*

Problem 5.8. Can we characterize hamiltonian-connected extended semicomplete digraphs?

6 Supereulerian digraphs

A digraph $D = (V, A)$ is **supereulerian** if it has a spanning closed trail or equivalently it contains a spanning eulerian subdigraph $D' = (V, A')$, $A' \subseteq A$. In particular, every hamiltonian digraph is supereulerian.

Theorem 6.1 (B-J, Bessy)**.** *It is NP-complete to decide whether a given digraph is supereulerian.*

Problem 6.2. Find classes of digraphs for which we can decide supereulerianity in polynomial time.

- It is polynomial for in-semicomplete digraphs as such a digraph is hamiltonian if and only if it is strongly connected.
- A semicomplete multipartite digraph is supereulerian if and only if it is strongly connected and has an eulerian factor (B-J and Maddaloni).
- What is the complexity for general split digraphs?

Definition 6.3. An **eulerian factor** of a digraph $D = (V, A)$ is a spanning subdigraph $D' = (V, A')$ such that each connected component of D' is non-trivial and is supereulerian. Equivalently A' is a set of arc-disjoint cycles such that every vertex is on at least one of these.

Proposition 6.4. *[2] One can check whether a given digraph D has an eulerian factor and produce such a factor in polynomial time when it exists.*

Observe that (while this holds for semicomplete split digraphs) it is NOT true that a split digraph is supereulerian if and only if it is strong and has an eulerian factor. To see this consider the semicomplete digraph D with vertices a, b, c, d and arcs ab, bc, cd, da, ac, bd and add two new vertices x, y and arcs bx, xa, dy, yc. This has a cycle factor (and hence an eulerian factor) but no spanning closed trail.

Theorem 6.5 (Chvátal and Erdös)**.** *Every k-connected graph with independence number at most k is hamiltonian*

Proposition 6.6. *Every graph G on at least 3 vertices whose edge-connectivity $\lambda(G)$ is at least its independence number $\alpha(G)$ has a spanning closed trail i.e. it is supereulerian*

Conjecture 6.7 (Bang-Jensen and Thomassé)**.** Every digraph D with $\lambda(D) \geq \alpha(D)$ is supereulerian

- true for symmetric digraphs
- true for semicomplete multipartite digraphs.
- true for quasi-transitive digraphs ($xy, yz \in A$ implies x and z adjacent).
- open whether there is any K such that $\lambda(D) \geq K$ and $\alpha(D) = 2$ implies that D is supereulerian.

Problem 6.8. Prove Conjecture 6.7 for split digraphs.

Let $\mathcal{D}_2$ denote the class of digraphs which can be obtained from two disjoint semicomplete digraphs D_1, D_2 by adding some arcs between these. Then $\alpha(D) \leq 2$ for every digraph in $\mathcal{D}_2$ so Conjecture 6.7 says that such digraphs are supereulerian as soon as the have arc-connectivity at least 2. Even this case is difficult but we can use results from [8] to obtain a proof when each of D_1, D_2 are 2-arc-strong:

Definition 6.9. A digraph $D = (V, A)$ is **eulerian-connected** if has a spanning (x, y)-trail for every pair of distinct vertices x, y. Here, as always, spanning means that every vertex of D is included.

Theorem 6.10. *[8] Every 2-arc-strong semicomplete digraph is eulerian-connected*

Using this result it follows that every digraph obtained from two 2-arc-strong semicomplete digraphs D_1 and D_2 by adding an arc xy from D_1 to D_2 and an arc uv in the other direction where x, y, u, v are all disjoint is supereulerian: Let T_1 be a spanning (v, x)-trail in D_1 and let T_2 be a spanning (y, u)-trail in D_2. Then T_1xyT_2uv is a spanning closed trail in D.

So there is only one case left when both D_1 and D_2 are 2-arc-strong and D is also 2-arc-strong, namely that there are exactly two arcs in both directions between D_1 and D_2 and they share the same endvertex in one of the sides. For example when they are joined by two 2-cycles xux and yuy where x and y are in D_1 and u is in D_2 and xy is an arc Now the result follows by using a hamiltonian cycle in each of D_1, D_2 and one of the two 2-cycles.

In order to confirm Conjecture 6.7 for all digraphs in $\mathcal{D}_2$ there are still the cases when D is 2-arc-strong but at least one of D_1, D_2 is not 2-arc-strong This case is MUCH harder, especially when none of D_1, D_2 are strong. I think we looked at it when we wrote the paper [8] but we did not quite finish it so this could be a challenging problem for you to try.

For this project the following results may be useful.

Theorem 6.11. *[8] Every 2-arc-strong semicomplete digraph has a spanning eulerian subdigraph which avoids any prescribed arc xy.*

Theorem 6.10 follows directly from the following result.

Theorem 6.12. *[8] Let D be a strong semicomplete digraph and let x, y be distinct vertices of D. If there are two arc-disjoint (x, y)-paths in D, then D has a spanning (x, y)-trail which can be chosen to avoid an arc from y to x.*

In [8] an important decomposition of semicomplete digraphs that are strong but not 2-arc-strong is given. We call this a **nice decomposition** (see Section 3 in [8]). This decomposition was used to classify all those arcs of a semicomplete digraph D that are contained in a spanning eulerian subdigraph of D (Theorem 23 in [8]). This classification could be useful when studying Conjecture **??** for the digraphs in the class $\mathcal{D}_2$.

In the view of Theorem 6.10 the following two problems seem interesting to solve

Problem 6.13. Classify those pairs x, y of distinct vertices in a semicomplete digraph D for which D has a spanning (x, y)-trail.

Problem 6.14. Find a polynomial algorithm for deciding whether a given semicomplete digraph is eulerian-connected.

Remark 6.15. Note the similarity between eulerian factors and cycle factors (spanning collection of disjoint cycles): an eulerian factor is a spanning collection of arc-disjoint cycles. So being supereulerian is a relaxation of being eulerian and of being hamiltonian.

Theorem 6.16 (Fraise,Thomassen)**.** *Every $(k + 1)$-strong tournament T has a hamiltonian cycle avoiding any prescribed set of k arcs of T.*

Conjecture 6.17 (B-J,Havet and Yeo)**.** For every semicomplete digraph $D = (V, A)$ with $\lambda(D) \geq k+1$ and every subset $A' \subset A$ with $|A'| = k$, the digraph $D - A'$ is supereulerian.

It follows from results in [4] that the conjecture holds for $k \leq 4$.

7 Linkages

k-LINKAGE PROBLEM
Input: D and distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of D.
Question: Does G contain disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i is an (s_i, t_i)-path for $i \in [k]$?

A digraph D is k**-linked** if it contains disjoint paths $P_1, \ldots, P_k$, where P_i is an (s_i, t_i)-path for every choice of distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$.

Theorem 7.1. *[1] Every 5-strong semicomplete is 2-linked and this is best possible even for tournaments*

Since the project started we have proved the following for split digraphs.

Theorem 7.2. *[21] Every 5-strong semicomplete split digraph is 2-linked.*

This is best possible as already for semicomplete digraphs 5-strong is the best bound.

Theorem 7.3. *[21] Every 6-strong split digraph is 2-linked.*

Question 7.4. Does there exist a 5-strong split digraph which is not 2-linked?

Theorem 7.5. *[21] Every 6-strong semicomplete multipartite digraph is 2-linked.*

Question 7.6. Does there exist a 5-strong semicomplete multipartite digraph which is not 2-linked?

7.1 Unilateral k-linkages

An $[x, y]$-path in a digraph is a directed path with endvertices x, y.

UNILATERAL k-LINKAGE PROBLEM
Input: D and distinct vertices $s_1, s_2, \dots, s_k, t_1, t_2, \dots, t_k$ of D.
Question: Does G contain disjoint paths $P_1, P_2, \dots, P_k$ such that P_i is an $[s_i, t_i]$-path for $i \in [k]$?

(1) NP-complete already for $k = 2$

(2) Trivial for tournaments as every tournament is a 'yes'-instance

(3) Does there exists a K such that every K-strong digraph D is unilaterally k-linked?

7.2 Mixed 2-linkages

MIXED 2-LINKAGE PROBLEM
Input: D and distinct vertices s_1, s_2, t_1, t_2 of D.
Question: Does D contain disjoint paths P_1, P_2 such that P_i is an (s_i, t_i)-path in $UG(D)$ and P_1 is a directed (s_1, t_1)-path in D?

(a) The problem is NP-complete.

(b) Does there exists a K such that every K-strong digraph D has a mixed 2-linkage?

7.3 Better than Menger?

Let D be 10^{10}-strong and let $A = \{a_1, a_2, \dots, a_{100}\}$ and $B = \{b_1, b_2, \dots, b_{100}\}$ be disjoint vertex sets in D

There are 100 factorial possible linkages from A to B.

Mengers theorem say that one of these exists (w.l.o.g) from a_i to b_i for $i \in [100]$.

Question 7.7 (Thomassen 2026)**.** Are we guaranteed one more linking?

7.4 An undirected variant

Let $s_1, s_2, s_3, t_1, t_2, t_3$ be distinct vertices in an undirected graph G.

There are 6 possible linkages from $\{s_1, s_2, s_3\}$ to $\{t_1, t_2, t_3\}$.

Question 7.8 (Thomassen 2026)**.** It is true that if G is 6-connected, then it has at least two of these 6 linkages?

5-connected is not enough as there are planar counterexamples.

7.5 Restricted linkages in Tournaments

Let T be a tournament and let $s_1, s_2, s_3, t_1, t_2, t_3$ be distinct vertices of T.

(a) Which connectivity do we need T to have to ensure that it has at least 2 of the 6 possible linkages from $\{s_1, s_2, s_3\}$ to $\{t_1, t_2, t_3\}$?

(b) Which connectivity do we need T to have to ensure that it has at least 3 of the 6 possible linkages from $\{s_1, s_2, s_3\}$ to $\{t_1, t_2, t_3\}$?

(c) Which connectivity do we need T to have to ensure that it has at least 4 of the 6 possible linkages from $\{s_1, s_2, s_3\}$ to $\{t_1, t_2, t_3\}$?

(d) Which connectivity do we need T to have to ensure that it has at least 5 of the 6 possible linkages from $\{s_1, s_2, s_3\}$ to $\{t_1, t_2, t_3\}$?

(e) Which connectivity do we need T to have to ensure that it has all 6 possible linkages from $\{s_1, s_2, s_3\}$ to $\{t_1, t_2, t_3\}$?

7.6 Many paths in linkages in tournaments

SEVERAL PATHS k-LINKAGE PROBLEM IN TOURNAMENTS
Input: distinct vertices $s_1, \dots, s_k, t_1, \dots, t_k$ in a tournament T and positive integers $r_1, \dots, r_k$ whose sum is K
Question: Does T have K paths $P_{(1,1)}, \dots, P_{(1,r_1)}, P_{(2,1)}, \dots, P_{(2,r_2)}, \dots, P_{(k,1)}, \dots, P_{(k,r_k)}$ so that $P_{(i,1)}, \dots, P_{(i,r_i)}$ are internally disjoint (s_i, t_i)-paths and $P_{(i,a)}$ is disjoint from $P_{(j,b)}$ when $i \neq j$.

Note that we can reduce the SEVERAL PATHS k-LINKAGE PROBLEM IN TOURNAMENTS to the K-linkage problem is tournaments. To see this we make r_i copies $s_i^1, \dots, s_i^{r_i}$ of s_i and r_i copies $t_i^1, \dots, t_i^{r_i}$ of t_i for each $i \in [k]$ and join the vertices $s_i^1, \dots, s_i^{r_i}$ $(t_i^1, \dots, t_i^{r_i})$ by the arcs of a transitive tournament on r_i vertices (these arcs play no role in the problem, they are just there to ensure we get a tournament). Call the new tournament T^*. Now it is easy to see that T^* has K disjoint paths so that these link the vertices $s_1^1, \dots, s_1^{r_1}, \dots, s_k^1, \dots, s_k^{r_k}$ to the vertices $t_1^1, \dots, t_1^{r_1}, \dots, t_k^1, \dots, t_k^{r_k}$ if and only if T has the desired paths $P_{1,1}, \dots, P_{(1,r_1)}, P_{(2,1)}, \dots, P_{(2,r_2)}, \dots, P_{(k,1)}, \dots, P_{(k,r_k)}$

Hence the following is a consequence of the polynomial algorithm for the K-linkage problem is tournaments in [22].

Theorem 7.9. *The* SEVERAL PATHS K-LINKAGE PROBLEM IN TOURNAMENTS *is polynomial for every fixed integer K.*

We say that a tournament T is **$(r_1, r_2, \dots, r_k)$-K-linked** if it has paths as above.

Question 7.10. Can we find precise connectivity bounds for small values of K? E.g. is every 7-strong tournament $(2, 1)$-3-linked?

8 Arc reversals

Let $a_k(D)$ denote the minimum number of new arcs we need to add to the digraph $D = (V, A)$ to obtain a k-strong superdigraph $D' = (V, A \cup F)$. Note that $a_k(D) < \infty$ precisely when D has at least $k + 1$ vertices as the complete digraph on $k + 1$ vertices is k-strong.

Let $r_k(D)$ denote the minimum number of arcs we need to reverse in $D = (V, A)$ so that the new digraph that we obtain is k-strong. Note that r_k may not exist and calculation r_k is NP-hard already for $k \geq 2$ [9, 23]. The number $r_1(D)$ is finite if and only if the underlying graph of D has a strong orientation and we can calculate $r_1(D)$ using submodular flows (so this is not trivial).

Also note that we always have $a_k(D) \leq r_k(D)$ for every digraph.

Conjecture 8.1 (Bang-Jensen, 1994)**.** For every tournament $T = (V, A)$ on at least $2k + 1$ vertices we have $r_k(T) \leq \binom{k+1}{2}$

The following result supports the conjecture.

Theorem 8.2. *[11] For every tournament $T = (V, A)$ on at least $2k+1$ vertices we have $a_k(T) \leq \binom{k+1}{2}$*

The following result shows that to prove Conjecture 8.1 we only have to consider tournaments on n vertices where $2k + 1 \leq n \leq 3k - 2$

Theorem 8.3. *[12] For every semicomplete digraph D on at least $3k-1$ vertices we have $a_K(D) = r_k(D)$ and $3k-1$ is best possible.*

9 2-partitions of semicomplete digraphs

A 2-partition (V_1, V_2) of a set V consists of two sets V_1, V_2 such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.

Theorem 9.1. *[28] There exists a function $f(k)$ such that every $f(k)$-strong tournament $T = (V, A)$ has a 2-partition (V_1, V_2) so that thw two induced subtournaments $T[V_1], T[V_2]$ and the bipartite subtournament $T[V_1, V_2]$ which is induced by the arcs between V_1 and V_2 are k-strong.*

Problem 9.2. What is the complexity of deciding whether a semicomplete digraph $D = (V, A)$ has a 2-partition (V_1, V_2) such that all three digraphs $D[V_1], D[V_2], D[V_1, V_2]$ are strong.

If we ignore $D[V_1, V_2]$ the problem is polynomial.

Theorem 9.3. *[15] There exists a polynomial algorith for deciding whether a semicomplete digraph $D = (V, A)$ has a 2-partition (V_1, V_2) so that each of $D[V_1]$ and $D[V_2]$ are strong.*

Theorem 9.4. *[3] It is NP-complete to decide whether a semicomplete digraph $D = (V, A)$ has a 2-partition (V_1, V_2) such that each of $D[V_1]$ and $D[V_2]$ are strong tournaments (so all 2-cycles must belong to $D[V_1, V_2]$).*

10 Acyclic dichromatic number of a digraph

The **(acyclic) dichromatic number** $\vec{\chi}_{\mathrm{a}}(D)$ of a digraph D is the minimum number of sets in a partition $V_1, \ldots, V_k$ of V into k subsets so that the induced subdigraph $D[V_i]$ is acyclic for each $i \in [k]$ (and each of the bipartite induced subdigraphs $D[V_i, V_j]$ is acyclic for each $1 \leq i < j \leq k$).

So the acyclic dichromatics number is a refinement of dichromatic number: $\vec{\chi}_{\mathrm{a}}(D) \geq \vec{\chi}(D)$ and it may become much larger.

It is similar to the definition of acyclic chromatic number of undirected graphs (any two sets induce a forrest)

Since an acyclic dicolouring is more restrictive than a dicolouring, the complexity may change for some digraphs.

ACYCLIC k-DICOLOURABILITY
Input: An oriented graph D.
Question: Does D admit an acyclic k-dicolouring?

While the dichromatic number of a bipartite digraph is clearly at most 2 the acyclic dichromatic number can be arbitrarily large.

Theorem 10.1. *[16] For every positive integer k there exists a bipartite digraph B such that $\vec{\chi}_{\mathrm{a}}(B) = k$*

Theorem 10.2. *[16] One can decide in poilynomial time whether a tournament has acyclic dichromatic number 2*

Conjecture 10.3. [16] For every positive integer k it is polynomial to decide whether a tournament has acyclic dichromatic number at most k

Problem 10.4. [16] Is there a polynomial algorithm for deciding whether a bipartite tournament has acyclic dichromatic number 2?

Theorem 10.5. *[16] It is NP-complete to decide whether a split digraph has acyclic dichromatic number at most 3*

Problem 10.6. [16] Is there a polynomial algorithm for deciding whether a split digraph has acyclic dichromatic number 2?

A digraph D is a **locally tournament digraph** if $D[N^+[v]]$ and $D[N^-[v]]$ are tournaments for every vertex v of D.

Problem 10.7. What is the complexity of deciding $\vec{\chi}_{\mathrm{a}}(D) \leq 2$ when D is a locally tournament digraph?

Another generalization of tournaments is the class of **multipartite tournaments** which are orientations of complete multipartite graphs.

Problem 10.8. What is the complexity of deciding whether a multipartite tournament has acyclic dichromatic number at most 2?

11 Decomposing k-arc-strong tournaments

One of the most famous conjectures on digraphs is Kelly's conjecture

Conjecture 11.1 (Kelly 1968)**.** The arc set of every k-regular tournament T can be decomposed into k-arc-disjoint hamiltonian cycles.

Theorem 11.2. *[27] The Kelly conjecture is true for very large tournaments, that is, there exist a* k_0 *so that for every* $k \geq k_0$ *every* k*-regular tournament has an arc-decomposition into* k *hamiltonian cycles.*

Theorem 11.3. *[19] Every 2-strong tournament has an arc-decomposition into two spanning strong subdigraphs.*

Theorem 11.4. *[19] Every* k*-arc-strong tournament* $T = (V, A)$ *with minimum semi-degree* $37k$ *has an arc-decomposition* $A = A_1 \cup A_2 \cup \ldots \cup A_k$ *so that each of the digraphs* $D = (V, A_i)$, $i \in [k]$ *are strong.*

These results inspired Bang-Jensen and Yeo to make the following conjecture which would imply Conjecture 11.1.

Conjecture 11.5. [19] Every k-arc-strong tournament $T = (V, A)$ has an arc-decomposition $A = A_1 \cup A_2 \cup \ldots \cup A_k$ so that each of the digraphs $D = (V, A_i)$, $i \in [k]$ are strong.

Note that each of the following conjectures are special cases of Conjecture

Conjecture 11.6. [19] For every $k \geq 2$ and positive integers s_1, s_2 with $s_1 + s_2 = k$ every k-arc-strong tournament $T = (V, A)$ has a 2-partition (A_1, A_2) of its arc set so that $D = (V, A_i)$ is s_i-arc-strong for $i = 1, 2$.

Conjecture 11.7. [19] Every k-arc-strong tournament $T = (V, A)$ contains a spanning strong subdigraph $D' = (V, A')$ so that the digraph $D = (V, A \setminus A')$ is $(k-1)$-arc-strong.

Thomassen [30] constructed tournament with arbitrarily high arc-strong connectivity with no two arc-disjoint hamiltonian cycles.

Problem 11.8. Is there a polynomial algorithm for deciding whether a tournament $T = (V, A)$ has a hamiltonian cycle C so that $D = (V, A \setminus A(C))$ is strong?

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